Research Statement

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My research is in the area of algebraic combinatorics, with my work having several focuses within the field. Particular projects that I have worked on include developing a lattice path interpretation of the diamond product of the $cd$-index of posets, explaining cyclotomic factors within the descent set polynomial, using the hyperpfaffian and a sign-reversing involution to extend a Pfaffian identity of Torelli, and computing the infinite $cd$-index for the extreme Coxeter group.

The following four sections discuss results on these four research topics, providing any necessary background information, prior results, and open questions or areas for further research. The final section describes my ideas for undergraduate research.

1 Lattice Path Interpretation of the Diamond Product

The $ab$-index of a poset $P$ is a polynomial in the non-commutative variables $a$ and $b$ that is a generating function for the number chains through the poset. If $P$ is Eulerian, i.e., its Möbius function satisfies the relation $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ for all intervals $[x, y]$ in $P$, then Bayer and Klapper [2] showed that this polynomial can be written in terms of the variables $c = a + b$ and $d = a \cdot b + b \cdot a$. This new polynomial, known as the $cd$-index and written as $\Psi(P)$, is an important tool for studying posets and polytopes since it is an efficient encoding of the data on chains through the poset. The $cd$-index is known to have non-negative coefficients for face lattices of convex polytopes, see [13], leading to many combinatorial questions, including finding combinatorial interpretations for these coefficients for particular types of posets or for the new coefficients once operations are applied to posets. One such operation is the diamond product, defined on posets $P$ and $Q$ in terms of the Cartesian product as $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$.

This operator is important due to its connection to polytopes, as the face lattice of the Cartesian product of two polytopes is the diamond product of their respective face lattices. Ehrenborg and Readdy [9] introduced a bilinear operator that describes the $cd$-index of $P \diamond Q$, and later Ehrenborg and H. Fox [4] developed a recursive formula for this operator. The diamond product operator, written as $u \diamond v$ when applied to $cd$-monomials $u$ and $v$, and its recursion are stated in terms of the underlying Newtonian coalgebra structure and are not straightforward to apply.

Slone [12] described a lattice path interpretation for this operator when focusing on the diamond product operator applied to powers of $c$. My research in [10] extends this interpretation for any two $cd$-monomials in order to develop a better understanding of how the diamond product affects $cd$-indices. I first defined the set of lattice paths $\Gamma$ to be words in letters that correspond to steps in the integer lattice including $R = (1, 0)$, $U = (0, 1)$, $D = (1, 1)$, $\overline{R} = (2, 0)$, and $\overline{U} = (0, 2)$. Then given $cd$-monomials $u$ of length $p$ and $v$ or length $q$, the set $\Gamma(u, v)$ consists of the lattice paths in $\Gamma$ from $(0, 0)$ to $(p, q)$ that meet the subsequent rules, where the horizontal axis is labeled by the monomial $u$ and the vertical axis by the monomial $v$.


1. No consecutive UR, UR, UR, or UR steps are allowed in the word $P$.

2. No $U$ step is allowed at the bottom of a $d$ label on the vertical axis.

3. Although an $R$ step is allowed along the first part of a $d$ label on the horizontal axis, two consecutive $R$ steps across such a $d$ label are not allowed.

4. A $U$ step is only allowed at the bottom of a $d$ label on the vertical axis, and similarly, an $R$ step is only allowed at the left of a $d$ label on the horizontal axis.

5. If a $D$ step is at the bottom of a $d$ label on the vertical axis, then the steps $DR$ above a $d$ label on the horizontal axis and within the top half of this $d$ label on the vertical axis are not allowed.

For each lattice path in $\Gamma(u,v)$, define a weight function as follows to assign to it a $cd$-monomial and coefficient.

**Definition 1.1.** Let $wt : \Gamma(u,v) \rightarrow \mathbb{Z} \langle c, d \rangle$ be the linear map determined by

$$
wt(R) = wt(U) = c, \quad wt(\overline{R}) = wt(\overline{U}) = d, \quad wt(D) = kd,
$$

where depending on the location of a diagonal step $D$, the scalar $k$ is given by

$$
k = \begin{cases}
2 & \text{if above a $c$ label and to the right of either a $c$ label or the bottom of a $d$ label} \\
2 & \text{if above the first part of a $d$ label, to the right of a $c$ label, and followed by a $U$ step,} \\
& \text{a $U$ step, or a $D$ step} \\
2 & \text{if above the first part of a $d$ label, to the right of the bottom of a $d$ label,} \\
& \text{and followed by a $U$ step} \\
1 & \text{otherwise.}
\end{cases}
$$

With the set of lattice paths and the weight function defined, the main result can be stated in the next theorem.

**Theorem 1.2** (Fox, [10]). For any two $cd$-monomials $u$ and $v$, the $cd$-polynomial $u \diamond v$ is given by the sum

$$u \diamond v = \sum_{P \in \Gamma(u,v)} wt(P).$$

Further research in this area is to consider the Cartesian product. In [4] the authors gave a recursion to compute the $cd$-index of the Cartesian product of two posets that is very similar to that of the diamond product. I have been trying to develop a lattice path interpretation for this product as well, but the product is not degree preserving, which is an added complication.

# 2 Cyclotomic Factors of the Descent Set Polynomial

For a permutation $\pi$ in the symmetric group $\mathfrak{S}_n$ define the descent set by $\text{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$. The descent set statistics $\beta_n(S)$ are defined for subsets $S$ of $\{1, 2, \ldots, n-1\} = [n-1]$ by $\beta_n(S) = |\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) = S\}|$. Chebikin, Ehrenborg, Pylyavskyy and Readdy [3] defined the $n$th descent set polynomial to be

$$Q_n(t) = \sum_{S \subseteq [n-1]} t^{\beta_n(S)}.$$
They observed that this polynomial has many factors that are cyclotomic polynomials, the most common of which being \( \Psi_2 = t + 1 \), which occurs if the proportion of odd entries in the descent set statistics is exactly 1/2. This proportion, written as \( \rho(n) \), is directly related to the number of 1’s in the binary expansion of \( n \), with \( \rho(n) = 1/2 \) when \( n \) has either two or three 1’s in its expansion. This proportion, however, is unknown when the number of 1’s is six or greater. Other factors that Chebekin et al. found can be seen in Table 2 in [3].

Using tools such as MacMahon’s Multiplication Theorem, the quasi-symmetric function of the Boolean algebra, and the Euler characteristic of a subcomplex of the dual of the permutahedron, Ehrenborg and I were able to continue their work in [6] at explaining cyclotomic factors, focusing on factors of the form \( \Phi_{2p} \) for a prime \( p \) as well as double factors of the form \( \Phi_{2}^2 \) or \( \Phi_{2p}^2 \). To state some of the results, the following definition is needed.

**Definition 2.1.** Let \( p \) be a prime and \( 1 \leq k \leq n - 1 \). We say \( k \) is essential for \( n \) in base \( p \) if we expand both \( n \) and \( k \) in base \( p \), that is, \( n = \sum_{i \geq 0} n_i \cdot p^i \) and \( k = \sum_{i \geq 0} k_i \cdot p^i \) where \( 0 \leq k_i, n_i < p \), and the inequality \( k_i \leq n_i \) holds for all indices \( i \). Otherwise we say \( k \) is non-essential for \( n \) in base \( p \).

This first theorem gives conditions for when \( \Phi_{2p} \) occurs as a factor of \( Q_n(t) \) when \( n \) has two 1’s in its binary expansion.

**Theorem 2.2** (Ehrenborg, Fox, [6]). Let \( n = 2^b + 2^a \), where \( b > a \) and \( p \) an odd prime. If \( 2^a \) is non-essential in base \( p \) and there is an integer \( k \) which is non-essential in both base 2 and base \( p \), then the cyclotomic polynomial \( \Phi_{2p} \) is a factor of \( Q_n(t) \).

A similar theorem holds for \( n \) with three binary digits, that is \( n = 2^c + 2^b + 2^a \), once you assume that all three of \( 2^c \), \( 2^b \) and \( 2^a \) are non-essential in base \( p \). Many results were found using these theorems to explain large classes of occurrences of the cyclotomic factor \( \Phi_{2p} \) based on assumptions on the exponents of the powers of 2, the prime \( p \), and the order \( m \) of 2 in the multiplicative group \( \mathbb{Z}_p^* \). Two of these results are stated below.

**Theorem 2.3** (Ehrenborg, Fox, [6]). Assume that 2 has order \( m \) in the multiplicative group \( \mathbb{Z}_p^* \) where \( m \) is even. Let \( n = 2^b + 2^a \) where we assume \( b > a \) and \( n \geq 9 \). If we have \( \{a, b\} \equiv \{0, m/2\} \mod m \), then \( 2^a \) is non-essential in base \( p \). Furthermore, the element 7 is non-essential for both base 2 and base \( p \). Hence the cyclotomic polynomial \( \Phi_{2p} \) is a factor of \( Q_n(t) \).

**Theorem 2.4** (Ehrenborg, Fox, [6]). Let \( n = 2^c + 2^b + 2^a \) where \( c > b > a \) and \( n > 7 \), and assume that \( p = 2^e + 2^d + 1 \) where \( e > d \). If \( \{a, b, c\} \equiv \{0, d, e\} \mod m \) where \( m \) is the multiplicative order of 2 in \( \mathbb{Z}_p^* \), then \( 2^c \), \( 2^b \) and \( 2^a \) are non-essential in base \( p \). Furthermore, at least one of the elements 7 or 13 is non-essential in both base 2 and base \( p \). Hence the cyclotomic polynomial \( \Phi_{2p} \) is a factor of \( Q_n(t) \).

Chebekin et al. also produced explanations for several cyclotomic polynomials that were double factors of the descent set polynomial, but their results were limited to only cases where \( n \) has two binary digits. The following is one of my results that extends their work to any \( n \) value with \( \rho(n) = 1/2 \).

**Theorem 2.5** (Ehrenborg, Fox, [6]). Assume \( q \) is an odd prime power, that is, \( q = p^r \) where \( p \) is an odd prime and \( r \) is a positive integer. If \( \rho(q) = 1/2 \), then the cyclotomic polynomial \( \Phi_{2p} \) is a double factor of the descent set polynomial \( Q_{2q}(t) \).

By observing Table 6 in [6], one sees that two factors remain unexplained, specifically \( \Phi_4 \) and \( \Phi_{28} \) as factors of \( Q_{14}(t) \). Additionally, all double factors that are known to exist have now been explained,
but another task for my research is to determine if any other multiple factors exist beyond ones found by Chebikin et al. Finally, Chebikin also examined cyclotomic factors of the signed descent set polynomial,

\[ Q_n^\pm(t) = \sum_{S \subseteq [n]} t^{\beta_n^\pm(S)}, \]

where \( \beta_n^\pm(S) \) is the number of signed permutations in \( \mathfrak{S}_n^\pm \) with descent set \( S \). A further avenue of my research is to determine if any of our techniques can be extended to explain cyclotomic factors in this polynomial, which contains many unexplained factors; see Table 3 in [3].

3 The Hyperpfaffian and Extending Torelli’s Identity

The Pfaffian of a skew-symmetric matrix is commonly defined as the square root of the determinant, but it can also be expressed as a sum over all perfect matchings of the complete graph. Barvinok [1] extended this notion to the hyperpfaffian by considering partitions of the vertex set \([n] = \{1, \ldots, n\}\) into blocks of size \( k \). Let \( \Pi_{n,k} \) denote the set of such set partitions, \( k \) be an even integer, \( n \) be a multiple of \( k \), and \( f \) be a \( k \)-ary skew symmetric function. For a \( k \)-element subset \( B = \{b_1 < b_2 < \cdots < b_k\} \) of \([n]\) write \( f(B) = f(b_1, b_2, \ldots, b_k) \). Lastly, define the sign \((-1)^\tau\) of a partition \( \tau = \{B_1, B_2, \ldots, B_{n/k}\} \) in \( \Pi_{n,k} \) to be the sign of the permutation \( b_1, b_2, \ldots, b_k, b_{k+1}, b_{k+2}, \ldots, b_{2k}, \ldots, b_{n/k}, b_{n/k+1}, \ldots, b_n \), where the \( i \)th block \( B_i \) is given by \( B_i = \{b_{i,1} < b_{i,2} < \cdots < b_{i,k}\} \). Then the hyperpfaffian is defined by the sum

\[
Pf(f) = \sum_{\tau} (-1)^\tau \cdot \prod_{i=1}^{n/k} f(B_i),
\]

where the sum is over all partitions \( \tau = \{B_1, B_2, \ldots, B_{n/k}\} \) in \( \Pi_{n,k} \).

Assume \( f \) is a \( k \)-ary skew symmetric polynomial of degree \( k/2 \cdot (n-1) \), let \( \Gamma_{n,k} \) be the set of increasing weak compositions of \( k/2 \cdot (n-1) \) into \( k \) distinct parts given by

\[
\Gamma_{n,k} = \left\{ (r_1, r_2, \ldots, r_k) \in \mathbb{N}^k : 0 \leq r_1 < r_2 < \cdots < r_k, \sum_{i=1}^{k} r_i = k/2 \cdot (n-1) \right\},
\]

and then let \( R_{n,k} \) denote the collection of sets of size \( n/k \) of compositions in \( \Gamma_{n,k} \) where all the parts of the compositions are distinct. Applying the hyperpfaffian to this function \( f \) yields the following theorem, which extends a classic Pfaffian result by Torelli [14] that involved applying the Pfaffian to a skew-symmetric polynomial \( f(x,y) \). With Ehrenborg, I was able to prove the result by assigning weights to oriented partitions and using a sign-reversing involution.

**Theorem 3.1** (Ehrenborg, Fox, [5]). *The hyperpfaffian \( Pf(f(x_S)) \) of order \( n \) is the product of the Vandermonde product with a signed sum of products of coefficients \( a_{\tau} \) of \( f \):

\[
Pf(f(x_S))_{S \subseteq [n]} = \left( \sum_{\beta} (-1)^\beta \cdot \prod_{i=1}^{n/k} a_{\tau_i} \right) \cdot \prod_{1 \leq i < j \leq n} (x_j - x_i),
\]

where the sum ranges over all partitions \( \beta \) in \( R_{n,k} \).

As a corollary we have the following identity that is a generalization of Torelli’s Pfaffian identity, see [14] or also [11, Equation (4.6)].
Corollary 3.2 (Ehrenborg, Fox, [5]). For a skew-symmetric polynomial $f(x, y) = \sum_{i=0}^{n-1} a_i \cdot x^i y^{n-1-i}$, the Pfaffian $\text{Pf}(f(x_i, x_j))$ of order $n$ is the product of the first $n/2$ of the coefficients $a_i$ times the Vandermonde product, that is

$$\text{Pf}(f(x_i, x_j))_{1 \leq i < j \leq n} = \prod_{i=0}^{n/2-1} a_i \cdot \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Many other identities exist that involve the Pfaffian such as those found in [11]. Other avenues of future research include determining if the hyperpfaffian can be used to generalize any such identities other than Torelli’s result.

4 The Infinite cd-index and the Extreme Coxeter Group

The study of Eulerian posets and the cd-index has thus far been limited to finite posets. Ehrenborg and I extended these notions to infinite ranked posets with a minimal element $\hat{0}$ such that each rank consists of a finite number of elements [7]. Two examples of such posets are level Eulerian posets, introduced by Ehrenborg, Hetyei and Readdy [8], and the strong Bruhat order of infinite Coxeter groups, which is the focus of our work.

Define the flag $f$-vector as the number of chains in $P$ whose elements have ranks given by the set $S = \{s_1 < \cdots < s_k\}$ of a finite subset of the positive integers $\mathbb{Z}_{>0}$. Then define the flag $h$-vector for finite subsets $S \subseteq \mathbb{Z}_{>0}$ by the relation $h_S = \sum_{T \subseteq S} (-1)^{|S\setminus T|} \cdot f_T$. The poset $P$ is said to be Eulerian if every interval of $P$ is Eulerian in the classical sense.

Similarly, for $S$ a subset of $\mathbb{Z}_{>0}$ we define the infinite $ab$-monomial $u_S = u_1 u_2 \cdots$, where $u_i = b$ if $i \in S$ and $u_i = a$ otherwise. Let $AB$ denote the set of all infinite monomials. Similarly, let $AB^=\infty$ denote the set of all monomials with an infinite number of $b$’s. Let $\mathbb{Z}[AB]$ and $\mathbb{Z}[AB^=\infty]$ denote all formal sums of monomials in $AB$, respectively, in $AB^=\infty$. Finally, define the quotient $\mathcal{A} = \mathbb{Z}[AB]/\mathbb{Z}[AB^=\infty]$. In effect, we are setting each monomial with an infinite number of $b$’s to be zero. A linear basis of $\mathcal{A}$ is given by all monomials with a finite number of $b$’s. Similarly we define $\mathcal{C}$ to be the quotient $\mathbb{Z}[CD]/\mathbb{Z}[CD^=\infty]$. We embed $\mathcal{C}$ into $\mathcal{A}$ by the map $c \mapsto a + b$ and $d \mapsto ab + ba$.

Define the $ab$-index of an infinite poset $P$ to be

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where the sum is over all finite subsets $S$ of $\mathbb{Z}_{>0}$. This motivates the following result that extends the classical result of Bayer–Klapper [2].

Theorem 4.1 (Ehrenborg, Fox, [7]). The $ab$-index of an Eulerian infinite poset belongs to the image of the quotient space $\mathcal{C} = \mathbb{Z}[CD]/\mathbb{Z}[CD^=\infty]$.

Let $(W, S)$ be the Coxeter system where the number of generators is $r$ and the only relations are $s^2 = 1$ for all generators $s$. This is the extreme Coxeter group. The Bruhat order of this group is an infinite Eulerian poset such that there are $r \cdot (r-1)^{n-1}$ elements of rank $n$.

Theorem 4.2 (Ehrenborg, Fox, [7]). Let $u$ be an element of length $k$ in the extreme Coxeter group $W$. Then the generating function of the elements not comparable to $u$ is given by

$$\sum_{u \not\leq w} x^{f(w)} = (1 + (r-1) \cdot z) \cdot (1 + z + \cdots + z^{k-1}),$$
where $z = x/(1 - (r - 2) \cdot x)$.

In particular, the number of elements of rank $n$ not above a given element $u$ only depends on the length of $u$. As a corollary Ehrenborg and I found that the order of the number of elements above $u$ is given by

$$\left|\{w \geq u : \ell(w) = n\}\right| = r \cdot (r - 1)^{n-1} + O((r - 2 + \varepsilon)^n).$$

An exact, but long expression can be given by expanding the generating function. At this moment, we believe that we can give the order of the coefficients of the infinite cd-index, shown in the following conjecture, although there is more work to do in this direction.

**Conjecture 4.3.** In the extreme Coxeter group there is a constant $C$ such that the order of the coefficients of the infinite cd-index is given by

$$[e^{\alpha_0}dc^{\alpha_1}d \cdots dc^{\alpha_s}dc^{\infty}] = C \cdot (r - 1)^{\alpha_0 + \cdots + \alpha_s} + O((r - 2 + \varepsilon)^{\alpha_0 + \cdots + \alpha_s}).$$

## 5 Undergraduate Research

In addition to my own research pursuits, I am excited about the opportunity to mentor undergraduate students through their first research experiences. As an undergraduate I was fortunate to have the chance to participate in two summer research programs on dynamic equations on time scales and on cryptography. Although my eventual research field as a graduate student went in a vastly different direction, these experiences allowed me to develop important mathematical skills and tools while showing me both the joy and the challenge of working on cutting edge mathematics. Whether it is coordinating an REU program or simply working with one of my students in a one-on-one setting, I will enjoy introducing students to the exciting world of research and hopefully encourage many of them to continue down this path toward graduate school and further research.

The broad field of combinatorics within which my research lies is a terrific field for undergraduate research that is filled with many open problems that are accessible for undergraduates to tackle. An ideal undergraduate research scenario, in my opinion, is for one of my students to find a research question to work on rather than for me to give them a particular problem. This allows the students to comb through the troves of open questions that exist in combinatorics to find a problem that intrigues them most. Even if this problem does not align perfectly with my own research projects, I feel qualified through my coursework and research experiences to guide them in their efforts within a number of subfields including graph theory, posets, polytopes, permutations, cryptography, and design theory, all of which are appropriate for undergraduates.

If a student needs help finding an unanswered question to approach, I would suggest the open problem discussed in Section 1 regarding a lattice path interpretation of the Cartesian product. After a little help establishing some background knowledge on coalgebras, a student could use my interpretation for the diamond product as a basis along with some creativity and ingenuity to develop their own combinatorial interpretation for the Cartesian product.

For a larger scale project with multiple students such as a summer REU, I believe the open questions in Section 2 regarding the remaining two cyclotomic factors of the descent set polynomial and the cyclotomic factors of the signed descent set polynomial would be a good research problem for this scenario. With the concrete nature of the problem and the accessibility of most of the tools used in our research, a group of undergraduates could make great progress toward explaining the existence of these factors.
References


