Example 1: An easy area problem

Find the area of the region in the $xy$-plane bounded above by the graph of $f(x) = 2$, below by the $x$-axis, on the left by the line $x = 1$ and on the right by the line $x = 5$.

This region is a rectangle of height 2 and base 4, so the area is 8.

Example 2: An easy distance problem

A car is travelling due east at a constant velocity of 55 miles per hour. How far does car travel between noon and 2:00 pm?

Distance Travelled = Velocity $\times$ Time Elapsed

so

Distance = (55 mph) $\times$ (2 hours) = 110 miles

Note that distance travelled is represented by area of a rectangle in time-velocity plane:
Example 3

Find the area under the graph of

\[ f(x) = x^2 + \frac{1}{2}x \]

for \( x \) between 0 and 2.

We can’t quite break the region into a number of pieces whose areas we can compute with basic geometry.

We’ll settle for an approximation.

Please consult the interactive graph at

math.uky.edu/~pkoester/teaching/Fall_2009/Math123/Notes

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Example 3 (continued)

First, divide the interval \([0, 2]\) into four equal subintervals,

\[ [0, 0.5], \ [0.5, 1], \ [1, 1.5], \ [1.5, 2] \]

First interval:

\[
\text{Height} = f(0) = 0^2 + \frac{1}{2} \cdot 0 = 0 \quad \text{Area} = f(0) \cdot 0.5 = 0
\]

Second interval:

\[
\text{Height} = f(0.5) = (0.5)^2 + \frac{1}{2} \cdot (0.5) = 0.5 \quad \text{Area} = f(0.5) \cdot 0.5 = 0.25
\]

Third interval:

\[
\text{Height} = f(1) = 1^2 + \frac{1}{2} \cdot 1 = 1.5 \quad \text{Area} = f(1) \cdot 0.5 = 0.75
\]

Fourth interval:

\[
\text{Height} = f(1.5) = (1.5)^2 + \frac{1}{2} \cdot 1.5 = 3 \quad \text{Area} = f(1.5) \cdot 0.5 = 1.5
\]

Total area of rectangles:

\[ 0 + 0.25 + 0.75 + 1.5 = 2.5 \]
Example 3 (different approximation)
Again, divide the interval \([0, 2]\) into four equal subintervals,
\[
[0, 0.5], \quad [0.5, 1] \quad [1, 1.5] \quad [1.5, 2]
\]
First interval:
Height \(= f(0.5) = 0.5^2 + \frac{1}{2} \cdot 0.5 = 0.5\)
Area \(= f(0.5) \cdot 0.5 = 0.25\)
Second interval:
Height \(= f(1) = 1^2 + \frac{1}{2} \cdot 1 = 1.5\)
Area \(= f(1) \cdot 0.5 = 0.75\)
Third interval:
Height \(= f(1.5) = 1.5^2 + \frac{1}{2} \cdot 1.5 = 3\)
Area \(= f(1.5) \cdot 0.5 = 1.5\)
Fourth interval:
Height \(= f(2) = 2^2 + \frac{1}{2} \cdot 2 = 5\)
Area \(= f(2) \cdot 0.5 = 2.5\)
Total area of rectangles:
\[
0.25 + 0.75 + 1.5 + 2.5 = 5
\]

Example 3 (continued)
Combining answers,
\[
2.5 < \text{Desired Area} < 5
\]
Difference between two estimates: \(5 - 2.5 = 2.5\).
This is same as
\[
\frac{(f(2) - f(0)) \cdot (2 - 0)}{4}
\]
Also, notice the first approximation was less than the actual area, whereas the second approximation was greater than the actual area.
Example 4

Estimate the area under the graph of \( f(x) = 3^x \) for \( x \) between 0 and 2. Use a partition that consists of four equal subintervals of \([0, 2]\). Use the left endpoint of each subinterval to determine the height of the rectangle.

Again, divide the interval \([0, 2]\) into four equal subintervals,

\[ [0, 0.5], \quad [0.5, 1], \quad [1, 1.5], \quad [1.5, 2] \]

First interval:

\[
\text{Height} = f(0) = 3^0 = 1 \quad \text{Area} = f(0) \cdot 0.5 = 0.5
\]

Second interval:

\[
\text{Height} = f(0.5) = 3^{1/2} \approx 1.732 \quad \text{Area} = f(0.5) \cdot 0.5 \approx 0.866
\]

Third interval:

\[
\text{Height} = f(1) = 3^1 = 3 \quad \text{Area} = f(1) \cdot 0.5 \approx 1.5
\]

Fourth interval:

\[
\text{Height} = f(1.5) = 3^{3/2} \approx 5.196 \quad \text{Area} = f(1.5) \cdot 0.5 \approx 2.598
\]

Total area of rectangles:

\[
0.5 + 0.866 + 1.5 + 2.598 = 5.464
\]

From the picture, area of these rectangles will be slightly less than desired area.

Example 4 (continued)

Third interval:

\[
\text{Height} = f(1) = 3^1 = 3 \quad \text{Area} = f(1) \cdot 0.5 \approx 1.5
\]

Fourth interval:

\[
\text{Height} = f(1.5) = 3^{3/2} \approx 5.196 \quad \text{Area} = f(1.5) \cdot 0.5 \approx 2.598
\]

Total area of rectangles:

\[
0.5 + 0.866 + 1.5 + 2.598 = 5.464
\]

From the picture, area of these rectangles will be slightly less than desired area.
Example 5
Estimate the area of the ellipse given by the equation

\[ 4x^2 + y^2 = 49 \]

The area of the ellipse is four times the area of the part of the ellipse in the first quadrant.
We thus need to estimate the area under the curve

\[ f(x) = \sqrt{49 - 4x^2} \]

from \( x = 0 \) to \( x = 3.5 \)
(Find right endpoint by setting \( 49 - 4x^2 = 0 \) and finding \( x \))
We’ll do this using four rectangles.

Divide the interval \([0, 3.5]\) into four equal subintervals,

\([0, 0.875], \ [0.875, 1.75], \ [1.75, 2.625], \ [2.625, 3.5]\)

Width of each rectangle is \( 3.5/4 = 0.875 \).

Example 5 (continued)

First interval:

Height \( = f(0) = \sqrt{49 - 4 \cdot 0^2} = 7 \)
Area \( = f(0) \cdot 0.875 = 7 \cdot 0.875 = 6.125 \)

Second interval:

Height \( = f(0.875) = \sqrt{49 - 4 \cdot 0.875^2} = 6.778 \)
Area \( = f(0.875) \cdot 0.875 = 6.778 \cdot 0.875 = 5.931 \)

Third interval:

Height \( = f(1.75) = \sqrt{49 - 4 \cdot 1.75^2} = 6.062 \)
Area \( = f(1.75) \cdot 0.875 = 6.062 \cdot 0.875 = 5.304 \)
Example 5 (continued)

Fourth interval:

\[ \text{Height} = f(2.625) = \sqrt{49 - 4 \cdot 2.625^2} = 4.630 \]

\[ \text{Area} = f(2.625) \cdot 0.875 = 4.630 \cdot 0.875 = 4.051 \]

Total area of rectangles:

\[ 6.125 + 5.931 + 5.304 + 4.051 = 21.411 \]

From the picture, area of these rectangles will be slightly more than desired area.

Recall that 21.411 was an approximation to the area of the part of the ellipse in the first quadrant.

An approximation for the area of the ellipse is then \( 4 \cdot 21.411 = 85.644 \)

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Example 6

A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>0</td>
<td>10</td>
<td>15</td>
<td>18</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

Approximate the distance travelled by assuming that the velocity of the train is a linear function between each measurement.

The assumption that the velocity is piecewise linear means that we should use trapezoids.

Each trapezoid has width 0.1.

\[ \text{Area of first trapezoid} = \frac{0 + 10}{2} \cdot 0.1 = 0.5 \]
Example 6 (continued)

Area of second trapezoid = \( \frac{10 + 15}{2} \cdot 0.1 = 1.25 \)

Area of third trapezoid = \( \frac{15 + 18}{2} \cdot 0.1 = 1.65 \)

Area of fourth trapezoid = \( \frac{18 + 20}{2} \cdot 0.1 = 1.9 \)

Area of fifth trapezoid = \( \frac{20 + 25}{2} \cdot 0.1 = 2.25 \)

Total distance is then

\[ 0.5 + 1.25 + 1.65 + 1.9 + 2.25 = 7.55 \text{ miles} \]

Example 7

Estimate the area under the graph of \( f(x) = \frac{1}{x} \) for \( x \) between 1 and 31 using 30 equal subintervals and using left endpoints.

Each rectangle will have width 1. The heights of the rectangles will be given by the sequence

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{30} \]

and so the area of the rectangles are also given by the same sequence. The area of the rectangles is thus

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{30} \approx 3.994987 \]

This area will be larger than the desired area.
Example 7 (continued)

Estimate the area under the graph of \( f(x) = \frac{1}{x} \) for \( x \) between 1 and 31 using 30 equal subintervals and using right endpoints.

Each rectangle will have width 1. The heights of the rectangles will be given by the sequence

\[
\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \cdots \cdot \frac{1}{31}
\]

and so the area of the rectangles are also given by the same sequence. The area of the rectangles is thus

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{31} \approx 3.027245
\]

This area will be smaller than the desired area.

Example 8

Estimate the area under the graph of \( f(x) = \frac{1}{x} \) for \( x \) between 1 and \( n \) where \( n \) is a positive integer. Use right endpoints for the rectangles. Make your estimate for several large values of \( n \), say \( n = 10, 20, 30 \), etc. What do you think happens as \( n \) tends to \( \infty \)?

For convenience use subintervals of lengths 1. Width of each rectangle is 1 and right endpoints are

\[
\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \cdots \cdot \frac{1}{n}
\]

The areas of the rectangles is thus

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}
\]

Estimating the area from \( x = 1 \) to \( x = n \) amounts to evaluating this sum up to \( n \):
Example 8 (continued)

For $x = 1$ to $x = 10$:
\[
\text{Area} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{10} = 1.928968
\]

For $x = 1$ to $x = 20$:
\[
\text{Area} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{20} = 2.597734
\]

For $x = 1$ to $x = 30$:
\[
\text{Area} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{30} = 2.994987
\]

Example 8 (continued)

What do you think happens as $n$ gets larger and larger?

Using a computer, I found:

For $x = 1$ to $x = 10^2 = 100$:
\[
\text{Area} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{100} = 4.187378
\]

For $x = 1$ to $x = 100^2 = 10,000$:
\[
\text{Area} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{10,000} = 8.787506
\]

For $x = 1$ to $x = (10,000)^2 = 100,000,000$:
\[
\text{Area} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{100,000,000} = 17.997896
\]
Example 8 (continued)

Doubling the number of digits of $n$ roughly doubled the area. Thus, the area grows without bound as $n$ goes to infinity.

**Discussion:** Recall $\ln(n^2) = 2\ln(n)$. Also, $n^2$ has roughly twice as many digits as $n$. Now

\[
\ln(n^2) = 2 \cdot \ln(n)
\]

Doubling the number of digits of $n$ doubles the logarithm. This suggests a relation between $\ln(x)$ and finding area under $y = 1/x$ from 1 to $n$.

Do we know another relation between $\ln(x)$ and $y = 1/x$???

Chapter 10 will tie things together.

Σ notation

We need a convenient shorthand for expressions like

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}
\]

We can describe the above sum verbally as:
“Add the reciprocals of each integer from 1 to 10”

We will express this symbolically as

\[
\sum_{k=1}^{10} \frac{1}{k}
\]
Σ notation

Σ is a command to add together whatever follows. The “$k = 1$” below and the “10” above tells us to start at the number 1 and go until we reach the number 10. The $\frac{1}{k}$ after the Σ tells us that we are adding the reciprocals. $k$ is refered to as a “dummy variable.”

\[
\sum_{k=1}^{10} \frac{1}{k} \quad \text{and} \quad \sum_{j=1}^{10} \frac{1}{j}
\]

mean the same thing. The choice of letter is unimportant here.

Note: Σ notation is nothing more than notation. It allows us to write sums in a compact form. It does not evaluate the sum for us.

Example 9

Evaluate the sum

\[
\sum_{k=1}^{5} (2k - 1)
\]

This means:

Evaluate $2k - 1$ at $k = 1, 2, 3, 4, 5$, then add those.

\[
\sum_{k=1}^{5} (2k - 1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1)
\]

\[
= 1 + 3 + 5 + 7 + 9 = 25
\]
Example 9 (continued)

Notice we could have grouped terms together as follows:

\[(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1)\]

\[= 2 \cdot (1 + 2 + 3 + 4 + 5) - (1 + 1 + 1 + 1 + 1)\]

and so we see

\[\sum_{k=1}^{5} (2k - 1) = 2 \cdot \sum_{k=1}^{5} k - \sum_{k=1}^{5} 1\]

Such manipulations can be helpful in computing large sums.

Example 10

Evaluate the sum

\[\sum_{k=2}^{6} (6k^3 + 3)\]

\[\sum_{k=2}^{6} (6k^3 + 3) = (6 \cdot 2^3 + 3) + (6 \cdot 3^3 + 3) + (6 \cdot 4^3 + 3) + (6 \cdot 5^3 + 3) + (6 \cdot 6^3 + 3)\]

\[= 51 + 165 + 387 + 753 + 1299 = 2655\]

Again, we note that

\[\sum_{k=2}^{6} (6k^3 + 3) = 6 \sum_{k=2}^{6} k^3 + 3 \sum_{k=2}^{6} 1\]
Example 11

Evaluate the sum

$$\sum_{k=1}^{5}(3k^2 + k)$$

This time lets rewrite the sum first:

$$\sum_{k=1}^{5}(3k^2 + k) = 3 \sum_{k=1}^{5}k^2 + \sum_{k=1}^{5}k$$

Now,

$$\sum_{k=1}^{5}k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

and

$$\sum_{k=1}^{5}k = 1 + 2 + 3 + 4 + 5 = 15$$

Example 11 (continued)

So

$$\sum_{k=1}^{5}(3k^2 + k) = 3 \sum_{k=1}^{5}k^2 + \sum_{k=1}^{5}k$$

$$= 3 \cdot 55 + 15 = 180$$
More on Summations (Preview Chapter 9)

Why do we bother writing

$$\sum_{k=1}^{5} (3k^2 + k) = 3 \sum_{k=1}^{5} k^2 + \sum_{k=1}^{5} k$$

and evaluating the two sums on the right? Why not just evaluate the sum on the left directly?

Certain sums (like the ones on the right) pop up so often that it is convenient to memorize formulas for them.

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Thus

$$\sum_{k=1}^{5} k^2 = \frac{5 \cdot (5+1) \cdot (2 \cdot 5 + 1)}{6} = 55$$

We’ll learn more summation formulas in Chapter 9.
Example 12

Evaluate the sum
\[
\sum_{k=1}^{112} 75
\]

This sum is
\[
\sum_{k=1}^{112} 75 = \underbrace{75 + 75 + 75 + \cdots + 75}_{112 \text{ times}}
\]

\[
= 75 \cdot 112
\]

\[
= 8400
\]

Example 3, Revisited

We wanted to estimate the area under \( f(x) = x^2 + \frac{1}{2}x \) from \( x = 0 \) to \( x = 2 \). We did so by subdividing \([0, 2]\) into 4 equal pieces, constructing rectangles on each of these, finding the area of each rectangle, then adding these together.

Using the left hand sums,

\[
\text{Area} \approx f(0) \cdot 0.5 + f(0.5) \cdot 0.5 + f(1) \cdot 0.5 + f(1.5) \cdot 0.5
\]

\[
= \sum_{k=1}^{4} \left( \frac{k - 1}{2} \right) \cdot 0.5
\]

Height of \( k \)th rectangle

Area of \( k \)th rectangle

Width of \( k \)th rectangle
Example 3, Revisited

Using the right hand sums,

\[
\text{Area} \approx f(0.5) \cdot 0.5 + f(1) \cdot 0.5 + f(1.5) \cdot 0.5 + f(2) \cdot 0.5
\]

\[
= \sum_{k=1}^{4} f \left( \frac{k}{2} \right) \cdot 0.5
\]

\[
\text{Height of } k\text{th rectangle} \quad \text{Width of } k\text{th rectangle}
\]

\[
\text{Area of } k\text{th rectangle}
\]

Riemann Sums

Recap of Examples 3 through 8:
Had a function \( f(x) \) defined on interval \([a, b]\).
Used rectangles to estimate area between \( y = f(x) \) and \( x\)-axis

Number of rectangles will be denoted by \( n \)

Width of each rectangle is \( \Delta x = \frac{b-a}{n} \)

Partition points are \( x_k = a + \Delta x \)
Then
\[a = x_0 < x_1 < x_2 < \cdots < x_n = b\]

Interval \([a, b]\) is divided into subintervals:
\[
[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \cdots, \quad [x_{n-1}, x_n]
\]

\( k\text{th} \) interval is \([x_{k-1}, x_k]\).
Riemann Sums

For each $k$, $p_k$ will denote a point from $[x_{k-1}, x_k]$ (We will only use $p_k = x_{k-1}$ or $p_k = x_k$.)

Height of $k^{th}$ rectangle will be $f(p_k)$.

A Riemann sum for $f(x)$ on $[a, b]$ is an expression of the form

$$\sum_{k=1}^{n} f(p_k) \Delta x$$

Remember:

- $f(p_k)$ is height of $k^{th}$ rectangle
- $\Delta x$ is width of $k^{th}$ rectangle
- So $f(p_k)\Delta x$ is area of $k^{th}$ rectangle

Left and Right Hand Sums

Taking $p_k = x_{k-1}$ gives the $n$th left hand sum of $f(x)$ on $[a, b]$,

$$\sum_{k=1}^{n} f(x_{k-1}) \Delta x = \sum_{k=1}^{n} f \left( a + (k - 1) \cdot \frac{b-a}{n} \right) \cdot \frac{b-a}{n}$$

Taking $p_k = x_k$ gives the $n$th right hand sum of $f(x)$ on $[a, b]$ is

$$\sum_{k=1}^{n} f(x_k) \Delta x = \sum_{k=1}^{n} f \left( a + k \cdot \frac{b-a}{n} \right) \cdot \frac{b-a}{n}$$
Better Approximations

Any particular choice of Riemann Sum for \( f(x) \) on \([a, b]\) gives an approximation to the area between \( f(x) \) and the \( x \)-axis.

The graphs and applets show that these approximations improve as the number of rectangles increase.

This suggests getting the exact area by taking a limit as the number of rectangles increases to infinity.

This will require a little more notation. (Don’t worry, we almost have everything in place)

The Definite Integral

**Theorem**

*Suppose \( f(x) \) is continuous on \([a, b]\). Then*

\[
\lim_{n \to \infty} \sum_{k=1}^{n} f(p_k) \cdot \Delta x_k
\]

*exists, and is independent of the choice of partition and the choice of sample points.*

*The value of this limit is called the **definite integral of** \( f(x) \) on \([a, b]\), and is denoted by*

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(p_k) \cdot \Delta x_k
\]

*a and \( b \) are called the **limits of integration**, \( f(x) \) is called the **integrand**, and \( dx \) is called the **differential**.*
Example 13:
Suppose we estimate the integral
\[ \int_{1}^{7} 8^x \, dx \]
using the sum
\[ \sum_{k=1}^{n} 8^{1+k \cdot \Delta x} \Delta x \]
If \( \Delta x = 0.2 \), what is the value of \( n \)?

\[ \Delta x = \frac{b - a}{n} \]

But \( a = 1 \), \( b = 7 \) and \( \Delta x = 0.2 \) so

\[ 0.2 = \frac{7 - 1}{n} = \frac{6}{n} \]

\[ n = \frac{6}{0.2} = 30 \]

Example 14
Suppose we estimate the integral
\[ \int_{2}^{10} x^2 \, dx \]
by evaluating the sum
\[ \sum_{k=1}^{n} (2 + k \cdot \Delta x)^2 \Delta x \]
If we use \( n = 10 \) intervals, what is the value of \( \Delta x \)?

We divide \([2, 10]\), an interval of width 8, into 10 equal pieces. Thus the width of each rectangle is

\[ \Delta x = \frac{8}{10} = 0.8 \]
Example 15
Suppose we estimate the integral
\[
\int_{-6}^{0} x^2 \, dx
\]
by the sum
\[
\sum_{k=1}^{n} \left[ A + B \cdot (k \cdot \Delta x) + C \cdot (k \cdot \Delta x)^2 \right] \Delta x
\]
where \( n = 30 \) and \( \Delta x = 0.2 \) The terms in the sum equal areas of rectangles obtained by using right endpoints of the subintervals of length \( \Delta x \) as sample points. What is the value of \( B \)?
The right hand sum, where \( n = 30 \) and \( \Delta x = 0.2 \) will be
\[
\sum_{k=1}^{30} (-6 + k \cdot \Delta x)^2 \Delta x
\]
Example 15 (continued)
Thus, for each \( k \) we have
\[
(-6 + k \cdot \Delta x)^2 = 36 + (-12) (k \cdot \Delta x) + (k \cdot \Delta x)^2
\]
Thus,
\[
A = 36
\]
\[
B = -12
\]
\[
C = 1
\]
Example 16
Suppose we estimate the integral
\[ \int_{5}^{15} x^3 \, dx \]
by the sum
\[ \sum_{k=1}^{n} (a + k \cdot \Delta x)^3 \Delta x \]
where \( n = 50 \) and \( \Delta x = 0.2 \). The terms in the sum equal areas of rectangles obtained by using right endpoints of the subintervals of length \( \Delta x \) as sample points. What is the value of \( a \)?

The right hand sum, with \( n = 50 \) and \( \Delta x = 0.2 \) is
\[ \sum_{k=1}^{50} (5 + k \cdot \Delta x)^3 \Delta x \]

Example 16 (continued)

Equating the summands in the two sums:
\[ (5 + k\Delta x)^3 = (a + k\Delta x)^3 \]
for each \( k \).

Therefore, \( a = 5 \).
Example 17

Suppose we estimate the integral

$$\int_{3}^{15} f(x) \, dx$$

by adding areas of $n$ rectangles of equal width, and use right endpoints to determine height of each rectangle. Determine $A$ if the sum we evaluate is

$$\sum_{k=1}^{n} f \left( 3 + k \cdot \frac{A}{n} \right) \cdot \frac{A}{n}$$

The term $A/n$ appears where we would place the $\Delta x$ so

$$\frac{A}{n} = \Delta x$$

Example 17 (continued)

But

$$\Delta x = \frac{b - a}{n} = \frac{15 - 3}{n} = \frac{12}{n}$$

Therefore

$$\frac{A}{n} = \frac{12}{n}$$

and so $A = 12$. 
Example 18
Suppose we estimate
\[ \int_3^9 f(x) \, dx \]
by evaluating a sum
\[ \sum_{k=1}^{n} f(3 + k \cdot \Delta x) \Delta x \]
If you use \( n = 6 \) intervals of equal length, what value should we use for \( \Delta x \)?

\[ \Delta x = \frac{b - a}{n} \]

but \( a = 3 \), \( b = 9 \) and \( n = 6 \) so

\[ \Delta x = \frac{9 - 3}{6} = 1 \]

Example 19
Suppose we estimate the area under the graph of \( f(x) = x^3 \) from \( x = 4 \) to \( x = 24 \) by adding the areas of rectangles as follows: partition the interval into 20 equal subintervals and use the right endpoint of each interval to determine the height of the rectangle. What is the area of the fifteenth rectangle?

First, \( a = 4 \), \( b = 24 \) and \( n = 20 \) so

\[ \Delta x = \frac{b - a}{n} = \frac{24 - 4}{20} = 1 \]

Next, the partition points are

\[ x_k = a + k \cdot \Delta x = 4 + k, \quad \text{for} \quad k = 0, 1, 2, 3 \ldots, 24 \]
Example 19 (continued)

The $k$th interval is

$$[x_{k-1}, x_k]$$

and so the 15th interval is

$$[x_{14}, x_{15}] = [4 + 14, 4 + 15] = [18, 19]$$

The height of the 15th rectangle is therefore

$$f(x_{15}) = f(19) = 19^3$$

and so the area of the 15th rectangle is

$$f(x_{15}) \cdot \Delta x = 19^3 \cdot 1 = 6859$$

Example 20

Suppose we estimate the area under the graph of

$$f(x) = \frac{1}{x}$$

from $x = 12$ to $x = 112$ by adding the areas of rectangles as follows: partition the interval into 50 subintervals and use the left endpoint of each interval to determine the height of the rectangle. What is the area of the 24th rectangle?

$n = 50$, $a = 12$, $b = 112$ and so

$$\Delta x = \frac{112 - 12}{50} = 2$$

Then the partition points are

$$x_k = a + k \Delta x = 12 + 2k$$
Example 20 (continued)

The 24th interval is

\[ [x_{23}, x_{24}] = [12 + 2 \cdot 23, 12 + 2 \cdot 24] = [58, 60] \]

We use the left endpoint to determine the height:

\[ f(58) = \frac{1}{58} \]

The area is then

\[ f(58) \cdot \Delta x = \frac{1}{58} \cdot 2 = \frac{1}{29} \]

Example 21

Suppose we are given the following data points for the function \( f(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

If \( f \) is a linear function on each interval between the given points, then find

\[ \int_{1}^{4} f(x) \, dx \]

The integral gives the area under the curve. Since curve is piecewise linear, trapezoids will give the exact area.
Example 21

Build a trapezoid on each of the intervals $[1, 2]$, $[2, 3]$, $[3, 4]$ Each has width 1. The areas of the trapezoids are

$$
\frac{2 + 5}{2} = \frac{7}{2}
$$

$$
\frac{5 + 8}{2} = \frac{13}{2}
$$

$$
\frac{8 + 12}{2} = 10
$$

Thus, the area of the three trapezoids is

$$
\frac{7}{2} + \frac{13}{2} + 10 = 20
$$

Thus,

$$
\int_{1}^{4} f(x) \, dx = 20
$$
Example 22

Suppose $f(x)$ is the greatest integer function (recall example 19 from Chapter 3)

Find

$$
\int_{6}^{10} f(x) \, dx
$$

$$
= 6 \cdot 1 + 7 \cdot 1 + 8 \cdot 1 + 9 \cdot 1
= 30
$$