Exercise 19 Suppose \( f \) is integrable on \( \mathbb{R}^d \). For each \( \alpha > 0 \), let \( E_\alpha = \{ x : |f(x)| > \alpha \} \). 

Prove that \( \int_{\mathbb{R}^d} |f(x)| \, dx = \int_0^{\infty} m(E_\alpha) \, d\alpha \).

Proof. First, we know \( E_\alpha \) is measurable for every \( \alpha \) since \( f \) is measurable,

\[
E_\alpha = \{ x : |f(x)| > \alpha \} = \{ x : f(x) > \alpha \} \cup \{ x : f(x) < -\alpha \}
\]

is the union of two measurable sets. Thus, \( m(E_\alpha) = \int_{\mathbb{R}^d} \chi_{E_\alpha(x)}(x) \, dx \). It follows that

\[
\int_0^{\infty} m(E_\alpha) \, d\alpha = \int_0^{\infty} \left( \int_{\mathbb{R}^d} \chi_{E_\alpha(x)}(x) \, dx \right) \, d\alpha.
\]

By Tonelli’s theorem, this equals

\[
\int_{\mathbb{R}^d} \left( \int_0^{\infty} \chi_{E_\alpha(x)}(x) \, d\alpha \right) \, dx.
\]

Now, for a fixed \( x \), if \( \alpha \geq |f(x)| \), then \( x \notin E_\alpha \), so \( \chi_{E_\alpha}(x) = 0 \). Further, if \( \alpha < |f(x)| \), then \( x \in E_\alpha \), so \( \chi_{E_\alpha}(x) = 1 \). As a result,

\[
\int_{\mathbb{R}^d} \left( \int_0^{\infty} \chi_{E_\alpha(x)}(x) \, d\alpha \right) \, dx = \int_{\mathbb{R}^d} \left( \int_0^{|f(x)|} 1 \, d\alpha \right) \, dx = \int_{\mathbb{R}^d} |f(x)| \, dx,
\]

which is the desired result. \( \blacksquare \)

Exercise 20 Prove that if \( E \subseteq \mathbb{R}^2 \) is a Borel set, then the slice \( E^y \subseteq \mathbb{R} \) is a Borel set \( \forall y \).

Proof. Let \( C = \{ E \subseteq \mathbb{R}^2 \mid E^y \text{ is a Borel set in } \mathbb{R} \text{ for every } y \in \mathbb{R} \} \). Suppose \( \{ E_i \}_{i=1}^{\infty} \) is a collection of sets in \( C \). Then for each \( y \in \mathbb{R} \),

\[
\left( \bigcup_{i=1}^{\infty} E_i \right)^y = \{ x \in \mathbb{R} \mid (x,y) \in \bigcup_{i=1}^{\infty} E_i \}
\]

\[
= \bigcup_{i=1}^{\infty} \{ x \in \mathbb{R} \mid (x,y) \in E_i \}
\]

\[
= \bigcup_{i=1}^{\infty} E_i^y.
\]

Since each \( E_i^y \) is a Borel set in \( \mathbb{R} \), so too is \( \bigcup_{i=1}^{\infty} E_i^y \). Consequently, \( \bigcup_{i=1}^{\infty} E_i \in C \). Similarly, for each \( y \in \mathbb{R} \),

\[
\left( \bigcap_{i=1}^{\infty} E_i \right)^y = \bigcap_{i=1}^{\infty} E_i^y, \text{ so } \bigcap_{i=1}^{\infty} E_i \in C. \text{ Finally, if } E \in C, \text{ then}
\]

\[
(E^c)^y = \{ x \in \mathbb{R} \mid (x,y) \in E^c \} = \{ x \in \mathbb{R} \mid (x,y) \in E \}^c = (E^y)^c,
\]
so $E^c \in \mathcal{C}$. Now, suppose $F \subset \mathbb{R}^2$ is closed. Since the standard topology on $\mathbb{R}$ is also the subspace topology which any horizontal line in $\mathbb{R}^2$ inherits from $\mathbb{R}^2$, and since lines are closed in $\mathbb{R}^2$, it follows that for any $y \in \mathbb{R}$, the intersection of $F$ with the horizontal line $L_y$ that goes through $(0, y)$ is closed in said vertical line. Further, $(x, y) \in F \cap L_y$ if and only if $x \in F^y$, so if we consider the homeomorphism $\varphi : L_y \to \mathbb{R}$ given by $\varphi((x, y)) = x$, we see that $\varphi(F \cap L_y) = F^y$ is closed in $\mathbb{R}$. Now, closed sets in $\mathbb{R}$ are Borel sets in $\mathbb{R}$, so $F \in \mathcal{C}$. Since $F$ was arbitrary, $\mathcal{C}$ contains the closed sets, and since we have shown that $\mathcal{C}$ is closed under complements, $\mathcal{C}$ contains the open sets as well. Hence, we have shown that $\mathcal{C}$ is a $\sigma$-algebra containing the open sets, so the Borel sets in $\mathbb{R}^2$ are contained in $\mathcal{C}$, thus completing the proof. ■

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Exercise 4 Prove that if $f$ is integrable on $\mathbb{R}^d$ and $f$ is not identically zero, then

$$f^*(x) \geq \frac{c}{|x|^d}, \quad \text{for some } c > 0 \text{ and all } |x| \geq 1.$$  

Conclude that $f^*$ is not integrable on $\mathbb{R}^d$. Then, show that if $f$ is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x \mid f^*(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

for some $c' > 0$ and all sufficiently small $\alpha$.

Proof.

■
Exercise 5 Consider the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 
\frac{1}{|x|\left(\log \frac{1}{|x|}\right)^2} & \text{if } |x| \leq \frac{1}{2}, \\
0 & \text{otherwise.}
\end{cases}$$

(a) Verify that $f$ is integrable.

(b) Establish the inequality

$$f^*(x) \geq \frac{c}{|x|\left(\log \frac{1}{|x|}\right)}$$

for some $c > 0$ and all $|x| \leq \frac{1}{2},$ to conclude that the maximal function $f^*$ is not locally integrable.

Proof.

(a) Note that $f$ is an even function, and that for $0 < |x| \leq \frac{1}{2},$ $f$ is finite, positive, and continuous, so $f$ can be integrated using an improper Riemann integral as follows:

$$\int_{|x| \leq x_0 \leq \frac{1}{2}} f(x)dx = 2 \int_0^{x_0} \frac{1}{x(\log \frac{1}{x})^2}dx$$

$$= -2 \int_{\log \frac{1}{x_0}}^{\infty} u^{-2}du, \quad u = \log \frac{1}{x}, \quad du = -\frac{1}{x}dx$$

$$= 2 \lim_{a \to \infty} \frac{1}{\log \frac{1}{x_0}}$$

$$= \frac{2}{\log \frac{1}{x_0}} - \lim_{a \to \infty} \frac{1}{a}$$

$$= \frac{2}{\log \frac{1}{x_0}}.$$

When $x_0 = \frac{1}{2},$ this reduces to $\int_{\mathbb{R}} |f| = \frac{2}{\log 2},$ which is finite, so $f$ is integrable.
(b) Let $|x| \leq \frac{1}{2}$. Then, the ball $[-x, x]$ contains $x$, so

$$f^*(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f|$$

$$\geq \frac{1}{2|x|} \int_{[-x,x]} \frac{1}{y(y \log \frac{1}{y})^2} dy$$

$$= \frac{1}{|x|} \int_0^x \frac{1}{y(y \log \frac{1}{y})^2} dy$$

$$= \frac{1}{|x|} \cdot 1 \log \frac{1}{x}.$$

Since $f$ is integrable, $f^*$ is measurable, so we can estimate its integral on $B = [-\frac{1}{2}, \frac{1}{2}]$:

$$\int_B |f^*| \geq \int_B \frac{1}{|x| \log \frac{1}{|x|}} dx$$

$$= 2 \int_0^{\frac{1}{2}} \frac{1}{x \log \frac{1}{x}} dx$$

$$= -2 \int_{\infty}^{\log 2} \frac{du}{u}, \quad u = \log \frac{1}{x}, \quad du = -\frac{1}{x} dx$$

$$= -2 \lim_{a \to \infty} \log u \bigg|_{\log a}^{\log 2}$$

$$= -2 \log(\log 2) + 2 \lim_{a \to \infty} \log(\log a) = \infty.$$

Since we have shown that $f^* \chi_B$ is not integrable, $f^*$ is not locally integrable.