Problem 2 Let \( f \) be given by \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) converging in some disk. Assume \( a_1 \neq 0 \) and fix \( r \) sufficiently small so that \( |a_1| > \sum_{n=2}^{\infty} n|a_n|r^{n-1} \). Show that \( f \) is one-to-one on \( |z| < r \).

**Proof.** Suppose \( z_1, z_2 \in \mathbb{C}, |z_1| < r, |z_2| < r \), and \( z_1 \neq z_2 \), i.e., \( |z_1 - z_2| \neq 0 \). It follows that
\[
|f(z_1) - f(z_2)| = \left| \sum_{n=1}^{\infty} a_n (z_1^n - z_2^n) \right|
\]
\[
= |z_1 - z_2| \left| a_1 + \sum_{n=2}^{\infty} a_n \left( z_1^{n-1} + z_1^{n-2} z_2 + z_1^{n-3} z_2^2 + \cdots + z_1 z_2^{n-2} + z_2^{n-1} \right) \right|
\]
\[
\geq |z_1 - z_2| \left| a_1 - \sum_{n=2}^{\infty} \left( a_n \sum_{k=1}^{n} z_1^{n-k} z_2^{k-1} \right) \right|
\]
\[
\geq |z_1 - z_2| \left| a_1 - \sum_{n=2}^{\infty} \left( a_n \sum_{k=1}^{n} r^{n-k} \right) \right|
\]
\[
= |z_1 - z_2| \left( |a_1| - \sum_{n=2}^{\infty} (|a_n| n r^{k-1}) \right)
\]
\[
> |z_1 - z_2| (|a_n| - |a_n|) = 0.
\]

Hence, if \( z_1 \neq z_2 \), then \( f(z_1) \neq f(z_2) \), so \( f \) is one-to-one on \( |z| < r \). ■

Problem 3 Does there exist a one-to-one analytic map \( g \) from \( 0 < |z| < 1 \) onto an annular region \( \Omega \) whose boundary \( B \) consists of two disjoint simple closed curves?

**Solution.** First, since \( g \) maps onto a bounded region, \( g \) is bounded, so \( g \) has a removable singularity at \( z_0 = 0 \). If it had a pole, then \( \lim_{z \to z_0} |g(z)| = \infty \) and \( g \) would be unbounded. Similarly, if it had an essential singularity then Picard’s theorem would also guarantee the unboundedness of \( g \).

Now, \( g \) has a removable singularity at \( z_0 = 0 \), so \( g \) extends analytically to a function \( G \) on \( |z| < 1 \). Since analytic functions map open sets to open sets, \( G(|z| < 1) = \Omega \cup \{G(0)\} \) is open. If \( G(0) \) were in either \( B \) or \( \mathbb{C} \setminus (\Omega \cup B) \), then \( G(0) \) would be either a boundary or isolated point of \( G(|z| < 1) \), respectively, which contradicts the fact that this set is open. Thus, we must have that \( G(0) \in \Omega \).

Finally, since \( g \) is one-to-one, there is a point \( z_1 \) in the punctured disk which maps to \( G(0) \). Let \( U, V \subset \{|z| < 1\} \) be disjoint disks containing \( 0 \) and \( z_1 \), respectively. Then \( G(U) \cap G(V) \) is a nontrivial open subset of \( \Omega \), contradicting the fact that \( g \) was one-to-one in the disk. Thus, no such map exists. ■
Problem 4 Use the residue problem to verify that

(a) \( \int_0^\infty \frac{1}{1+x^n} \, dx = \frac{\pi}{n \sin \frac{\pi}{n}}, n \geq 2. \)

(b) \( \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \frac{\pi}{2}. \)

Solution.

(a) The singularities of \( f(z) = \frac{1}{1+z^n} \) are at the points \( \{e^{\frac{2\pi i k}{n}} \}_{k=0}^{n-1} \), since \( 1 + z^n = 0 \) whenever \( z^n = e^{i\theta} = -1 \), i.e., \( n\theta = \pi + 2\pi k \). Now, let \( S_R = \{ Re^{i\theta} \mid 0 \leq \theta \leq \frac{2\pi}{n} \} \) and \( L_R = \{ re^{\frac{2\pi i}{n}} \mid 0 \leq r \leq R \} \). Then for \( R > 1 \), \( z_0 = e^{\frac{i\pi}{n}} \) is the only singularity of \( f \) interior to the simple closed curve \( C_R \) which consists of \( \{ z = x \mid x \in [0, R] \} \), \( S_R \) and \( L_R \) traversed counterclockwise. Further, \( f \) has a simple pole at \( z_0 = e^{\frac{i\pi}{n}} \), so\n
\[
\text{Res}_{z=z_0} \left( \frac{1}{1+z} \right) = \frac{1}{nz_0^{n-1}} = -\frac{1}{n} \cos \frac{\pi}{n} - \frac{i}{n} \sin \frac{\pi}{n}.
\]

Hence, we have that

\[
\int_{C_R} \frac{dz}{1+z^n} = \int_0^R \frac{dx}{1+x^n} + \int_{S_R} \frac{dz}{1+z^n} + \int_{L_R} \frac{dz}{1+z^n} = \frac{2\pi}{n} \sin \frac{\pi}{n} - \frac{2i\pi}{n} \cos \frac{\pi}{n}.
\]

Now,

\[
\left| \int_{S_R} \frac{dz}{1+z^n} \right| = \left| \int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta} \, d\theta}{1+R^n e^{i\theta}} \right| \leq \int_0^{\frac{2\pi}{n}} \frac{Rd\theta}{1+R^n} \leq \frac{2\pi}{n} \frac{R}{n - nR^n},
\]

which goes to 0 as \( R \) goes to infinity. Further,

\[
\int_{L_R} \frac{dz}{1+z^n} = \int_0^R \frac{e^{\frac{2\pi i}{n}} \, dr}{1+r^n e^{2i\pi}} = \left( -\cos \left( \frac{2\pi}{n} \right) - i \sin \left( \frac{2\pi}{n} \right) \right) \int_0^R \frac{dx}{1+x^n}.
\]

Combining these results, we see that as \( R \) tends to infinity,

\[
\int_{C_R} \frac{dz}{1+z^n} \to \int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi}{n} \sin \frac{\pi}{n} - \frac{2i\pi}{n} \cos \frac{\pi}{n},
\]

Finally, recalling that \( 1 - \cos \frac{2\pi}{n} = 2 \sin^2 \left( \frac{\pi}{n} \right) \) and equating the real parts of the expressions in the above equations, we obtain

\[
2 \sin^2 \left( \frac{\pi}{n} \right) \int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi}{n} \sin \left( \frac{\pi}{n} \right) \iff \int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n \sin \left( \frac{\pi}{n} \right)},
\]

which was the desired result.
(b) First, note that \(e^z + e^{-z} = 0\) if and only if \(z = i \left( \frac{\pi}{2} + n\pi \right)\). Thus for any \(R > 0\), the point \(z_0 = \frac{i\pi}{2}\) is the only singularity of \(f(z) = \frac{1}{e^z + e^{-z}}\) interior to the rectangle \(C_R\) formed by the lines \(x = \pm R\), \(y = 0\) and \(y = i\pi\). Since \(f\) has a simple pole at \(z_0 = \frac{i\pi}{2}\), we know that

\[
\text{Res}_{z=z_0} f(z) = \frac{1}{e^{z_0} - e^{-z_0}} = \frac{1}{i - (-i)} = -\frac{i}{2}.
\]

Thus, \(\int_{C_R} \frac{dz}{e^z + e^{-z}} = 2i\pi \left( -\frac{i}{2} \right) = \pi\). Consider the integral of \(f\) on \(x = R\): \(z = R + iy\), so \(dz = idy\), and

\[
\left| \int_{x=R} \frac{dz}{e^z + e^{-z}} \right| = \left| \int_0^\pi \frac{id y}{e^{R+iy} + e^{-R-iy}} \right| \leq \int_0^\pi \frac{dy}{e^R - e^{-R}} = \frac{1}{e^R - e^{-R}}.
\]

which tends to 0 as \(R\) tends to infinity. A similar result holds for \(x = -R\). On the line \(y = \pi\) we have \(z = x + i\pi\) and \(dz = dx\), so

\[
\int_{y=\pi} \frac{dz}{e^z + e^{-z}} = \int_R^{-R} \frac{dx}{e^x e^{i\pi} + e^{-x} e^{-i\pi}} = - \int_{-R}^R \frac{dx}{-1(e^x + e^{-x})} = \int_{-R}^R \frac{dx}{e^x + e^{-x}}.
\]

Hence, as \(R \to \infty\), we have that

\[
\int_{C_R} \frac{dz}{e^z + e^{-z}} = 2 \int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}} = \pi.
\]

Division by 2 gives the desired result. \(\blacksquare\)
Problem 5 Prove that \( \cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2 \pi^2} \) and use this to show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

Proof. First, we need a result involving converging sequences of analytic functions.

Lemma 6 Suppose \( \{f_n\}_{n=1}^{\infty} \) are analytic in a region \( \Omega \) and let \( g_n = \sum_{k=1}^{n} f_k \) converge uniformly to a function \( f \) on compact subsets of \( \Omega \). Then \( f \) is analytic in \( \Omega \).

Proof of lemma. Since the \( f_k \) are analytic in \( \Omega \), each \( g_n \) is also analytic in \( \Omega \). Let \( z_0 \in \Omega \) and let \( R \) be such that \( K = \{ z \mid |z - z_0| \leq R \} \subset \Omega \). For some positive \( r < R \) let \( C_r \) and \( C_R \) be the circles of radius \( r \) and \( R \), respectively, centered at \( z_0 \). The set \( C_R \) is compact, so by hypothesis, for arbitrary \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( n > N \) implies \( |g_n(c) - f(c)| < \varepsilon \) for all \( c \in C_R \). Now, for \( z \in K \setminus C_R \) we observe the following:

1. \( g_n(z) = \frac{1}{2i\pi} \int_{C_R} \frac{g_n(\zeta)}{\zeta - z} d\zeta \),
2. \( h(z) = \frac{1}{2i\pi} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta \) exists, and
3. \( |\zeta - z| > R - r \) for all \( \zeta \in C_R \).

It follows that for \( n > N \),

\[
|g_n(z) - h(z)| \leq \frac{1}{2\pi} \int_{C_R} \frac{|g_n(\zeta) - f(\zeta)|}{|\zeta - z|} d\zeta
\leq \frac{1}{2\pi} \frac{\varepsilon}{R - r} \int_{C_r} |d\zeta|
= \frac{R\varepsilon}{R - r}.
\]

This tells us that \( g_n \) converges uniformly to \( h \). Therefore, \( f(z) = \frac{1}{2i\pi} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta \) and is thus analytic on \( \Omega \).

Now, to prove the main result, let \( \Omega = \mathbb{C} \setminus \{ \pm n\pi \} \), \( g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2 \pi^2} \), and \( g_n \) be \( \frac{1}{z} \) plus the \( n \)th partial sum of the series in \( g \). For a fixed \( z \in \Omega \), the series in \( g \) is on the order of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), which converges absolutely, so \( g_n \to g \) on \( \Omega \). Let \( \varepsilon > 0 \) be given and let \( K \subset \Omega \) be compact. Since \( |z| \) is continuous, there is an \( M > 0 \) so that \( |c| \leq M \) for all \( c \in K \). Further, there exists an \( N \in \mathbb{N} \) so that \( n > N \) implies \( \sum_{k=n+1}^{\infty} \frac{1}{z^2 - k^2 \pi^2} < \frac{\varepsilon}{2M} \). For \( n > N \),

\[
|g_n(z) - g(z)| = \left| 2z \sum_{k=n+1}^{\infty} \frac{1}{z^2 - k^2 \pi^2} \right| \leq 2M \sum_{k=n+1}^{\infty} \left| \frac{1}{z^2 - k^2 \pi^2} \right| < 2M \frac{\varepsilon}{2M} = \varepsilon.
\]

Hence, \( g_n \to g \) uniformly on compact subsets of \( \Omega \), so by our lemma, \( g \) is analytic on \( \Omega \).
Next, we will show that \( g \) has period \( \pi \). For any \( z \in \Omega \), we have
\[
g(z + \pi) = \frac{1}{z + \pi} + \sum_{k=1}^{\infty} \frac{1}{(z + \pi) - k\pi} + \sum_{k=1}^{\infty} \frac{1}{(z + \pi) + k\pi}
\]
\[
= \frac{1}{z + \pi} + \left( \frac{1}{z} + \frac{1}{z + 2\pi} \right) + \sum_{k=2}^{\infty} \frac{1}{z - (k-1)\pi} + \sum_{k=2}^{\infty} \frac{1}{z + (k+1)\pi}
\]
\[
= \frac{1}{z} + \frac{1}{z + \pi} + \frac{1}{z + 2\pi} + \sum_{k=1}^{\infty} \frac{1}{z - k\pi} + \sum_{k=1}^{\infty} \frac{1}{z + k\pi}
\]
\[
= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{z - k\pi} + \sum_{k=1}^{\infty} \frac{1}{z + k\pi}
\]
\[
= g(z).
\]

We now wish to compute \( g \left( \frac{\pi}{2} \right) \) to ensure that \( \cot z \) and \( g \) agree on at least one point.
\[
g \left( \frac{\pi}{2} \right) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \left( \frac{1}{\frac{\pi}{2} - k\pi} + \frac{1}{\frac{\pi}{2} + k\pi} \right)
\]
\[
= \frac{2}{\pi} - \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}} \right)
\]
\[
= \frac{2}{\pi} - \frac{1}{\pi} \left( \frac{1}{1 - \frac{1}{2}} - \lim_{k \to \infty} \frac{1}{k + \frac{1}{2}} \right)
\]
\[
= \frac{2}{\pi} - \frac{1}{\pi} (2 - 0)
\]
\[
= 0,
\]
which agrees with \( \cot \frac{\pi}{2} = 0 \). Now, \( g - \frac{1}{z} \) is analytic at 0, so \( g \) has a simple pole there. By the periodicity of \( g \), it follows that all singularities of \( g \) are simple poles. Moreover, since \( \cot z = \frac{\cos z}{\sin z} \), where \( \sin \) and \( \cos \) are analytic at 0, \( \cos 0 \neq 0 \), and \( \sin z \) has a zero of order 1 at \( z = 0 \), \( \cot z \) has a simple pole at 0 and, by periodicity, at \( z = n\pi \) for all \( n \in \mathbb{N} \). Hence, \( \cot g \) is entire, so if we can show that it is identically zero, we will have our result. Since the functions are periodic, it suffices to show that for \( x \in [0, \pi] \) each function is bounded as \( y \) goes to \( \infty \), for then \( \cot g \) will be bounded and entire. At this point, a combination of Liouville’s theorem and the maximum modulus principle will guarantee that \( \cot g \) is identically constant. We know this constant is zero since the functions agree at \( \frac{\pi}{2} \).

First, observe that
\[
\cot z = \frac{e^{iz} + e^{-iz}}{2} = \frac{2i}{e^{iz} - e^{-iz}} = i \frac{e^{iz} + e^{-iz}}{e^{2iz} - 1} = i + \frac{2i}{e^{2iz} - 1}.
\]
Next, \( |e^{2iz}| = |e^{2i(x+iy)}| = |e^{2xi}e^{-2y}| = e^{-2y} \), so \( i + \frac{2i}{e^{2ix}} \leq 1 + \frac{2}{e^{-2y-1}} \). This quantity approaches 3 as \( y \to +\infty \) and 1 as \( y \to -\infty \), bounding \( |\cot z| \).
Let \( z = x + iy \), where \( x \in [0, \pi] \) and \(|y| \in (1, \infty)\). Then

\[
|z^2 - k^2\pi^2| \geq |z|^2 - k^2\pi^2 \\
= x^2 + y^2 - k^2\pi^2 \\
\geq y^2 - k^2\pi^2
\]

This leads us to the following estimate:

\[
|g(z)| \leq \frac{1}{|y|} + 2(|y| + 1) \sum_{k=1}^{\infty} \frac{1}{y^2 - k^2\pi^2}.
\]

If we let \( d = \lfloor|y|\rfloor\), a slightly modified division algorithm allows us to write \( k = qd + r \) for unique nonnegative integers \( q \) and \( r, 0 < r \leq q \). Hence,

\[
\sum_{k=1}^{\infty} \frac{1}{y^2 - k^2\pi^2} = \sum_{q=1}^{\infty} \sum_{r=0}^{q} \frac{1}{y^2 - \pi^2(qd + r)^2}.
\]

Given a particular \( q \),

\[
\sum_{r=1}^{q} \frac{1}{y^2 - \pi^2(qd + r)^2} \leq \sum_{r=1}^{q} \frac{1}{y^2 - \pi^2(qd)^2} \\
\leq \sum_{r=1}^{d} \frac{1}{d^2 - \pi^2 q^2 d^2} \\
= \frac{d}{d^2(1 - \pi^2 q^2)} \\
= \frac{1}{d(1 - \pi^2 q^2)}.
\]

Therefore,

\[
|g(z)| \leq \frac{1}{|y|} + 2(|y| + 1) \sum_{q=0}^{\infty} \frac{1}{d(1 - \pi^2 q^2)} = \frac{1}{|y|} + 2 \frac{|y| + 1}{\lfloor|y|\rfloor} \sum_{q=0}^{\infty} \frac{1}{1 - \pi^2 q^2}.
\]

Although \( \frac{1}{1 - \pi^2 q^2} < 0 \) when \( q \neq 0 \), the above quantity is nonnegative, since \( \frac{1}{1 - \pi^2 q^2} > \frac{1}{8} \) for \( q = 1, 2, 3, \) and \( 2^q > q^2 \) when \( q \geq 4 \), so \( 1 - \pi^2 q^2 < -9 \cdot 2^q \). It follows that

\[
\sum_{q=0}^{\infty} \frac{1}{1 - \pi^2 q^2} > 1 - \frac{3}{8} - \frac{1}{9} \sum_{q=4}^{\infty} \frac{1}{2^q} = \frac{5}{8} - \frac{1}{72} > 0.
\]

Hence, our bound is not contradictory, so as \(|y| \to \infty\), \(|g|\) is bounded above by 2, thus completing the proof that \( \cot z = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{2z}{z^2 - k^2\pi^2} \).
To prove the other result, we now have that
\[
\frac{\cot z}{z} = g(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{2}{z^2 - n^2 \pi^2}.
\]

In particular, we know \(\frac{z \cot z - 1}{z^2}\) has a removable discontinuity at \(z = 0\), so \(\lim_{z \to 0} \frac{z \cot z - 1}{z^2}\) exists and can be found by a few iterations of L'Hôpital's Rule and the fact that \(\frac{z \cot z - 1}{z^2} = \frac{z \cos z - \sin z}{z^2 \sin z}\).

\[
\lim_{z \to 0} \frac{z \cos z - \sin z}{z^2 \sin z} = \lim_{z \to 0} \frac{\cos z - z \sin z - \cos z}{z^2 \cos z + 2z \sin z} = \lim_{z \to 0} \frac{-\sin z}{z^2 \cos z + 2z \sin z} = \lim_{z \to 0} \frac{-\cos z}{z \cos z + 2 \sin z} = \lim_{z \to 0} \frac{-1}{z \cos z - z \sin z + 2 \cos z} = \lim_{z \to 0} \frac{-1}{1 - 0 + 2} = -\frac{1}{3}.
\]

As a direct result, \(\sum_{n=1}^{\infty} \frac{2}{-k^2 \pi^2} = -\frac{1}{3}\), so \(\sum_{n=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\). \(\blacksquare\)

**Problem 7** Determine the number of roots of \(2z^5 - 6z^2 + z + 1 = 0\) in the annulus \(1 < |z| < 2\).

**Solution.** Let \(C_k = \{z \mid |z| = k\}\). On the contour \(C_1\), we have

\[
|2z^5 + z + 1| \leq 2|z|^5 + |z| + 1 \quad \text{and} \quad | - 6z^2 | = 6|z|^2
\]

\[
= 2 + 1 + 1 = 4 \quad \text{and} \quad 6
\]

By Rouche’s Theorem, \(f(z) = 2z^5 - 6z^2 + z + 1\) has the same number of zeroes (including multiplicity) interior to \(C_1\) as \(-6z^2\) does, so \(f\) has two zeroes interior to \(C_1\). Similarly, on the contour \(C_2\) we have

\[
| - 6z^2 + z + 1| \leq 6|z|^2 + |z| + 1 \quad \text{and} \quad |2z^5| = 2|z|^5
\]

\[
= 6 \cdot 2^2 + 2 + 1 = 27 \quad \text{and} \quad 32
\]

By Rouche’s Theorem, \(f\) has the same number of zeroes interior to \(C_2\) as \(2z^5\) does, so \(f\) has five zeroes interior to \(C_2\). By a bit of arithmetic, this leaves three zeroes of \(f\) in the annular region \(1 \leq |z| < 2\). Finally, notice that a change of 0.1 or less in either direction in the radius of the first circle we used will not change the number of zeroes interior to said circle, so none of the zeroes of \(f\) lie on \(C_1\). Hence, three zeroes of \(f\) lie in the annulus \(1 < |z| < 2\), and by extension three solutions to the equation \(2z^5 + z + 1 = 0\) lie in this region. \(\blacksquare\)