1 Lecture 37 and 38: The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral—a new name for anti-derivative.
- Differentiating integrals.

Today we provide the connection between the two main ideas of the course. The integral and the derivative.

**Theorem 1** (FTC I) Suppose $f$ is a continuous function on $[a, b]$. If $F$ is an anti-derivative of $f$, then

$$\int_a^b f(t) \, dt = F(b) - F(a).$$

(FTC II) Assume $f$ is continuous on an open interval $I$ and $a$ is in $I$. Then the area function

$$A(x) = \int_a^x f(t) \, dt$$

is an anti-derivative of $f$ and thus $A' = f$.

**Example.** Compute $\frac{d}{dx} \int_1^x \frac{1}{t} \, dt$.

Compute $\int_0^3 x^3 \, dx$.

**Proof.** An idea of the proofs.

FTC I: We let $F$ be an anti-derivative of $f$ and let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$. We will express the change of $F$, $F(b) - F(a)$, as a Riemann sum for this partition. Letting the size of the largest interval in the partition tend to zero, we obtain the integral is equal to the change in $F$.

We begin by writing

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \ldots + F(x_i) - F(x_{i-1}) + \ldots + F(x_1) - F(x_0).$$

We recall that $F$ is an anti-derivative of $f$ and apply the mean value theorem on each interval $[x_{i-1}, x_i]$ and find a value $c_i$ so that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$. Thus, we have

$$F(b) - F(a) = \sum_{i=1}^n f(c_i)(x_i - x_{i-1}).$$
Since the right-hand side is a Riemann sum for the integral, we may let the width of the largest subinterval tend to zero and obtain

\[ F(b) - F(a) = \int_a^b f(s) \, ds. \]

**FTC II:**

Write

\[ \frac{A(x + h) - A(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt. \]

We will show

\[ \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x). \]

The reader should write out a similar argument for the limit from the left.

If \( f \) is continuous, then \( f \) has maximum and minimum values \( M_h \) and \( m_h \) on the interval \([x, x+h]\). Using the order property of the integral,

\[ m_h \leq \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq M_h. \]

As \( h \) tends to 0, we have \( \lim_{h \to 0^+} M_h = \lim_{h \to 0^+} m_h = f(x) \) since \( f \) is continuous. It follows that

\[ \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x). \]

### 1.1 Indefinite integrals.

We use the symbol

\[ \int f(x) \, dx \]

to denote the indefinite integral or anti-derivative of \( f \).

The indefinite integral is a function. The definite integral is a number. According FTC I, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this evaluation.

\[ F(x)|_{x=a}^b = F(b) - F(a). \]

**Example.** Find

\[ xa|_{x=a}^b \quad \text{and} \quad xa|_{a=x}^y \]

**Solution.**

\[ ba - a^2 \quad xy - x^2 \]
According to FTC II, anti-derivatives exist provided $f$ is continuous.
The box on page 351 should be memorized. (In fact, you should already have memorized this information when we studied derivatives in Chapter 3 and when we studied anti-derivatives in Chapter 4.)

**Example.** Verify
\[
\int x \cos(x^2) \, dx = \frac{1}{2} \sin(x^2).
\]

**Solution.** According to the definition of anti-derivative, we need to see if
\[
\frac{d}{dx} \frac{1}{2} \sin(x^2) = x \cos(x^2).
\]
This holds, by the chain rule.

### 1.2 Computing Integrals.

The main use of FTC I is to simplify the evaluation of integrals.

We give a few examples.

**Example.** a) Compute
\[
\int_0^\pi \sin(x) \, dx.
\]
b) Compute
\[
\int_1^4 \frac{2x^2 + 1}{\sqrt{x}} \, dx.
\]

**Solution.** a) Since \(\frac{d}{dx}(-\cos(x)) = \sin(x)\), we have \(-\cos(x)\) is an anti-derivative of \(\sin(x)\). Using the second part of the fundamental theorem of calculus gives,
\[
\int_0^\pi \sin(x) \, dx = -\cos(x)|_0^\pi = 2.
\]

b) We first find an anti-derivative. As the indefinite integral is linear, we write
\[
\int \frac{2x^2 + 1}{\sqrt{x}} \, dx = \int 2x^{3/2} + x^{-1/2} \, dx = 2 \int x^{3/2} \, dx + \int x^{-1/2} \, dx = \frac{4}{5} x^{5/2} + 2x^{1/2} + C.
\]
With this anti-derivative, we may then use FTC I to find
\[
\int_1^4 \frac{2x^2 + 1}{\sqrt{x}} \, dx = \left. \frac{4}{5} x^{5/2} + 2x^{1/2} \right|_1^4 = \frac{4}{5} 4^{5/2} + 24^{1/2} - \left( \frac{4}{5} + 2 \right) = 128/5 + 20/5 - (4/5 + 10/5) = 134/5.
\]
Example. Find

\[ \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx. \]

Solution. We recognize that \( \sin(x^2) \) is an anti-derivative of \( 2x \cos(x^2) \),

\[ \int 2x \cos(x^2) \, dx = \sin(x^2) + C. \]

Thus,

\[ \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx = \sin(x^2)|_{x=0}^{\sqrt{\pi}} = 0 - 0. \]

Here, is a more involved example that illustrates the progress we have made.

Example. Find

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sin(k/n). \]

Solution. We recognize that

\[ \frac{1}{n} \sum_{k=1}^{n} \sin(k/n) \]

is a Riemann sum for an integral. The points \( x_k, k = 0, \ldots, n \) divide the interval \([0, 1]\) into \( n \) equal sub-intervals of length \( 1/n \). Thus, we may write the limit as an integral

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sin(k/n) = \int_0^1 \sin(x) \, dx. \]

To evaluate the resulting integral, we use FTC I. An anti-derivative of \( \sin(x) \) is \( -\cos(x) \), thus

\[ \int_0^1 \sin(x) \, dx = -\cos(x)|_0^1 = 1 - \cos(1). \]

1.3 Differentiating integrals.

FTC II shows that any continuous function has an anti-derivative and can be used to find the derivatives of integrals.

Example. Find

\[ \frac{d}{dx} \int_0^x \sin(t^2) \, dt \quad L'(x) \text{if } L(x) = \int_1^x \frac{1}{t} \, dt \quad \frac{d}{dx} \int_0^x \sin(t^2) \, dt \]

Is the function \( L(x) = \int_1^x \frac{1}{t} \, dt \) increasing or decreasing? Is the graph of \( L \) concave up or concave down?
**Solution.** The first one is a straightforward application of the second part of the fundamental theorem. The function \( \sin(x^2) \) is continuous everywhere and thus we have

\[
\frac{d}{dx} \int_0^x \sin(t^2) \, dt = \sin(x^2).
\]

The second one is also straightforward,

\[
\frac{d}{dx} \int_0^x \frac{1}{t} \, dt = \frac{1}{x}, \quad x > 0.
\]

Taking another derivative, we find that

\[
\frac{d^2}{dx^2} \int_0^x \frac{1}{t} \, dt = -\frac{1}{x^2}.
\]

Thus this function is concave down for \( x > 0 \).

Of course we can also use FTC I to see that \( \int_0^x \frac{1}{t} \, dt = \ln(x) - \ln(1) \) and then apply the differentiation rules to compute the derivative. Note that we could not use this approach in the first example since we do not know an anti-derivative for \( \sin(x^2) \).

Finally, the third one requires us to use the properties of the integral to put it in a form where we can use FTC II. We can write

\[
\int_{x^2}^x \sin(t^2) \, dt = \int_0^0 \sin(t^2) \, dt + \int_0^x \sin(t^2) \, dt = -\int_0^{x^2} \sin(t^2) \, dt + \int_0^x \sin(t^2) \, dt.
\]

Now applying FTC II and using the chain rule for the first integral gives

\[
\frac{d}{dx} (-\int_0^{x^2} \sin(t^2) \, dt + \int_0^x \sin(t^2) \, dt) = -2x \sin(x^4) + \sin(x^2).
\]

Our second example shows that it is necessary to assume that \( f \) is continuous in FTC II.

**Example.** Let \( f \) be the function given by

\[
f(x) = \begin{cases} 0, & x < 2 \\ 1, & x \geq 2 \end{cases}
\]

Find \( F(x) = \int_0^x f(x) \, dx \) and determine where \( F \) is differentiable.

**Solution.** We have that the integral is given by

\[
F(x) = \begin{cases} 0, & x < 2 \\ (x - 2), & x \geq 0 \end{cases}
\]

It is pretty clear that \( F \) is differentiable everywhere except at 2. At 2, we can compute the left and right limits of the difference quotient and find

\[
\lim_{h \to 0^-} \frac{F(2 + h) - F(2)}{h} = 0 \quad \lim_{h \to 0^+} \frac{F(2 + h) - F(2)}{h} = 1.
\]

Thus \( F'(2) \) does not exist.
1.4 The net change theorem

Since $F$ is always an anti-derivative of $F'$, one consequence of part I of the fundamental theorem of calculus is that if $F'$ is continuous on the interval $[a, b]$, then

$$\int_a^b F'(t) \, dt = F(b) - F(a).$$

This helps us to understand some common physical interpretations of the integral.

*Example.* An object falls with constant acceleration $g$, at $t = 1$ its height is $h_1$ and its velocity is $v_1$. Find its position at all times.

*Solution.* By the net change theorem,

$$v(t) - v(1) = \int_1^t g \, ds = g(t - 1).$$

Thus $v(t) = g(t - 1) + v_1$. Applying the net change theorem again we have the height at time $t$, $h(t)$ is

$$h(t) - h(1) = \int_1^t g(s - 1) + v_1 \, ds = \frac{1}{2} g(s - 1)^2|_{s=1}^{t} + v_1 s|_{s=1}^{t} = \frac{1}{2} g(t - 1)^2 + v_1 (t - 1).$$

Thus

$$h(t) = \frac{1}{2} g(t - 1)^2 + v_1 (t - 1) + h_1.$$

Note this gives a different version of the equations for a falling object.

To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let $m(x)$ denote the mass in kilograms of the rope from $0$ to $x$ meters along the rope. If we take the derivative, $\frac{dm}{dx} = \lim_{h \to 0} (m(x+h) - m(x))/h$, then this represents mass per unit length (or linear density) of the rope near $x$ and the units are kilograms/meter. If we integrate this linear density and observe that $m(0) = 0$, then we recover the mass

$$m(x) = \int_0^x \frac{dm}{dx} \, dx.$$

This is another example of the net change theorem.

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