Homework 06 Solutions
Math 123

These are solutions to the common version problems. The problems on your personal version should be similar, but usually with different numbers.

2. Solution:
f′(1/2) is the slope of the tangent line to y = f(x) at x = 1/2. But f(x) is a linear function and so f′(1/2) is just the slope of the graph y = f(x). From the graph we can read off the rise and the run. The points (−0.5, 0) and (0, 1) are on the graph and so the rise is 1 − 0 and the run is 0 − (−0.5) = 0.5, so rise over run is 1/0.5 = 2, thus f′(1/2) = 2.

3. The graph of the function f(x) (solid) and its tangent at (1, 2) (dotted) is given above. Find f′(1).
Solution:
f′(1) is the slope of the tangent line to y = f(x) at x = 1. Thus, f′(1) is the slope of the dotted line. Reading off the rise and the run from the graph we see that the slope is $\frac{2 - (-4)}{1 - 0} = 6$ and so f′(1) = 6.

4. For the function f(x) = 8x^2 + 4, find the equation of the tangent line to the graph of f(x) at x = 1. If the equation of the tangent line is y = mx + b, what are m and b?
Solution:
f′(x) = 16x and so the slope of the tangent line at x = 1 is f′(1) = 16. The point of tangency is (1, f(1)) = (1, 8 · 1^2 + 4) = (1, 12). The equation of the tangent line is then

$$y - 12 = 16(x - 1).$$

Writing this is slope-intercept form:

$$y = 16x - 4$$

and so m = 16 and b = −4.

5. For the function f(x) = (x + 8)^2, find the equation of the tangent line to the graph of f(x) at x = −3. If the equation of the tangent line is y = mx + b, what are m and b?
Solution:
f′(x) = (x + 8)^2 = x^2 + 16x + 64 and so f′(x) = 2x + 16 and so the slope of the tangent line to f(x) at x = −3 is f′(−3) = 2 · (−3) + 16 = 10. The point of tangency is (−3, f(−3)) = (−3, (−3 + 8)^2) = (−3, 25). The equation of the tangent line is then

$$y - 25 = 10(x - (-3)).$$

Writing this is slope-intercept form:

$$y = 10x + 55$$

and so m = 10 and b = 55.

6. Consider the function f(x) = 2x^2 + 8. Its tangent line at x = 2 goes through the points (2, y_1) and (−1, y_2). Find y_1 and y_2.
Solution:
First we find the tangent line. The slope is f′(2), but f′(x) = 4x and so f′(2) = 8. The point of tangency is (2, f(2)) = (2, 16) and so the equation of the tangent line is

$$y - 16 = 8(x - 2).$$
Putting this in slope-intercept form, 
\[ y = 8x. \]

Now we want to find the \( y \) values belonging to \( x = 2 \) and \( x = -1 \). For \( x = 2 \) we find \( y_1 = 8 \cdot 2 = 16 \). For \( x = -1 \) we find \( y_2 = 8 \cdot (-1) = -8 \).

7. If the tangent line to the function \( f(x) = \sqrt{x} + a \) at \( x = 16 \) has equation \( y = mx + 1 \), find \( a \) and \( m \). You may use \( f'(x) = \frac{1}{2\sqrt{x}} \).

Solution:
The slope of the tangent line is \( f'(16) \).
Using the hint, \( f'(x) = \frac{1}{2\sqrt{x}} \) and so \( f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8} \).

Therefore \( m = \frac{1}{8} \) and the tangent line is \( y = \frac{1}{8}x + 1 \). Now, the point of tangency is \( (16, \sqrt{16} + a) = (16, 4 + a) \). But this can also be obtained by plugging in \( x = 16 \) into the tangent line so \( y = \frac{1}{8} \cdot 16 + 1 = 3 \). Therefore, \( 4 + a = 3 \) and so \( a = -1 \).

8. The graph of \( f(x) \) is given. Find all values of \( x \) where the derivative \( f'(x) \) is not defined.

Solution:
The derivative fails to exist at any point where the graph has no tangent line, as well as any point where the tangent line is vertical. There does not appear to be any points with vertical tangent lines. The tangent line fails to exist at any point with a corner or any point where the graph has a jump. The graph has a corner at \( x = -2 \) and a jump at \( x = 0 \), so \( f'(x) \) fails to exist at \( x = -2 \) and \( x = 0 \).

9. Find all values of \( x \) where the derivative of the function
\[ f(x) = |x^2 - 9x + 18| \]
is not defined. Solution:
We graph \( f(x) \) and look for points where graph has a sharp corner. These occur at any point where \( x^2 - 9x + 18 \) changes from positive to negative or vice versa. But \( x^2 - 9x + 8 = (x - 3)(x - 6) \), and so \( x^2 - 9x + 18 \) changes signs at \( x = 3 \) and \( x = 6 \). Thus, the derivative is not defined at \( x = 3 \) and \( x = 9 \).

10. Find all values of \( x \) where the function
\[ f(x) = \begin{cases} 
  x - 4, & \text{if } x \leq 4 \\
  4 - x, & \text{if } 4 < x < 6 \\
  x^2 + 6, & \text{if } 6 \leq x
\end{cases} \]
is continuous but not differentiable.

Solution:
The only points that might possibly be points of nondifferentiability are \( x = 4 \) and \( x = 6 \). It seems easiest to answer this question if we can produce the graph first.

Now, plugging in \( x = 6 \) into \( y = 4 - x \) gives \( y = -2 \) whereas plugging \( x = 6 \) into \( y = x^2 + 5 \) gives \( y = 41 \) so \( f(x) \) is neither continuous nor differentiable at \( x = 6 \).
On the other hand, plugging in $x = 4$ into either $y = x - 4$ or $y = 4 - x$ gives $y = 0$ and so $f(x)$ is continuous at $x = 4$. However, if you graph $y = f(x)$ you should see a sharp corner at $x = 4$. Thus, $f(x)$ is continuous but not differentiable at $x = 4$. 