27.) A subgroup \( H \) is **conjugate to a subgroup** \( K \) of a group \( G \) if there exists an inner automorphism \( i_g \) of \( G \) such that \( i_g[H] = K \). Show that conjugacy is an equivalence relation on the collection of subgroups of \( G \).

So again, this problem is saying to show that the relation on subgroups defined by \( H \sim K \) if and only if there exists \( g \in G \) such that \( i_g[H] = gHg^{-1} = K \) is an equivalence relation. So we need to show the three properties of an equivalence relation.

- **reflexive:** Here we want to show that \( H \sim H \). Consider the identity \( e \in G \). Then \( i_e[H] = eHe^{-1} = H \). Therefore the relation is indeed reflexive.

- **symmetric:** Here we want to show that if \( H \sim K \) then we also have \( K \sim H \). By assumption, there exists \( g \in G \) such that \( i_g[H] = gHg^{-1} = K \). Therefore we have that
  \[
  H = eHe^{-1} = (g^{-1}g)H(g^{-1}g) = g^{-1}(gHg^{-1})g = g^{-1}Kg
  \]
  and so \( i_{g^{-1}}[K] = H \) giving us that \( K \sim H \) and so the relation is indeed symmetric.

- **transitive:** Assume that \( H \sim K \) and \( K \sim L \) then we wish to show that \( H \sim L \). By assumption, there exists \( g_1, g_2 \in G \) such that \( i_{g_1}[H] = g_1Hg_1^{-1} = K \) and \( i_{g_2}[K] = g_2Kg_2^{-1} = L \). Therefore by substitution we have that
  \[
  L = g_2Kg_2^{-1} = g_2(g_1Hg_1^{-1})g_2^{-1} = (g_2g_1)H(g_2g_1)^{-1} = i_{g_1g_2}[H]
  \]
  and so indeed \( H \sim L \).

35.) Show that if \( H \) and \( N \) are subgroups of a group \( G \), and \( N \) is normal in \( G \), then \( H \cap N \) is normal in \( H \). Show by an example that \( H \cap N \) need not be normal in \( G \).

You showed for homework last semester that the intersection of two subgroups is itself a subgroup. But let’s go over that again for practice. To show that \( H \cap N \) is a subgroup, it is sufficient to show closure and that \( a^{-1} \) is an element of \( H \cap N \) for all \( a \) in \( H \cap N \).

- **closure:** Let \( a, b \in H \cap N \). Then since both \( H \) and \( N \) are themselves subgroups we have that \( ab \in H \) and \( \in N \) and therefore \( ab \in H \cap N \).

- **inverses:** Let \( a \in H \cap N \). Again, since both \( H \) and \( N \) are themselves subgroups, we must have that \( a^{-1} \in H \) and \( a^{-1} \in N \) and so \( a^{-1} \in H \cap N \).

Now we need only show that \( H \cap N \) is normal in \( H \). To do this, we will show that for all \( h \in H \) and \( g \in H \cap N \) we must have \( hgh^{-1} \in H \cap N \). First note that since \( g \in H \cap N \) we have that in particular \( g \in H \) and so since \( H \) is a subgroup we must have that \( hgh^{-1} \in H \). Furthermore, since \( N \) is normal in \( G \) we have that \( hgh^{-1} \in N \). Therefore \( hgh^{-1} \in H \cap N \) giving us that the intersection is normal in \( H \).
37.) Let $\text{Aut}(G)$ denote the set of automorphisms of a group $G$.

a Show that $\text{Aut}(G)$ is a group under function composition.

Here we need to show that $\text{Aut}(G)$ satisfies the properties of a group.

- **closure:** Let $\phi, \psi \in \text{Aut}(G)$. Then we need to show that $\phi \circ \psi$ is also an automorphism. There are then three things to check.

  - **homomorphism:** We need to make sure that $\phi \circ \psi$ is a homomorphism. Let $g_1, g_2 \in G$. Then because both $\phi$ and $\psi$ are themselves homomorphisms, we have
    \[ \phi \circ \psi(g_1g_2) = \phi(\psi(g_1)\psi(g_2)) = \phi(\psi(g_1))\phi(\psi(g_2)) = \phi \circ \psi(g_1)\phi \circ \psi(g_2). \]
    And so the composition also satisfies the homomorphism property.

  - **one-to-one:** Here we need to show that $\phi \circ \psi$ is also one-to-one. Note that since $\phi$ and $\psi$ are injective then we have $\ker \phi = \ker \psi = \{e\}$ where $e \in G$ is the identity. By definition
    \[ \ker \phi \circ \psi = \{ g \in G \mid \phi \circ \psi(g) = \phi(\psi(g)) = e \}. \]
    So if $\phi(\psi(g)) = e$ then we must have that $\psi(g) = e$ since $\ker \phi = \{e\}$ and similarly we must then have that $g = e$ since $\ker \psi = \{e\}$. And so we indeed have that $\ker \phi \circ \psi = \{e\}$ giving us that the composition is an injection.

  - **onto:** Here we want to make sure that under the composition, for every $g_1 \in G$ there exists $g' \in G$ such that $\phi \circ \psi(g') = g_1$. Note that since $\phi$ is surjective, there exists $g_2 \in G$ such that $\phi(g_2) = g_1$. Furthermore, since $\psi$ is also surjective, there exists $g_3$ such that $\psi(g_3) = g_2$. Therefore
    \[ \phi \circ \psi(g_3) = \phi(\psi(g_3)) = \phi(g_2) = g_1 \]
    giving us that the composition is also surjective.

And so we have that $\phi \circ \psi \in \text{Aut}(G)$.

b Show that the inner automorphisms of a group $G$ form a normal subgroup of $\text{Aut}(G)$ under function composition.

For ease of notation, let $\text{Inn}(G)$ denote the set of inner automorphisms of $G$,

$$ \text{Inn}(G) = \{ i_g \mid g \in G \}. $$

The name of these maps tells us that they are indeed automorphisms and therefore $\text{Inn}(G) \subset \text{Aut}(G)$. (We should have shown this is true in class but that’s okay.) We need to show two things for this problem. First we need to show that $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$. Then we will show that it is a normal subgroup.
• **subgroup:** Again, to show that it is a subgroup, we need to show that $\text{Inn}(G)$ is closed under function composition and for all $i_g \in \text{Inn}(G)$ that $(i_g)^{-1} \in \text{Inn}(G)$.

  - **closure:** By the proof in part (a) we know that for all $g_1, g_2 \in G$ that $i_{g_1} \circ i_{g_2}$ is again an automorphism and so we just need to show that it is an inner automorphism. Note that for all $h \in G$

    
    
    $i_{g_1} \circ i_{g_2}(h) = i_{g_1}(i_{g_2}(h)) = i_{g_1}(g_2hg_2^{-1}) = g_1(g_2hg_2^{-1})g_1^{-1} = (g_1g_2)h(g_1g_2)^{-1} = i_{g_1g_2}(h)$

    
    giving us that $\text{Inn}(G)$ is closed under composition.

  - **inverses:** Now let $g \in G$ and consider $g^{-1} \in G$. Then for all $h \in G$

    
    
    $i_g \circ i_{g^{-1}}(h) = i_g(i_{g^{-1}}(h)) = i_g(g^{-1}hg) = g(g^{-1}hg)g^{-1} = h = i_e(h)$

    
    Where $i_e$ is just the identity map which is the identity element in $\text{Aut}(G)$. A similar argument can be made to show $i_{g^{-1}} \circ i_g = i_e$ and so every inner automorphism as an inverse inner automorphism. Therefore $\text{Inn}(G)$ is indeed a subgroup of $\text{Aut}(G)$.

• **normality:** So the easiest way to show that $\text{Inn}(G)$ is normal in $\text{Aut}(G)$ is to show that $\phi \circ i_g \circ \phi^{-1} \in \text{Inn}(G)$ for all $\phi \in \text{Aut}(G)$, $i_g \in \text{Inn}(G)$. Let $\phi \in \text{Aut}(G)$ and $i_g \in \text{Inn}(G)$ and $h \in G$. Using the homomorphism property of $\phi$

    
    
    $\phi \circ i_g \circ \phi^{-1}(h) = \phi(i_g(\phi^{-1}(h))) = \phi(g\phi^{-1}(h)g^{-1})$

    
    
    $= \phi(g)\phi(\phi^{-1}(h))\phi(g^{-1}) = \phi(g)h[\phi(g)]^{-1}$

    
    
    $= i_{\phi(g)}(h)$

    
    which is an inner automorphism of $G$ since $\phi(g) \in G$. 
