1. (a) (6 pts) Write out the form of the partial fraction decomposition of the function \( \frac{2x - 3}{(x - 1)(x^2 + x + 1)^2} \). Don’t determine the numerical values of the coefficients.

\[
\frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1} + \frac{Dx + E}{(x^2 + x + 1)^2}
\]

(b) (12 pts) Use the method of partial fractions to evaluate the integral

\[
I = \int_0^1 \frac{2}{(x + 1)(x^2 + 1)} \, dx.
\]

\[
\frac{2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}
\]

\[
2 = A(x^2+1) + (Bx+C)(x+1) \quad \cdots \quad (1)
\]

Put \( x = -1 \). We get \( 2 = 2A \Rightarrow A = 1 \)

Comparing coefficients on both sides of (1), we obtain

\[
A + B = 0 \quad \Rightarrow \quad B = -1,
\]

\[
A + C = 2 \quad \Rightarrow \quad C = 1.
\]

\[
\int_0^1 \frac{1}{x+1} \, dx = \left[ \ln |x+1| \right]_0^1 = \ln 2
\]

\[
- \int_0^1 \frac{x}{x^2+1} \, dx = - \left[ \frac{1}{2} \ln (x^2+1) \right]_0^1 = -\frac{1}{2} \ln 2
\]

\[
\int_0^1 \frac{1}{x^2+1} \, dx = \left[ \tan^{-1} x \right]_0^1 = \frac{\pi}{4}
\]

\[\therefore \quad I = \frac{1}{2} \ln 2 + \frac{\pi}{4}. \]
2. Determine whether each improper integral below is convergent or divergent. If it is convergent, then evaluate it.

(a) (10 pts) $\int_{1}^{\infty} \frac{dx}{x(x+1)}$

\[
\int_{1}^{\infty} \frac{dx}{x(x+1)} = \lim_{t \to \infty} \int_{1}^{t} \left( \frac{1}{x} - \frac{1}{x+1} \right) dx
\]

\[
= \lim_{t \to \infty} \left[ \ln x - \ln(x+1) \right]_{1}^{t}
\]

\[
= \lim_{t \to \infty} \left( \ln t - \ln(t+1) \right) + \ln 2
\]

\[
= \lim_{t \to \infty} \ln \left( \frac{t}{t+1} \right) + \ln 2
\]

\[
= \ln \left( \lim_{t \to \infty} \frac{t}{t+1} \right) + \ln 2
\]

\[
= \ln 1 + \ln 2
\]

\[
= \ln 2.
\]

\[
\therefore \text{The given improper integral is convergent.}
\]
(b) (10 pts) \( \int_{1}^{2} \frac{x}{\sqrt{x-1}} \, dx \).

\[
I = \lim_{t \to 1^+} \int_{1}^{2} \frac{x}{\sqrt{x-1}} \, dx = \lim_{t \to 1^+} \int_{t}^{2} \frac{x}{\sqrt{x-1}} \, dx.
\]

Let \( u = x - 1 \). Then \( du = dx \), \( x = 1 + u \). When \( x = t \), \( u = t - 1 \). When \( x = 2 \), \( u = 1 \). Hence

\[
I = \lim_{t \to 1^+} \int_{t-1}^{1} \frac{1+u}{\sqrt{u}} \, du
\]

\[
= \lim_{t \to 1^+} \left( \int_{t-1}^{1} u^{-\frac{1}{2}} \, du + \int_{t-1}^{1} u^{\frac{2}{3}} \, du \right)
\]

\[
= \lim_{t \to 1^+} \left( \left[ 2u^{\frac{1}{2}} \right]_{t-1}^{1} + \left[ \frac{3}{2} u^{\frac{5}{3}} \right]_{t-1}^{1} \right)
\]

\[
= \lim_{t \to 1^+} \left( 2(1 - \sqrt{t-1}) + \frac{2}{3} (1 - (t-1)^{3/2}) \right)
\]

\[
= 2 + \frac{2}{3} = \frac{8}{3}.
\]

Thus, the given improper integral is convergent.
3. (16 pts) Use the integral test to show that the infinite series \( \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \) is convergent. You need to verify that your choice of \( f \) satisfies the hypotheses of this test.

Let \( f(x) = \frac{\ln x}{x^2} \). Then \( f'(x) = \frac{1 - 2 \ln x}{x^3} \).

Since \( f(x) > 0 \) and \( f'(x) < 0 \) on \([1, \infty)\),
the function \( f \) satisfies the hypotheses of the integral test.

\[
\int_1^\infty \frac{\ln x}{x^2} \, dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x^2} \, dx
\]

\[
= \lim_{t \to \infty} \left( \left[ -\frac{\ln x}{x} \right]_1^t + \int_1^t \frac{1}{x} \cdot \frac{1}{x} \, dx \right)
\]

\[
= \lim_{t \to \infty} \left( -\frac{\ln t}{t} + \left[ -\frac{1}{x^2} \right]_1^t \right)
\]

\[
= -\lim_{t \to \infty} \frac{\ln t}{t} - \lim_{t \to \infty} \frac{1}{t} + 1
\]

\[
= 0 + 0 + 1
\]

\[
= 1.
\]

By the integral test, the series \( \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \) is convergent.
4. (a) The following table gives the values of a function $W$ at the given points in the interval $[0, H]$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$H/4$</th>
<th>$H/2$</th>
<th>$3H/4$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W(x)$</td>
<td>0.093</td>
<td>0.067</td>
<td>0.082</td>
<td>0.030</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Let $I = \int_{0}^{H} W(x)dx$. In each part below, you may leave your answer as an expression that involves sums and products of numbers or decimals.

(i) (5 pts) Use the Trapezoidal Rule with $n = 4$ to obtain an approximate value of $I$ expressed in terms of $H$.

$$T_4 = \frac{H}{8} \left( 0.093 + 2 \times 0.067 + 2 \times 0.082 + 2 \times 0.030 + 0.009 \right)$$

$$= (0.0575) H$$

(ii) (5 pts) Use Simpson’s Rule with $n = 4$ to obtain an approximate value of $I$ expressed in terms of $H$.

$$S_4 = \frac{H}{12} \left( 0.093 + 4 \times 0.067 + 2 \times 0.082 + 4 \times 0.030 + 0.009 \right)$$

$$= (0.0545) H$$

(iii) (2 pts) Use the approximate value of $I$ obtained either in (i) or in (ii) to get an estimate for the average value (or integral average) of $W$ on $[0, H]$.

$$\frac{T_4}{H} = 0.0575 \quad \text{or} \quad \frac{S_4}{H} = 0.0545$$
4. (b) (8 pts) How large should we take \( n \) in order to guarantee that the approximation by Simpson’s Rule for \( \int_2^3 \ln x \, dx \) is accurate to within \( 10^{-4} \)? Recall that the error bound for Simpson’s rule for \( \int_a^b f(x) \, dx \) is given by

\[
|E_S(n)| \leq \frac{K(b - a)^5}{180n^4}, \quad \text{with} \quad K = \max\{|f^{(4)}(x)| : a \leq x \leq b\}.
\]

\[
f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2},
\]

\[
f'''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4}
\]

\[
K = \max\left\{ \frac{6}{x^4} : 1 \leq x \leq 2 \right\}
\]

\[
= 6
\]

\[
\frac{K (b - a)^5}{180n^4} < 10^{-4} \iff \frac{1}{30n^4} < 10^{-4}
\]

\[
n > \sqrt[4]{\frac{1}{30} \times 10} \approx 4.28 \quad \text{and} \quad n \text{ is even.}
\]

\[
\Rightarrow \quad n = 6.
\]
5. (10 pts) Set up, but do not evaluate, an integral for the length of the curve

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \ a > 0, \ b > 0.
\]

The given curve can be described parametrically by

\[x = a \cos \theta, \quad y = b \sin \theta, \quad 0 \leq \theta \leq 2\pi.\]

\[L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \ d\theta = \int_0^{2\pi} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \ d\theta\]

Alternate Solution

For the part of the given ellipse above the x-axis,

\[y = b \sqrt{1 - \frac{x^2}{a^2}}.\]

\[\frac{dy}{dx} = -\frac{b}{a^2} \frac{x}{\sqrt{1 - \frac{x^2}{a^2}}}, \quad \left(\frac{dy}{dx}\right)^2 = \frac{b^2 x^2}{a^2(a^2-x^2)}\]

\[L = 2 \int_{-a}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx = 2 \int_{-a}^{a} \sqrt{1 + \frac{b^2 x^2}{a^2(a^2-x^2)}} \ dx\]

\[= 4 \int_0^{a} \sqrt{1 + \frac{b^2 x^2}{a^2(a^2-x^2)}} \ dx.\]
6. A curve $C$ is defined by the parametric equations

$$x = t^3 - 3t, \quad y = 3t^2 - 9.$$ 

(a) (10 pts) Find the Cartesian coordinates of the points on $C$ where (i) the tangent is horizontal and (ii) the tangent is vertical.

$$\frac{dx}{dt} = 3t^2 - 3 = 3(t^2 - 1), \quad \frac{dy}{dt} = 6t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{t^2 - 1}$$

$$\frac{dy}{dt} = 0 \Leftrightarrow t = 0. \quad \text{At } t = 0, \quad \frac{dx}{dt} \neq 0$$

and \((x, y) = (0, -9)\).

\therefore The tangent is horizontal at \((x, y) = (0, -9)\).

$$\frac{dx}{dt} = 0 \Leftrightarrow t = \pm 1. \quad \text{At } t = 1, \quad \frac{dy}{dt} \neq 0$$

and \((x, y) = (-2, -6)\). \quad \text{At } t = -1, \quad \frac{dy}{dt} \neq 0$$

and \((x, y) = (2, -6)\).

\therefore The tangent is vertical at \((x, y) = (-2, -6)\) and \((x, y) = (2, -6)\).
6. (b) (6 pts) Determine the values of $t$ for which (i) the curve is concave upward and (ii) the curve is concave downward.

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{2t}{t^2-1} \right) \frac{1}{3(t^2-1)}$$

$$= \frac{2}{3} \cdot \frac{-(1+t^2)}{(t^2-1)^3}$$

$$\therefore \frac{d^2y}{dx^2} > 0 \quad \text{if} \quad t^2 - 1 < 0$$

$$\frac{d^2y}{dx^2} < 0 \quad \text{if} \quad t^2 - 1 > 0$$

The curve is concave upward when $-1 < t < 1$.

It is concave downward when $t > 1$ or $t < -1$. 