14 Polynomials

Concepts:

- Polynomial Functions
- The Definition of a Polynomial
- Identifying Polynomials
- Leading Coefficient, Leading Term, and the Degree of a Polynomial
- Constant Term of a Polynomial
- Polynomial Division - The Division Algorithm
- Roots and Zeros of a Polynomial
- The Remainder Theorem
- The Factor Theorem
- The Number of Roots of a Polynomial
- Finding Real Roots of a Polynomial
- Applications

(Sections 2.9)
We have already studied linear functions and power functions. Linear and power functions are special types of functions known as polynomial functions. Polynomial functions have several very nice properties. The domain of a polynomial function is \((-\infty, \infty)\). The graph of a polynomial function is continuous and smooth. Intuitively, this means that you can sketch the graph without picking up your pencil and that there are no sharp corners on the graph. (The graph of the greatest integer function is not continuous because there are breaks in the graph. The graph of the absolute value function is not smooth because there is a sharp corner at the tip of the vee.)

We will study several classes of functions in this course including polynomial, rational, exponential, logarithmic, and trigonometric functions. Functions in the same class have similar formulas or common origins. Polynomial functions have similar formulas. Trigonometric functions were all derived from ratios of right triangles. The graphs of functions in each class are distinctive. By the end of the semester, you should be able to look at a graph and identify its class.

We begin with polynomial functions because they can be simply built from the power functions we have already studied. Linear functions are the simplest of all polynomial functions. Quadratic functions are next in the line of polynomial functions. Although quadratic functions are a bit more complicated than linear functions, we gain a lot by moving up a level as you will see when we discuss optimization.

### 14.1 Polynomials

**Definition 14.1**

A polynomial in \(x\) is an algebraic expression that is equivalent to an expression of the form

\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

where \(n\) is a non-negative integer, \(x\) is a variable, and the \(a_i\)'s are all constants.

In a polynomial, \(a_i\) is called the coefficient of \(x^i\) and \(a_0\) is called the constant term of the polynomial. If the polynomial contains only a constant term, it is called a constant polynomial. If the polynomial equals 0, then it is called the zero polynomial.

To find the degree of a polynomial, find all \(x^i\)'s with nonzero coefficients. List their exponents. (The exponent corresponding to a nonzero constant term is zero since \(a_0 = a_0 x^0\).) The degree is the largest exponent in the list. The degree of the zero polynomial is undefined since your list will be empty. The degree of a constant polynomial that is not the zero polynomial is zero. If \(k\) is the degree of the polynomial, then the coefficient of \(x^k\) is the leading coefficient of the polynomial and \(a_k x^k\) is the leading term of the polynomial.
Example 14.2 (Polynomials)
Which of the following are polynomials? If the expression is a polynomial, find its degree and leading term.

- $x^3 + 3x^4 + 2$
- $x^3 + 0x^4 + 2$
- $x^{-3} + 3x^4 + 2$
- $\sqrt{2x} + 1$
- $\sqrt{2x} + 1$
- $\frac{1}{5}x^7 + 2x^3 + \frac{2}{9}x + x^8 + 3x^4 + 5x^8$
- $\frac{1}{5x^7} + 2x^3 + \frac{2}{9}x + x^8 + 3x^4 + 5x^8$
- $x^{\frac{1}{3}} + 3x^4 + 2$
- $\frac{7}{2}$
- $0$

Example 14.3 (Review of Long Division)
Use long division to find the quotient and the remainder.

\[
\begin{array}{c|ccccc}
7843 \\
\hline
3 & \\
\end{array}
\]

Example 14.4 (Polynomial Division)
Find the quotient and the remainder.

\[
\frac{3x^3 - 2x^2 + 4x - 3}{x + 4}
\]

3
Example 14.5 (Polynomial Division)
Find the quotient and the remainder.

\[
x^5 + 3x^2 + 1 \\
\hline
x^2 + 3
\]
Theorem 14.6 (The Division Algorithm)
Let \( P(x) \) and \( D(x) \) be polynomials. Then there are unique polynomials \( Q(x) \) and \( R(x) \) such that
\[
P(x) = D(x)Q(x) + R(x)
\]
and either \( R(x) \) is the zero polynomial or the degree of \( R(x) \) is less than the degree of \( D(x) \).

Definition 14.7
In the Division Algorithm:
\begin{itemize}
  \item \( P(x) \) is the \textit{dividend}.
  \item \( D(x) \) is the \textit{divisor}.
  \item \( Q(x) \) is the \textit{quotient}.
  \item \( R(x) \) is the \textit{remainder}.
\end{itemize}

Example 14.8 (A Preview of the Factor and Remainder Theorems)
Let \( P(x) = x^2 + 5x + 6 \). Find the quotient and remainder of
\[
\frac{P(x)}{x + 3}.
\]

\begin{itemize}
  \item What does the remainder tell you about the factors of \( P \)?
  \item What does the remainder tell you about \( P(3) \)?
  \item What does the remainder tell you about the graph of \( P \)?
\end{itemize}
Definition 14.9 (Roots and Zeros)
Let $P(x)$ be a polynomial. The number $c$ is called a root or a zero of $P$ if and only if $P(c) = 0$.

Example 14.10 (A Preview of the Factor and Remainder Theorems)
Let $P(x) = 2x^4 + 1$. Find the quotient and remainder of

$$\frac{P(x)}{x+2}.$$

- What does the remainder tell you about the factors of $P$?
- What does the remainder tell you about $P(-2)$?
- What does the remainder tell you about the graph of $P$?

Theorem 14.11
Let $P(x)$ and $D(x)$ be polynomials. Then $D(x)$ is a factor of $P(x)$ if and only if the remainder of the division problem $\frac{P(x)}{D(x)}$ is the zero polynomial.

Theorem 14.12 (The Remainder Theorem)
Let $P(x)$ be a polynomial. Then

$$P(c) = \text{the remainder of the division problem } \frac{P(x)}{x-c}.$$
The next theorem includes the phrase, “The following are equivalent. . .” This means that all the statements are true or all of them are false. It is never the case that some are true and some are false.

**Theorem 14.13**

The following are equivalent for the polynomial $P(x)$:

- $(x - c)$ is a factor of $P(x)$.
- __________ is a root of $P(x)$.
- $P(c) = __________$.
- __________ is an $x$-intercept of the graph of $P$.

**Theorem 14.14 (Number of Roots)**

A polynomial of degree $n$ has at most $n$ distinct roots.

### 14.2 Application: Roots of Polynomials

Use a calculator to completely factor the polynomial by approximating the roots.

$$2x^3 + 5x^2 - 1$$
14.3 Application: Max and Mins of Quadratics

**Definition 14.15**
A quadratic function is a function that is equivalent to a function of the form

\[ q(x) = ax^2 + bx + c \]

where \(a\), \(b\), and \(c\) are constants and \(a \neq 0\).

**Definition 14.16**
The graph of a quadratic function is called a parabola.

Parabolas are important because they have either an absolute minimum value (a smallest output value) or an absolute maximum value (a largest output value). The point on the graph that corresponds to the absolute minimum or absolute maximum value is called the vertex of the parabola. For the graph above, the absolute minimum value is 0 and the vertex is \((0,0)\).

The graph of every quadratic function can be obtained by transforming the graph of \(y = f(x) = x^2\).

**Example 14.17 (Transformations and Quadratic Functions)**
Let \(f(x) = x^2\) and \(g(x) = 2x^2 + 4x - 5\).

- Describe the transformations that could be applied to the graph of \(f\) to obtain the graph of \(g\).
- Sketch the graph of \(g\).
- What is the vertex of the graph of \(g\)?
- Does the graph of \(g\) have an absolute minimum or an absolute maximum? What is it?
Example 14.18 (Optimization)
A farmer has 200 feet of fencing to construct five rectangular pens, as shown in the diagram below.
What is the maximum possible area of all five pens?

Example 14.19 (Min or Max?)
- When does a quadratic function have an absolute maximum?
- When does a quadratic function have an absolute minimum?