Solutions for Assignment 3
MA 113 – Calculus I – Fall 2012

Question A: If the function \( f \) is continuous on \([-1, 1]\) and \( f(-1) = -1, f(0) = 1 \) and \( f(1) = 0 \), is it possible that \( f \) is one-to-one?
If the answer is yes, give an example of a one-to-one function satisfying these conditions.
If the answer is no, provide a justification that any function satisfying these conditions cannot be one-to-one.

Solutions: (5 points)
No. Since \( f(-1) < 0 \) and \( f(0) > 0 \), by the Intermediate Value Theorem, there must be a real number \( c \in (-1, 0) \) so that \( f(c) = 0 \). Since \( f(1) = 0 \), the function is not one-to-one.

Question B: Consider the graphs of the two functions \( f(x) = 1 - x^2 \) for \(-1 \leq x \leq 1\) and \( g(x) = 4 - (x - 4)^2 \) for \(2 \leq x \leq 6\). These two functions have a common tangent line. Find the slope of that tangent line. Include a graph of the functions and the common tangent line. Hint: Find the tangent line to \( y = f(x) \) that intersects \( y = g(x) \) in exactly one point.

Solutions: (5 points)
The tangent line to \( y = f(x) \) has slope at \( x = a \) given by \( m = f'(a) = -2a \). The equation of the tangent line to this curve at this point is \( y = f(a) + f'(a)(x - a) \). \( f(a) = 1 - a^2 \) and \( f'(a) = -2a \), so the equation of this line is
\[
y = f(a) + f'(a)(x - a) \\
= (1 - a^2) - 2a(x - a) \\
= 1 - 2ax + a^2
\]
Now, we need to find where this intersects the graph of \( g(x) \) in only one point. The intersection is found by solving the equation
\[
1 - 2ax + a^2 = 4 - (x - 4)^2.
\]
\[
(8 + 2a)^2 - 4(1)(13 + a^2) = 0 \\
32a + 12 = 0 \\
a = -\frac{3}{8}
\]
Thus, the point on the tangent line to the first equation has $x$ coordinate $a = -\frac{3}{8}$. The slope of the line is then $m = \frac{3}{4}$. 

![Graph showing the tangent line and its slope](image-url)