11-14 Homework

Presentation Problems:

1. Show that $\mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$ is a field but $\mathbb{Z}_3[x]/\langle x^3 + x + 1 \rangle$ is not a field.

2. Let $F$ be a field and $f(x), g(x) \in F[x]$ be non-zero polynomials. Show that $\langle f(x) \rangle + \langle g(x) \rangle = \langle \gcd(f(x), g(x)) \rangle$.

Problems to be turned in:

1. Let $\varphi : R \rightarrow R'$ be a homomorphism of commutative rings.
   
   (a) Show that if $p \subset \varphi[R]$ is a prime ideal and $\varphi^{-1}[p] \neq R$ then $\varphi^{-1}[p]$ is a prime ideal of $R$.

   (b) If $m \subset \varphi[R]$ is a maximal ideal and $\varphi^{-1}[m] \neq R$, then $\varphi^{-1}[m]$ is a maximal ideal in $R$.

2. (a) Let $F$ be a field and $p(x) \in F[x]$ an irreducible polynomial. Prove that the ideal $\langle p(x) \rangle$ is a maximal ideal in $F[x]$. [HINT: You want to make a similar argument as we did in class to show that $p\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$.]

   (b) Use the Fundamental Homomorphism Theorem for Rings to show that $\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$. [HINT: You will have to use Problem 1 to argue that the kernel of your map is indeed $\langle x^2 + 1 \rangle$.]