MathExcel Worksheet # 19: Exponential Growth and Decay; Rates of Change

Reminders: Worksheet 4 is due Weds. in lecture.

1. A population of protozoa grows with at a constant relative rate of .475 per day. On day zero, the population consists of two members. Find the population size after 10 days.

\[ p(t) = 2 \cdot e^{.475t} \]

\[ p(10) = 2 \cdot e^{.475 \cdot 10} \approx 231 \text{ protozoa} \]

2. An explorer brought two rabbits (male and female) to a small island. Based on 32 years of data, the Rabbit Research Group has concluded that the population on the island doubles every year. Set up the proportional growth rate population equation and use it to predict the number of rabbits 1000 years later. What might be wrong with using this equation to predict population values for large \( t \)?

\[ P(t) = 2 \cdot 2^t = 2^{t+1} \]

\[ P(000) = 2 \cdot 2^{1000} \approx 2 \cdot 10^{301} \]

\[ P(100) = 2^{100 + 1} \approx 1.2 \times 10^{30} \]

\[ P(50) = 2^{50} \approx 2.25 \times 10^{15} \]

\[ P(t) = 2^t \]

\[ P(100) = 2^{100} \approx 2048 \]

\[ P(100) = 2^{100} \approx 2048 \]

\[ P(100) = 2^{100} \approx 2048 \]

\[ P(100) = 2^{100} \approx 2048 \]

It's very unlikely that the environment will support an unlimited growth in population. Hence, growth rates should change our long-term.
3. Suppose you borrow $1000 at an interest rate of 8% to be repaid at the end of 5 years. How much interest do you pay if your interest is compounded a.) yearly, b.) monthly, c.) weekly and d.) continuously?

<table>
<thead>
<tr>
<th>N</th>
<th>Balance</th>
<th>Principle</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1469.33</td>
<td>1000</td>
<td>469.33</td>
</tr>
<tr>
<td>12</td>
<td>1489.85</td>
<td></td>
<td>489.85</td>
</tr>
<tr>
<td>52</td>
<td>1491.27</td>
<td></td>
<td>491.27</td>
</tr>
<tr>
<td></td>
<td>1491.82</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. With continuous compounding at a rate of 6%, how long will it take for an investment to double in value? What is the equivalent annual interest rate?

\[ 2A_0 = A_0 e^{0.06t} \]
\[ 2 = e^{0.06t} \]
\[ \ln 2 = 0.06t \]
\[ t = \frac{\ln 2}{0.06} \approx 11.55 \text{ yrs.} \]

\[ A_0 (1.18) = A_0 e^{0.06} \]
\[ 1.18 = e^{0.06} \]
\[ 0.18 \approx 6.18\% \]

5. Let \( A(t) \) be the balance of an account which pays interest rate \( r \) compounded continuously. Let \( A_0 \) be the initial investment. What differential equation does \( A(t) \) satisfy? What is the initial condition satisfied by \( A(t) \)?

\[ \left\{ \begin{array}{l}
\frac{dA}{dt} = r \ A(t) \\
A(0) = A_0
\end{array} \right. \]

6. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) Find an expression for the number of bacteria after \( t \) hours.

\[ A(t) = 100 \cdot e^{rt} \]

\[ \frac{420}{100} = e^{rt} \]
\[ 4.2 = e^{r} \]
\[ r = \ln 4.2 \]

\[ A (1) = 100 \cdot (4.2)^t \]
(b) Find the number of bacteria after 3 hours.

\[ A(3) = 100 \times 4.2^3 \approx 7409 \text{ bacteria} \]

(c) Find the rate of growth after 3 hours.

\[ \frac{dA}{dt} = \ln 4.2 \times 7409 \approx 10632.5 \frac{\text{bacteria}}{\text{hr}} \]

(d) When will the population reach 10000.

\[ 10000 = 100 \times 4.2^t \]
\[ 100 = 4.2^t \]
\[ \ln(100) \approx 3.21 \text{ hr} \]

7. A sample of a chemical compound decayed to 95.47% of its original mass after one year. What is the half-life of the compound? How long would it take for the compound to decay to 20% of its original mass.

\[ 0.9547A_0 = A_0 e^{-0.41676t} \]
\[ \ln 0.9547 = -0.41676t \]
\[ \frac{1}{2} = e^{-0.41676t} \]
\[ \ln \left( \frac{1}{2} \right) = -0.41676t \]
\[ t_{1/2} = \frac{-0.41676}{-0.5} \text{ years} \]
\[ t_{20\%} = 34.72 \text{ years} \]

8. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

(a) Find the mass that remains after \( t \) years.

\[ \frac{1}{2} = e^{-0.0231 \times 30} \]
\[ \ln \left( \frac{1}{2} \right) = r \approx -0.231 \]

\[ R(t) = 100 \times e^{-0.0231 \times t} \]
(b) How much of the sample remains after 100 years?
\[ A(100) = 9.92 \text{ mg} \]

(c) After how long will only 1 mg remain?
\[ 1 = 100 \cdot e^{-0.2311 \cdot t} \]
\[ \ln\left(\frac{1}{100}\right) = -0.2311 \cdot t \]
\[ t \approx 199 \text{ yrs} \]

9. A particle is traveling along a straight line. Its position after \( t \) seconds is given by \( s(t) = 2t^3 - 21t^2 + 72t + 10 \) meters.

(a) Find the velocity and acceleration of the particle at time \( t \).
\[ s'(t) = 6t^2 - 42t + 72 \]
\[ s''(t) = 12t - 42 \]

(b) Find all times \( t \) where the velocity is zero.
\[ s'(t) = 0 \]
\[ 0 = 6t^2 - 42t + 72 \]
\[ 0 = (t^2 - 7t + 12) \]
\[ 0 = (t-3)(t-4) \]
\[ t = 3 \text{ sec}, \quad t = 4 \text{ sec} \]

10. Show that the rate of change of the area of a square with respect to its side length is half its perimeter.
\[ P = 4x \quad \rightarrow \quad \frac{1}{2} \cdot P = 2x \]
\[ A = x^2 \quad \frac{dA}{dx} = 2x = \frac{1}{2} P \]
11. Show that the rate of change of the volume of a cube with respect to its side lengths is equal to half of its surface area.

\[ V = x^3 \quad \frac{dV}{dx} = 3x^2 \]

So

\[ \frac{dV}{dx} = \frac{1}{2} \cdot 6x^2 \]

\[ SA = 6x^2 \implies \frac{1}{2} \cdot SA = 3x^2 \]

12. Let \( P(x) \) be the total value of production when there are \( x \) workers in a plant. Then the average productivity of labor at the plant is \( AP(x) = \frac{P(x)}{x} \)

(a) Find \( AP'(x) \). Why does the company want to hire more labor if \( AP'(x) > 0 \)?

(b) Show that \( AP'(x) > 0 \) when \( P'(x) \) is greater than the average productivity of labor.

\[ AP'(x) = \frac{xP'(x) - P(x)}{x^2} \]

If \( AP'(x) > 0 \) then \( AP \) will increase w/ more workers.

\[ \text{Suppose} \quad P'(x) > AP(x) = \frac{P(x)}{x} \]

Then:

\[ AP'(x) = \frac{xP'(x) - P(x)}{x^2} > 0 \]