1. Given

\[ y'' - 4y' + 4y = te^t. \]

(a) Find the complimentary solution.

\[ r^2 - 4r + 4 = 0 \]

\[ (r - 2)^2 = 0 \]

\[ r = 2, \text{ repeated} \]

Fundamental set:

\[ \{ e^{2t}, te^{2t} \} \]

(b) Calculate the Wronskian of the fundamental set of solutions determined from part (a).

\[
W = \begin{vmatrix}
  e^{2t} & te^{2t} \\
  2e^{2t} & e^{2t} + 2te^{2t}
\end{vmatrix} = e^{4t} + 2te^{4t} - 2te^{4t} = e^{4t} \neq 0
\]

(c) Find the general solution using variation of parameters. Do not use undetermined coefficients! You may assume a starting \( t_0 \) but it is not necessary.

\[ u_1 = -\int \frac{te^{2t}e^{2t}}{e^{2t}} dt = -\int 2e^t dt = -\frac{e^t}{3} \]

\[ g(t) = te^{2t} \]

\[ u_2 = \int \frac{e^{2t}e^{2t}}{e^{4t}} dt = \int e^{-t} dt = -\frac{t}{2} \]

\[ y_p = u_1y_1 + u_2y_2 = -\frac{t^3}{3} e^t + \frac{t^2}{2} te^{3t} = \frac{t^3}{6} e^{3t} \]

\[ y(t) = C_1 e^t + C_2 t e^t + \frac{t^3}{6} e^{3t} \]
2. Let

\[ y'' + 4y' + 3y = \sin(4t) \]

model a mass on a spring with damping and forcing function.

(a) Find the complimentary solution.

\[
\begin{align*}
&\lambda^2 + 4\lambda + 3 = 0 \\
&(\lambda + 3)(\lambda + 1) = 0 \\
&\lambda_1 = -3, \quad \lambda_2 = -1 \\
&y_c = c_1 e^{-3t} + c_2 e^{-t}
\end{align*}
\]

(b) Find a particular solution using undetermined coefficients.

\[
\begin{align*}
3. \quad y_p &= A \cos(4t) + B \sin(4t) \\
4. \quad y_p' &= -4A \sin(4t) + 4B \cos(4t) \\
5. \quad y_p'' &= -16A \cos(4t) - 16B \sin(4t) \\
\end{align*}
\]

\[
\begin{align*}
&16A + 16B = 0 \\
&-16A - 13B = 1
\end{align*}
\]

\[
\begin{align*}
&A = \frac{16}{13} \\
&B = \frac{-13}{425}
\end{align*}
\]

(c) Find the general solution.

\[
y = y_c + y_p = c_1 e^{-3t} + c_2 e^{-t} - \frac{16}{425} \cos(4t) - \frac{13}{425} \sin(4t)
\]

(d) Describe the behavior of the general solution as \( t \to \infty \).

\[
y = \underbrace{\frac{16}{425} \cos(4t) - \frac{13}{425} \sin(4t)}_{\text{steady state}} + \underbrace{\frac{16}{425} e^{-3t} + \frac{16}{425} e^{-t}}_{\text{transient}}
\]

\[y \to \frac{16}{425} \cos(4t) - \frac{13}{425} \sin(4t) \quad \text{as} \quad t \to \infty\]
3. Given the differential equation

\[ y'' - 4y' + 3y = u_6(t) \]

with \( y(0) = 0 \) and \( y'(0) = 0 \)

(a) Apply the Laplace transform to the equation and solve for the Laplace transform \( Y(s) \) of \( y(t) \). Do not find the inverse transform.

\[
\begin{align*}
S^2 Y(s) - 3y(0) - y'(0) - 4\left(S \cdot Y(s) - y(0)\right) + 3Y(s) &= \frac{e^{-6s}}{s} \\
(S^2 - 4S + 3) Y(s) &= e^{-\frac{6s}{5}} \\
X(s) &= e^{-\frac{6s}{5}}\left(\frac{1}{s} - \frac{1}{s - 3} - \frac{s}{s^2 - 4s + 3}\right)
\end{align*}
\]

(b) Find the solution \( y(t) \) by taking the inverse transform of \( Y(s) \).

\[
\begin{align*}
\frac{1}{s} \cdot \frac{1}{s - 3} \cdot \frac{1}{s - 1} &= A \cdot \frac{B}{s - 3} + C \cdot \frac{1}{s - 1} \\
(5 - 3)(s - 1) + 8s(s - 1) + e5(s - 3) &= 1 \\
A &= \frac{1}{5}, \quad B = \frac{1}{6}, \quad C = -\frac{1}{2} \\
Y(s) &= \frac{e^{-6s}}{s} \left(\frac{1}{5} - \frac{e^{-\frac{6s}{5}}}{\frac{5}{6}} + \frac{L}{s - 3} - \frac{e^{-\frac{6s}{5}}}{\frac{1}{6}}\right) \\
\tau &= t - \frac{3}{2} \\
y(t) &= \frac{1}{3} u_6(t) + \frac{1}{6} u_1(t) e^{-(t-6)} - \frac{1}{2} u_6(t) e^{-(t-6)}
\end{align*}
\]
4. Multiple choice. Circle the letter of your answer. Five points each.

1. Given the equation \( ay'' + by' + cy = 0 \) where \( a, b, c > 0 \) and \( \{y_1, y_2\} \) is a fundamental set of solutions to the equation. Which one of the following statements is true?
   (a) The roots of the corresponding characteristic equation always have negative real parts.
   (b) \( \lim_{t \to \infty} (c_1y_1 + c_2y_2) \) depends on the initial conditions.
   (c) The general solution may oscillate with out decaying to 0 as \( t \to \infty \).
   (d) The roots of the corresponding characteristic equation are always complex conjugates.
   (e) The general solution is always \( y(t) = c_1e^{r_1t} + c_2e^{r_2t} \) where \( r_1, r_2 \) are roots, possibly repeated, of the corresponding characteristic equation.

2. Given a physical system modeled by \( y'' + 2y' + y = 0 \). Which of one of the following statements is true?
   (a) The system is critically damped.
   (b) The system is under-damped.
   (c) The system is over-damped.
   (d) The general solution grows with out bound (resonance).
   (e) There is no damping because there is no forcing function.

3. Given the equation \( y'' + p(t)y' + q(t)y = g(t) \) with \( p(t), q(t) \) and \( g(t) \) continuous on \( (a, b) \), Which of the following statements is true.
   (a) The interval of existence of solutions depends on the initial conditions.
   (b) A specific solution \( y(t) \) with \( y(t_0), y'(t_0) \) specified for \( a < t_0 < b \) exists and is continuous on all of \( (a, b) \).
   (c) The Wronskian of a fundamental set of solutions may be 0 at a single point \( t_0 \in (a, b) \).
   (d) Given initial conditions, a particular solution will exist but not be defined on all of \( (a, b) \).
   (e) \( y(t) = 0 \) is a solution for any \( g(t) \).

4. Given the equation \( y'' + p(t)y' + q(t)y = g(t) \) with \( p(t), q(t) \) and \( g(t) \) continuous on \( (a, b) \) and \( g(t) \neq 0 \). Which of the following statements is true.
   (a) A fundamental set of solutions can always be found.
   (b) The behavior of a specific solution as \( t \to \infty \) does not depend on the initial conditions.
   (c) The equation may be non-linear depending on \( p(t), q(t) \) and \( g(t) \) (Note that \( p, q, \) and \( q \) are functions of \( t \) only.)
   (d) The method of variation of parameters always gives a solution even if the fundamental set of solutions is not known.
   (e) The method of variation of parameters always gives a solution provided a fundamental set of equations to the homogeneous equation is known.
5. Which of one of the following statements is true.

(a) The Laplace transform of a function \( f(x) \) is \( \int_0^\infty f(t)e^{-st}dt \).

(b) The Laplace transform of a discontinuous function can never be continuous.

(c) \( \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \)

(d) \( \mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \)

(e) The Laplace transform \( F(s) \) of a function \( f(t) \) is always defined for \( s > 0 \).