Daily Quiz Items and Exam 1 Review Problems

1 Quiz Problems By Date

The daily 10-minute quizzes will be taken for a given date will be adaptations of the items listed under that date. This document may be updated at any time but the source problems for a particular date will be set 48 hrs before class time.

Note that some of the dates on these are different from those on the original list. Expect adjustments of this type as the course progresses.

This document was last updated September 21, 2015.

1.1 August 26

1. Complete the matrix below to an augmented matrix for the following linear system:
\[
\begin{cases}
x - 4y + 1 = y, \\
2x + 3 = 11y + 4
\end{cases}
\]

\[
\begin{bmatrix}
x & y \\
\_ & \_ & \_
\end{bmatrix}
\]

\[
\begin{bmatrix}
x & y & \text{RHS} \\
1 & -5 & -1 \\
2 & -11 & 1
\end{bmatrix}
\]

- Modify your augmented matrix by one elementary row operation which results in an equivalent matrix with 0 in the lower left corner. Be sure to indicate the operation by one of \( P_{ij} \), \( kR_j \), or \( R_i + cR_j \) with appropriate values for the \( i, j, \) or \( c \) for your operation.

\[
\text{ans } R_2 - 2R_1,
\]

\[
\begin{bmatrix}
x & y & \text{RHS} \\
1 & -5 & -1 \\
0 & -1 & 3
\end{bmatrix}
\]

2. Write down an augmented matrix for the following linear system:
\[
\begin{cases}
x - 4y + 1 = y, \\
2x + 3 = 11y + 4
\end{cases}
\]

\[
\begin{bmatrix}
x & y & \text{RHS} \\
1 & -5 & -1 \\
2 & -11 & 1
\end{bmatrix}
\]

1.2 August 28

1. Definition: A Linear Equation of (or in) the variables \( x_1, \ldots, x_n \) is an equation which can be written in the form \( a_1x_1 + \cdots + a_nx_n = b \).

2. Definition: A Linear Equation is homogenous if it can be written in the form \( a_1x_1 + \cdots + a_nx_n = 0 \).

3. Definition: A solution to a linear equation \( a_1x_1 + \cdots + a_nx_n = b \) is an ordered list of scalars \( (s_1, s_2, \ldots, s_n) \) such that the equation becomes a true statement when \( s_i \) is substituted for \( x_i \) for each \( i = 1, \ldots, n \). An ordered list \( (s_1, s_2, \ldots, s_n) \) is a solution to the linear system \( S \) if it is a solution to each linear equation in \( S \).
4. **Definition:** Two linear systems are **equivalent** if they have exactly the same solutions.

5. Explain why \( x - 3y = 2 + 5y + z \) is a linear equation?
   **Ans** Because it can be written as \( x - 8y - z = 2 \).

6. Why is \( x + 3xy = 1 \) not a linear equation?
   **Ans:** Because it cannot be written in the form \( ax + by = c \).

7. Show that \((1, 2)\) is a solution to the linear equation \( 2x - y = 0 \) while \((1, 1)\) is not.
   **Ans**
   - From the definition, \((1, 2)\) is a solution since \(2(1) - 2 = 0\) is true; \((1, 1)\) is not a solution since \(2(1) - 1 = 0\) is not true.

8. Show that \((1, 1)\) is a solution to the linear equation \( 2x + y = 3 \) while \((1, 2)\) is not.
   **Ans**
   - Also checked by direct substitution

9. Show that neither \((1, 2)\) nor \((1, 1)\) is a solution to the linear system \( S = \{2x - y = 0, 2x + y = 3\} \) but each of them is a solution to all but one of the equations in \( S \).
   **Ans**
   - \( (\frac{3}{4}, \frac{3}{2}) \) is a solution to the system \( S \) because it is a solution to every equation in \( S \).
   \[
   2\left(\frac{3}{4}\right) - \frac{3}{2} = 0, \quad \text{and} \quad 2\left(\frac{3}{4}\right) + \frac{3}{2} = 3
   \]
   - As seen in the previous two problems, each is a solution to one of the two equations in \( S \) but not the other.

10. Show that the linear system \( S = \{x + y + z = 0, y - z = 0\} \) is **not** equivalent to the linear system \( T = \{x + y + z = 0, x - z = 0\} \)
    **Ans:**
    We must show that one of these has a solution which is not a solution of the other. For instance \((x, y, z) = (−2, 1, 1)\) is a solution to \( S \) but not a solution to \( T \) since it is not a solution to \( x - z = 0 \) since \(-2 - 1 = 0\) is not true.

    (b) Modify your augmented matrix by one elementary row operation. Be sure to indicate the operation by one of \( P_{ij} \), \( kR_j \), or \( R_i + cR_j \) with appropriate values for the \( i, j, \) or \( c \) for your operation.
    **Ans:** \( R_2 - 2R_1, \)
    \[
    \begin{array}{ccc|c}
    x & y & \text{RHS} \\
    1 & \begin{array}{c} \frac{1}{3} \end{array} & -5 & -1 \\
    0 & \begin{array}{c} \frac{1}{3} \end{array} & -1 & 3 \\
    \end{array}
    \]

11. **Definition:** The linear system \( S \) is **consistent** if it has at least one solution.
1.3 August 31

1. **Definition:** If \(A\) is an \(m \times n\) matrix with rows \([R_1, R_2, \cdots, R_m]\) then the **Pivot Column List** (PC List) of \(A\) is the ordered list \([c_1, c_2, \cdots, c_m]\) where \(c_i\) is the pivot location of row \(i\).

2. **Definition:** The matrix \(A\) is in **Row Echelon Form (REF)** if the pivot location of any row is less than that of any row below it. (by convention \(\infty < \infty\)). Equivalently, \(A\) is in \(REF\) if its PC List is strictly increasing or **strict**.

3. **Definition:** If \(A\) is a matrix in \(REF\) then the **rank** of \(A\) is the number of non-zero rows in \(A\).

4. For the following system of linear equations in \(x, y\), write down the augmented matrix and carry out the necessary elementary row operation to solve the system.
   Be sure to identify the operations in correct notation introduced in class.
   
   \[
   x + 2y = 5, \quad 2x + 5y = 4, \quad 3x + 7y = 9
   \]

   (a) The augmented matrix is \(A = \)

   \[
   \begin{pmatrix}
   x & y & | & RHS \\
   1 & 2 & | & 5 \\
   2 & 5 & | & 4 \\
   3 & 7 & | & 9 \\
   \end{pmatrix}
   \]

   (b) The row reduced form (REF) for \(A\) is:

   \[
   \begin{pmatrix}
   1 & 2 & | & 5 \\
   0 & 1 & | & -6 \\
   0 & 1 & | & -6 \\
   \end{pmatrix}
   \]

   First \(R_2 - 2R_1\) and \(R_3 - 3R_1\) give:

   (c) What is the equivalent system of linear equations given by the REF of \(A\)?

   **Ans:** \(x + 2y = 5\) and \(y = -6\).
(d) The solution to the linear system is:

**Ans:**

Using backsubstitution:

From the second equation we have \( y = -6 \).

Substituting for \( y \) in the first equation: \( x + 2(-6) = 5 \) gives \( x = 17 \) so the solution is \( x = 17, y = -6 \).

(e) The pc list for \( A \) is:

**Ans:** 1, 1, 1

(f) The pc list for the REF of \( A \) is:

**Ans:** 1, 2, \( \infty \).

(g) The rank of the matrix \( A \) is:

**Ans:**

2, since the REF has two pivots.

1.4 September 2

1. **Definition:** \( \mathbb{R}^n \) is the set of all column vectors of length \( n \) with entries in \( \mathbb{R} \).

2. **Definition:** The elementary row operations on a matrix are:

   (a) exchange two rows
   (b) multiply any row by a non-zero scalar
   (c) add a multiple of any row to a different row.

The linear system \( \{-x + 3y + 5z = 5, 2x + y + 7z = 7t, 3x + 4y + 12z = -9\} \) has the following augmented matrix and associated REF:

\[
(A|B) = \begin{bmatrix}
  x & y & z & \text{RHS} \\
  1 & 3 & 5 & 5 \\
  2 & 1 & 7 & 7t \\
  3 & 4 & 12 & -9 \\
\end{bmatrix} \\
R = \begin{bmatrix}
  x & y & z & \text{RHS} \\
  1 & 3 & 5 & 5 \\
  0 & -5 & -3 & 7t - 10 \\
  0 & 0 & 0 & -14 - 7t \\
\end{bmatrix}
\]

where \( t \) is an unknown number.

(a) Depending on the value of \( t \), what are the ranks of \( A \) and \( (A|B) \). Explain.

**Ans** Regardless of the value of \( t \) the matrix \( A \) has rank 2 because there are 2 pivot columns in its REF. If \( t = -2 \) then \( (A|B) \) has two pivot columns and therefore has rank 2. Otherwise, \( (A|B) \) has three pivot columns and therefore has rank 3.

(b) For which values of \( t \) is the system consistent?

**Ans** The rank of the coefficient matrix is equal to the rank of the augmented matrix if and only if \( t = -2 \). Therefore the system is consistent if and only if \( t = -2 \).

(c) For values of \( t \) for which the system is consistent, which are the pivot variables and which the free variables?

**Ans** The pivot variables are \( x \) and \( y \); the free variable is \( z \).
1. **Definition:** If \((A|B)\) is an augmented matrix in REF form which has a “title row” of variables then the variables above pivot columns are called the **pivot variables** and the non-pivot variables are called the **free variables**.

2. **Definition:** A set \(V \subset \mathbb{R}^n\) is a **vector subspace** of \(\mathbb{R}^n\) if it is closed under vector addition and scalar multiplication.

3. **Definition:** Let \(A\) be an \(m \times n\) matrix with columns \(A_1, A_2, \ldots, A_n\). If \(X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n\) then

\[
AX = x_1A_1 + x_2A_2 + \cdots + x_nA_n
\]

4. **Definition:** (Multiplication of a matrix times a vector) If \(M\) is an \(m \times n\) matrix, written as a row of its columns \((M = [C_1, \ldots, C_n])\) and \(X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\) a vector of length \(n\), then

\[
MX = x_1C_1 + \cdots + x_mC_n
\]

5. \((A|B) = \begin{vmatrix} x & y & \text{RHS} \\ 1 & 2 & 8 \\ -1 & 4 & 10 \\ 2 & -3 & -5 \end{vmatrix}\) and \(M = \begin{vmatrix} x & y & \text{RHS} \\ 1 & 2 & 8 \\ 0 & 6 & 18 \\ 0 & 0 & 0 \end{vmatrix}\) are the augmented matrix and REF of a linear system \(S\).

1) Express the linear system \(S\) as an equivalent vector equation \(x\vec{C}_1 + y\vec{V}_2 = \vec{B}\).

2) Use \(M\) to solve the system \(S\).

3) Use your result from (2) to express your \(\vec{B}\) in (1) as a linear combination of your vectors \(\vec{C}_1\) and \(\vec{C}_2\) in (1).

**Ans:**

1) \[
\begin{vmatrix} x & 1 \\ y & -2 \\ -3 & 2 \\ -5 \end{vmatrix} + \begin{vmatrix} 2 \\ 4 \\ 0 \end{vmatrix} = \begin{vmatrix} 8 \\ 10 \end{vmatrix}.
\]

2) \(\{x + 2y = 8, 6y = 18\}, \{y = 3, x = 8 - 2(3)\}, \{x = 2, y = 3\}\)

3) \[
\begin{vmatrix} x & 1 \\ y & -1 \\ -3 & 2 \\ -5 \end{vmatrix} + \begin{vmatrix} 2 \\ 4 \\ 0 \end{vmatrix} = \begin{vmatrix} 8 \\ 10 \end{vmatrix}.
\]
1.6 September 9

1. \((A|B) = \begin{bmatrix} x_1 & x_2 & x_3 & RHS \\ 1 & 3 & -1 & -1 \\ 2 & 6 & -3 & 0 \\ 1 & -3 & 1 & 1 \end{bmatrix}\) and \(R = \begin{bmatrix} x_1 & x_2 & x_3 & RHS \\ 1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\) are the augmented matrix and its REF of a linear system \(S\)

(a) Identify the free variables.
Ans \(x_2\)

(b) Find the general solution, written as a vector equation.
Ans \(\begin{cases} x_1 - x_3 = -1 - 3x_2, x_3 = -2 \{x_1 = -3x_2 + x_3 - 1, x_2 = x_2, x_3 = -2 \} \\
x_2 = -3x_2 - 3, x_1 = -3, x_2 = 0 + x_2, 1 \\
x_3 = -2, x_3 = -2, 0 \end{cases}\)

(c) Find the general solution to the associated homogeneous system, also written as a vector equation.
Ans \(\begin{cases} x_1 \\
x_2 = x_2 \\
x_3 \end{cases} = \begin{cases} -3 \\
1 \\
0 \end{cases}\)

2. Find the general solutions of the system whose augmented matrix is given below. You must reduce the system to REF using the standard algorithm. The final answer may be given by back substitution or producing RREF.

Be sure to identify the operations in correct notation introduced in class, namely \(kR_i, R_i + cR_j\) or \(P_{ij}\).

\[(**) \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ 1 & 2 & -1 & 0 & 1 & 1 \\ 2 & 4 & -2 & 1 & 3 & 2 \\ -1 & 1 & 1 & 1 & -1 & 2 \end{bmatrix}\]

The REF and RREF are respectively:
\[
\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 & 0 & 5/3 & -1 \\ 0 & 1 & 0 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.
\]

(a) Reduce the system to REF using the standard algorithm.
Ans:
\[
\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 1 \\ 2 & 4 & -2 & 1 & 3 & 2 \\ -1 & 1 & 1 & 1 & -1 & 2 \end{bmatrix} R_2 \rightarrow R_2 - 2R_1 \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 5/3 & -1 \\ 0 & 1 & 0 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} R_3 \rightarrow R_3 + R_1
\]
\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 3 & 0 & 1 & 0
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
1 & 2 & -1 & 0 & 1 \\
0 & 3 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

(b) Find the general solution, written as a system of linear equations. Identify the free variables.

\text{Ans:}

From the given RREF the solution is:

\[x_1 = -1 + x_3 - \frac{5}{3}x_5, x_2 = 1 + \frac{1}{3}x_5, x_4 = -x_5 \text{ with } x_3, x_5 \text{ free.}\]

(c) The system \((**\)) can be written as \(AX = B\). What are \(A\) and \(B\)?

\text{Ans:}

\[A = \begin{bmatrix}
1 & 2 & -1 & 0 & 1 \\
2 & 4 & -2 & 1 & 3 \\
-1 & 1 & 1 & 1 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix}.
\]

(d) The solution \(X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}\) can be written as \(X = X_p + X_h\). What are \(X_p, X_h\)?

\text{Ans:}

\[X_p = \begin{bmatrix}
-1 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}, \quad X_h = x_3 \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
\frac{-5}{3} \\
\frac{1}{3} \\
-1 \\
1
\end{bmatrix}.
\]

(e) Write down a vector equation equivalent to the given system of equations in the above questions.

\[x_1 \begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix} + x_2 \begin{bmatrix}
2 \\
4 \\
1
\end{bmatrix} + x_3 \begin{bmatrix}
-1 \\
-2 \\
1
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} + x_5 \begin{bmatrix}
1 \\
3 \\
-1
\end{bmatrix} = \begin{bmatrix}
1 \\
2
\end{bmatrix}.
\]

1.7 September 11

1. \textbf{Definition:} The set of vectors \(\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^n\) is \textbf{linearly dependent} if there are scalars \(a_1, a_2, \ldots, a_n\) which are \textbf{not all zero} such that \(a_1v_1 + a_2v_2 + \cdots + a_nv_n = O\), the zero vector.

2. \textbf{Definition:} The set of vectors \(\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^n\) is \textbf{independent} if it is not dependent. An arbitrary (possibly infinite) subset of \(\mathbb{R}^n\) is independent if every finite subset is independent.

3. Consider the vectors \(V_1 = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad V_2 = \begin{bmatrix}
0 \\
-2 \\
6
\end{bmatrix}, \quad V_3 = \begin{bmatrix}
-2 \\
-6 \\
0
\end{bmatrix}\).

Show that \(\{V_1, V_2, V_3\}\) is a linearly dependent subset of \(\mathbb{R}^3\).
Alternatively, calculate homogenous solution \( AX = O \). From an REF this matrix has two pivots so there is a free variable which guarantees that the homogenous equation \( AX = O \) has a non-trivial solution. Alternatively, calculate homogenous solution \( \{ x = 2z, y = -z, z = z \} \) and choose a non-zero value of \( z \), say \( z = 1 \) to get a non-trivial solution \((2, -1, 1)\) which says that \( 2V_1 - V_2 + V_3 = O \).

1.8 September 14

(a) **Definition:** A linear transformation is a function \( \mathbb{L} : \mathbb{R}^n \to \mathbb{R}^m \) which satisfies

\[
\mathbb{L}(\alpha v + \beta w) = \alpha \mathbb{L}v + \mathbb{L}w
\]

for every \( v, w \in \mathbb{R}^n \) and every \( \alpha, \beta \in \mathbb{R} \).

(b) **Definition:** If \( \mathbb{L} : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation then the kernel of \( \mathbb{L} \) is \( \{ X \in \mathbb{R}^n \mid \mathbb{L}(X) = O \} \).

(c) Let \( e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), \( e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \), and \( e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). Suppose \( T \) is a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) such that

\[
T(e_1 + e_2) = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix},
T(e_1 + 3e_2) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},
T(e_1 + e_2 + e_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Calculate \( T_A \), the standard matrix for \( T \).

(d) Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be defined by

\[
T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - 3y \\ 5x + y \\ -x \end{bmatrix}
\]

a) If \( V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \), \( V_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \), show that \( T(V_1 + V_2) = T(V_1) + T(V_2) \)

b) If \( V = \begin{bmatrix} x \\ y \end{bmatrix} \), \( \alpha \in \mathbb{R} \), show that \( T(\alpha V) = \alpha T(V) \)

c) Is \( T \) a linear transformation? Why or why not?

d) Calculate the matrix \( M_T \) such that \( T(X) = M_T X \) for every \( X \in \mathbb{R}^2 \).

**Ans:** (a) \( T(V_1) + T(V_2) = T(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) + T(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) =
\begin{bmatrix} 2x_1 - 3y_1 \\ 5x_1 + y_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 - 3y_2 \\ 5x_2 + y_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 2(x_1 + x_2) - 3(y_1 + y_2) \\ 5(x_1 + x_2) + (y_1 + y_2) \\ -(x_1 + x_2) \end{bmatrix} = T(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}) = T(V_1 + V_2) \)

(b) \( T(\alpha V) = T(\alpha \begin{bmatrix} x \\ y \end{bmatrix}) = T(\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}) = \begin{bmatrix} 2\alpha x - 3\alpha y \\ 5\alpha x + \alpha y \\ -\alpha x \end{bmatrix} = \alpha \begin{bmatrix} 2x - 3y \\ 5x + y \\ -x \end{bmatrix} = \alpha T(V) \)
(c) Yes, by (a) and (b).

(d) If \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) then \( M_T = [T(e_1), T(e_2)] = \begin{pmatrix} 2*1 - 3*0 \\ 5*1 + 0 \\ -1 \end{pmatrix} , \begin{pmatrix} 2*0 - 3*1 \\ 5*0 + 1 \\ -0 \end{pmatrix} \)

\[= \begin{pmatrix} 2 & -3 \\ 5 & 1 \\ -1 & 0 \end{pmatrix} \]

(e) Let \( A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \). Show that the columns of \( A \) are linearly dependent according to the definition by finding a non-trivial linear combination of these vectors that is the zero vector in \( \mathbb{R}^3 \).

(f) **Definition:** If \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) then the image of \( L \) is \( \{L(X) \mid X \in \mathbb{R}^n\} \).

### 1.9 September 16

(a) **Definition:** If \( A \) is an \( m \times n \) matrix then \( \text{Null}(A) \), the null space of \( A \) is \( \{X \in \mathbb{R}^m \mid AX = O\} \).

(b) **Definition:** If \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) then the kernel of \( L \) is \( \{X \in \mathbb{R}^n \mid L(X) = O\} \).

(c) **Definition:** A transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one to one or injective if \( L(u_1) = L(u_2) \) implies that \( u_1 = u_2 \).

(d) **Definition:** A transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is onto or surjective if \( \mathbb{R}^n = \text{image}(L) \).

(e) **Definition:** A transformation \( L \) is an isomorphism if it is both injective and surjective.

(f) **Definition:** An \( m \times n \) matrix, \( A \), is a rectangular array of numbers (scalars) with \( m \) rows and \( n \) columns.

### 1.10 September 18

(a) **Definition:** If \( A \) is an \( m \times n \) matrix then \( \text{Col}(A) \), the column space of \( A \) is the linear span of the columns of \( A \). Equivalently, \( \text{Col}(A) = \{AX \mid X \in \mathbb{R}^n\} \). (Note that \( \text{Col}(A) \subset \mathbb{R}^m \))

(b) \( \text{rownum}(A) \) denotes the number of rows of the matrix \( A \).

(c) Suppose that \( A \) is a matrix of rank \( d \) with **3 rows and 5 columns**. Answer using complete, meaningful sentences. All examples requested need have only 0 and 1 as entries.

   i. What is the largest possible value for \( d \)? Give an example of a 3 by 5 matrix \( A \) which has this largest possible rank.

   **Ans:** The largest possible rank is 3 because \( \text{rank}(A) \leq \min(\text{rownum}(A), \text{colnum}(A)) \) = \( \min(3, 5) = 3 \). As an example, consider

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
ii. What is the smallest possible value for $d$? Give an example of a 3 by 5 matrix which has this least possible rank.

**Ans:** The smallest possible rank is 0. There is only one example

\[
\begin{vmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\]

(d) For each of the following, provide a matrix that fits the description or explain why none exists. All examples requested need have only 0 and 1 as entries.

a) A 3 by 5 matrix $A$ such that the columns of $A$ span all of $\mathbb{R}^5$

**Ans:** It is certainly not possible because the columns of $A$ are in $\mathbb{R}^3$ so their linear span is in $\mathbb{R}^3$ which contains no elements of $\mathbb{R}^5$.

### 1.11 September 21

i. $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

\[
\begin{align*}
T(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}) &= \begin{vmatrix} 3 \\ 4 \\ -2 \end{vmatrix}, \\
T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) &= \begin{vmatrix} 2 \\ -3 \\ 0 \end{vmatrix}, \\
T(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}) &= \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}
\end{align*}
\]

Calculate the standard matrix for $T$.

**Ans:** The standard matrix is $[T(e_1), T(e_2), T(e_3)]$ where $e_1, e_2, e_3$ are the columns of the $3 \times 3$ identity matrix. We note that

\[
\begin{align*}
T(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}) &= T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) = T(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}) - T(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}) = \begin{vmatrix} 3 \\ 4 \\ -2 \end{vmatrix} - \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 3 \\ 4 \\ -3 \end{vmatrix} \\
T(\begin{vmatrix} 0 \\ 1 \\ 1 \end{vmatrix}) &= T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) = T(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}) - T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) = \begin{vmatrix} 3 \\ 4 \\ -2 \end{vmatrix} - \begin{vmatrix} 0 \\ 1 \\ -2 \end{vmatrix} = \begin{vmatrix} 3 \\ 3 \\ -3 \end{vmatrix} \\
T(\begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}) &= T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) - T(\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}) = \begin{vmatrix} 3 \\ 4 \\ -2 \end{vmatrix} - \begin{vmatrix} 1 \\ 0 \\ -2 \end{vmatrix} = \begin{vmatrix} 2 \\ 4 \end{vmatrix} \\
T(\begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}) &= T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) - T(\begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}) = \begin{vmatrix} 3 \\ 4 \\ -2 \end{vmatrix} - \begin{vmatrix} 0 \\ 3 \\ 3 \end{vmatrix} = \begin{vmatrix} 3 \\ 1 \end{vmatrix} \\
T(\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}) &= T(\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}) - T(\begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}) = \begin{vmatrix} 3 \\ 4 \\ -2 \end{vmatrix} - \begin{vmatrix} 0 \\ 0 \\ -3 \end{vmatrix} = \begin{vmatrix} 3 \\ 4 \end{vmatrix}
\end{align*}
\]
so \( T = T_A \) where \( A = \begin{vmatrix} 3 & -1 & 1 \\ 4 & -7 & 7 \\ -3 & 3 & -2 \end{vmatrix} \)

b) A 3 by 5 matrix \( A \) such that the columns of \( A \) are linearly independent?

Ans: It is not possible. The rank of \( A \) is less than its number of columns so there will be free variables in the solution to the homogenous linear system \((A|O)\). This implies that there will be non-zero vectors in the nullspace of \( A \). If \( X \) is any one of these then \( AX = O \) is the expression of \( O \) as a non-trivial linear combination of the columns of \( A \).

c) A 3 by 5 matrix \( A \) such that the linear system \((A|B)\) is consistent for all \( B \) in \( \mathbb{R}^3 \)?

Ans: Yes, this is possible. Consider \( A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \). Then if \( B = \begin{vmatrix} a \\ b \\ c \end{vmatrix} \) then

\[
\begin{vmatrix} a \\ b \\ c \\ x \\ y \end{vmatrix} = B\] for any choice of \( x, y \).

ii. The matrix \( A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 7 & 12 \end{vmatrix} \) is the standard matrix of the linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \).

a) In this case \( m = \) _____ and \( n = \) _____

b) Explain why the transformation \( T \) is or is not injective (one-to-one).

c) Explain why the transformation \( T \) is or is not surjective (onto).

Ans: (a) \( m = 3, n = 4 \),

(b) The elementary row operations \( R_2 \rightarrow R_2 + R_1, R_3 \rightarrow 3R_1 \) reduce \( A \) to \( R = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{vmatrix} \)

so \( A \) has rank 2. Therefore the rank of \( A \) is less than the number of columns so \( T \) is not injective.

(c) The rank of \( A \) is less than the number of rows so the transformation is not surjective.

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(e) You are given an augmented matrix \( M \) and a matrix \( R \) which is an REF of \( M \).

\[
M = \begin{bmatrix}
  x & y & z & w & \text{RHS} \\
  2 & 1 & 2 & 5 & 3 \\
-2 & 0 & -1 & 0 & 1 \\
 6 & 2 & 5 & 10 & 5
\end{bmatrix},
W = \begin{bmatrix}
  x & y & z & w & \text{RHS} \\
  2 & 1 & 2 & 5 & 3 \\
 0 & 1 & 1 & 5 & 4 \\
 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The sequence of elementary row operations converting \( M \) to \( W \) are, in order:

\( R_2 \rightarrow R_2 + R_1, R_3 \rightarrow R_3 - 3R_1, R_3 \rightarrow R_3 + R_1 \)

i. Write down the complete solution in parametric form of the linear system described by \( M \) using the given variable names.

ii. Write down the complete solution in parametric form of the associated homogenous system.
iii. Find a vector $B$ such that the linear system $(A|B)$ is inconsistent.

\textbf{Ans:} \\

iv. Find a matrix $Q$ such that $W = QM$. (Keep in mind that, for this purpose, neither the top row with the variable names nor the vertical bar separating the coefficient matrix from the rightmost column are considered to be part of $M$ and $W$.

(f) If $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & 0 \\ 0 & 6 & -1 \\ 7 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 6 & 2 & 5 & 8 \\ 3 & 1 & 0 & 5 & 1 \\ -4 & 5 & 4 & -1 & 1 \end{bmatrix}$, then the number in the fourth row and third column of $AB$ is _________ and the number in the third row, second column of $BA$ is _________.

Provide the calculations that produced your answers.

\textbf{Ans:} \[
\begin{vmatrix} 0 & 6 & -1 \\ 2 \\ 0 \\ 4 \end{vmatrix} = 0 \cdot 2 + 6 \cdot 0 + (-1) \cdot 4 = -4, \quad \begin{vmatrix} -4 & 0 & 0 \\ 6 \\ 1 \\ 5 \end{vmatrix} = -4 \cdot 6 + 0 \cdot 1 + 0 \cdot 5 = -24
\]

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