ABSOLUTE CONTINUITY OF PERIODIC SCHRÖDINGER OPERATORS WITH POTENTIALS IN THE KATO CLASS

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Abstract. We consider the Schrödinger operator $-\Delta + V$ in $\mathbb{R}^d$ with periodic potential $V$ in the Kato class. We show that, if $d = 2$ or $d = 3$, the spectrum of $-\Delta + V$ is purely absolutely continuous.

1. Introduction

Let $V$ be a real valued measurable function on $\mathbb{R}^d$, $d \geq 2$. $V$ is said to belong to the Kato class $K_d$ if

\begin{equation}
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq r} \frac{|V(y)| dy}{|y-x|^{d-2}} = 0, \quad \text{for } d \geq 3,
\end{equation}

\begin{equation}
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq r} |V(y)| \ln \{|y-x|^{-1}\} dy = 0, \quad \text{for } d = 2.
\end{equation}

It is well known that, if $V \in K_d$, then the quadratic form associated with $-\Delta + V$ defines a unique self-adjoint operator which we also denote by $-\Delta + V$ [7]. We refer the reader to [18] for the naturalness of the Kato class in the study of $L^p$ properties of the semigroup $e^{-t(\Delta + V)}$. The purpose of this paper is to show that, if $d = 2$ or $d = 3$ and $V \in K_d$ is a real periodic function on $\mathbb{R}^d$, then the spectrum of $-\Delta + V$ is purely absolutely continuous.

Main Theorem. Let $A = (a_{jk})_{d \times d}$ be a symmetric, positive-definite matrix with real constant entries. Let $V \in K_d$ be a real periodic function on $\mathbb{R}^d$. If $d = 2$ or $d = 3$, then the spectrum of operator $DAD^T + V$ is purely absolutely continuous where $D = -i\nabla$ and $DAD^T = \sum_{j,k} D_j a_{jk} D_k$.

A few remarks are in order.

Remark 1.3. For a Schrödinger operator $-\Delta + V$ with periodic potential $V$, the absolute continuity of the spectrum was first established by

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L. Thomas [21] in $\mathbb{R}^3$ under the assumption $V \in L^2_{\text{loc}}(\mathbb{R}^3)$. Thomas’ result was subsequently extended to $\mathbb{R}^d$ by M. Reed and B. Simon [13] under the assumption $V \in L^r_{\text{loc}}(\mathbb{R}^d)$, where $r > d - 1$ if $d \geq 4$ and $r = 2$ if $d = 2$ or $d = 3$. In [4] L. Danilov applied the approach of Thomas to the Dirac operator with a periodic potential. Recently, the absolute continuity of the magnetic Schrödinger operator $(-i \nabla - A(x))^2 + V(x)$ with periodic potentials $A$ and $V$ was investigated by R. Hempel and I. Herbst [5], [6], M. Birman and T. Suslina [1], [2], [3], A. Morame [12], and A. Sobolev [19]. In particular, the results in [2] and [3], pertaining to the case $-\Delta + V$, give the absolute continuity for $V \in L^p_{\text{loc}}(\mathbb{R}^d)$, where $p > 1$ if $d = 2$, $p = d/2$ if $d = 3$ or $d = 4$, and $p = d - 2$ if $d \geq 5$. In [16], the author established the absolute continuity of $-\Delta + V$ under the condition $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$, $d \geq 3$. This is best possible in the context of the $L^p$ spaces, in the sense that, under the periodicity condition, $L^{d/2}_{\text{loc}}$ is the largest space for which the self-adjoint operator $-\Delta + V$ may be defined by a quadratic form. The case $V \in \text{weak-}L^{d/2}$ was also studied in [16].

**Remark 1.4.** In [17] the author investigated the periodic Schrödinger operator $-\Delta + V$ with potential $V$ in the Morrey–Campanato class. The author showed that, if $d \geq 3$, $p \in ((d - 1)/2, d/2]$, and

$$
\limsup_{r \to 0} \sup_{x \in \Omega} r^2 \left\{ \frac{1}{r^d} \int_{|y - x| \leq r} |V(y)|^p \, dy \right\}^{1/p} < \varepsilon(p, d, \Omega),
$$

where $\varepsilon(p, d, \Omega) > 0$ and $\Omega$ is a periodic cell for $V$, then $-\Delta + V$ has purely absolutely continuous spectrum. This improves the $L^{d/2}$ and weak-$L^{d/2}$ results in [16]. We point out that the Kato class considered in this paper is not comparable with the Morrey–Campanato class for $d \geq 3$ and $p > 1$. Indeed, one can construct a periodic potential $V$ in $\mathbb{R}^3$ such that

$$
|V(x)| \sim \frac{1}{|x'|^2 |\ln(|x'|)|^\delta} \quad \text{as} \quad |x| \to 0,
$$

where $x' = (x_2, x_3)$. Then $V \in K_3$ if $\delta > 2$, but $V$ does not satisfy (1.5) since $V \notin L^p_{\text{loc}}(\mathbb{R}^3)$ for any $p > 1$. On the other hand, if

$$
|V(x)| \sim \frac{1}{|x|^2 |\ln(|x|)|^\delta} \quad \text{as} \quad |x| \to 0,
$$

then $V$ satisfies (1.5) for $1 < p < 3/2$, if $\delta > 0$. However, $V \in K_3$ if and only if $\delta > 1$. Clearly, in the two-dimensional case, our result improves the $L^p$ ($p > 1$) result in [2].

**Remark 1.8.** By a change of coordinates, we may assume that $V$ is periodic with respect to the lattice $(2\pi \mathbb{Z})^d$.

Our main theorem is proved by using the approach of L. Thomas [21] and a new pointwise estimate on the kernel function of a certain integral operator.
To be more precise, let $\Omega = [0, 2\pi)^d \simeq \mathbb{R}^d / (2\pi\mathbb{Z})^d = T^d$. We consider a family of operators

$$H_V(za + b) = (D + za + b)A(D + za + b)^T + V, \quad z \in \mathbb{C},$$

defined on $L^2(T^d)$ with $a, b \in \mathbb{R}^d$ fixed. Using the Floquet decomposition and Thomas’ argument, we may reduce the main theorem to the problem of showing that the family of operators $\{H_V(za + b) : z \in \mathbb{C}\}$ has no common eigenvalues. To this end, we will show that, for some appropriately chosen $a \in \mathbb{R}^d$,

$$\|\{H_V(za + b)\}^{-1}\|_{L^1(T^d) \to L^1(T^d)} \to 0 \quad \text{as} \quad \rho \to \infty,$$

where $\langle b, a \rangle = 0$, $z = \delta + i\rho$, and $\delta$ is some fixed number depending on $a$ and $b$.

To prove (1.10), the key step is to show that

$$\|V\{H_0(za + b)\}^{-1}\|_{L^1(T^d) \to L^1(T^d)} \leq \begin{cases} C \sup_{x \in \Omega} \int_{\Omega} \frac{|V(y)| \, dy}{|y - x|}, & \text{if } d = 3, \\ C \sup_{x \in \Omega} \int_{\Omega} \{1 + |\ln|x - y||\} \, |V(y)| \, dy, & \text{if } d = 2, \end{cases}$$

where $H_0(za + b) = (D + za + b)A(D + za + b)^T$. This will be done by establishing the following pointwise estimate on the kernel function $G_\rho(x, y)$ of the operator $\{H_0(za + b)\}^{-1}$:

$$|G_\rho(x, y)| \leq \begin{cases} \frac{C}{|x - y|}, & \text{if } d = 3, \\ C \{1 + |\ln|x - y||\}, & \text{if } d = 2. \end{cases}$$

This paper is organized as follows. In Sections 2 and 3 we prove the kernel function estimate (1.12), and in Section 4 we prove the Main Theorem.

Throughout the rest of this paper we assume that $d = 2$ or $d = 3$, and that $V \in K_d$ is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. We use $\|\cdot\|_p$ to denote the norm in $L^p(T^d)$. Finally we use $C$ and $c$ to denote positive constants, which may depend on the matrix $A$, and which are not necessarily the same at each occurrence.

2. Some preliminaries

We begin by choosing $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ such that

$$|a| = 1, \quad aA = (s_0, 0, \ldots, 0), \quad s_0 > 0.$$
For \( \mathbf{b} = (b_1, \cdots, b_d) \in \mathbb{R}^d \) with \( \langle \mathbf{b}, \mathbf{a} \rangle = 0 \) and \( |\mathbf{b}| \leq \sqrt{d} \), let
\[
(2.2) \quad \delta = \frac{1}{a_1} \left( \frac{1}{2} - b_1 \right).
\]
Note that \( a_1 > 0 \) since \( \mathbf{a} \mathbf{a}^T = s_0 a_1 > 0 \). We consider the operator
\[
(2.3) \quad \mathbb{H}_0(k) = (\mathbf{D} + k) A (\mathbf{D} + k)^T
\]
defined on \( L^2(\mathbb{T}^d) \), where
\[
(2.4) \quad k = (\delta + i\rho) \mathbf{a} + \mathbf{b} \quad \text{and} \quad \rho \geq 1.
\]
For \( \psi \in L^1(\Omega) \), let
\[
(2.5) \quad \hat{\psi}(n) = \frac{1}{(2\pi)^d} \int_\Omega e^{-i(n, \mathbf{y})} \psi(\mathbf{y}) \, d\mathbf{y}.
\]
We may write
\[
(2.6) \quad \{\mathbb{H}_0(\mathbf{k})\}^{-1} \psi(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n}) e^{i(n, \mathbf{x})}}{(\mathbf{n} + \mathbf{k}) A (\mathbf{n} + \mathbf{k})^T}
\]
for \( \psi \in C^\infty(\mathbb{T}^d) \). Using (2.4) and (2.1), it is easy to see that
\[
(2.7) \quad (\mathbf{n} + \mathbf{k}) A (\mathbf{n} + \mathbf{k})^T = (\mathbf{n} + \delta \mathbf{a} + \mathbf{b}) A (\mathbf{n} + \delta \mathbf{a} + \mathbf{b})^T - \rho^2 s_0 a_1 + 2i\rho s_0 (n_1 + \delta a_1 + b_1).
\]
By (2.2) we have
\[
|\mathbf{n} + \mathbf{k}) A (\mathbf{n} + \mathbf{k})^T| \geq 2\rho s_0 |n_1 + \delta a_1 + b_1|
= 2\rho s_0 \left| n_1 + \frac{1}{2} \right| \geq \rho s_0,
\]
since \( n_1 \) is an integer.

We now choose \( \eta \in C^\infty(\mathbb{R}_+) \) such that \( \eta(r) = 1 \) if \( r \geq s_0^2 \), and \( \eta(r) = 0 \) if \( 0 < r < s_0^2/2 \). Then,
\[
\eta \left( \frac{|(\mathbf{n} + \mathbf{k}) A (\mathbf{n} + \mathbf{k})^T|^2}{\rho^2} \right) = 1 \quad \text{for any} \ \mathbf{n} \in \mathbb{Z}^d.
\]
It follows that
\[
(2.8) \quad \{\mathbb{H}_0(\mathbf{k})\}^{-1} \psi(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{e^{i(n, \mathbf{x})} \hat{\psi}(\mathbf{n}) \eta \left( \frac{|(\mathbf{n} + \mathbf{k}) A (\mathbf{n} + \mathbf{k})^T|^2}{\rho^2} \right)}{(\mathbf{n} + \mathbf{k}) A (\mathbf{n} + \mathbf{k})^T}
= \int_\Omega G_\rho(x - \mathbf{y}) \psi(\mathbf{y}) \, d\mathbf{y},
\]
where

\[ G_\rho(x) = \frac{1}{(2\pi)^d} \sum_{\mathbf{n} \in \mathbb{Z}^d} e^{i\langle \mathbf{n}, x \rangle} \eta \left( \frac{|(\mathbf{n} + \mathbf{k}) A(\mathbf{n} + \mathbf{k})^T|^2 / \rho^2}{(\mathbf{n} + \mathbf{k}) A(\mathbf{n} + \mathbf{k})^T} \right). \]

Note that, by the Plancherel theorem, we have \( G_\rho \in L^2(\Omega) \) if \( d = 2 \) or \( d = 3 \).

Let

\[ \varphi(\xi, \rho) = \xi A\xi^T - \rho^2 s_0 a_1 + 2i\rho s_0 \xi_1, \text{ where } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d. \]

Then,

\[ h_\rho(\xi) = \frac{\eta \left( |\varphi(\xi, \rho)|^2 / \rho^2 \right)}{\varphi(\xi, \rho)} \in L^2(\mathbb{R}^d). \]

We denote its inverse Fourier transform by \( F_\rho(x) \), i.e.,

\[ F_\rho(x) = (h_\rho)^\vee(x) = \int_{\mathbb{R}^d} e^{ix\xi} h_\rho(\xi) d\xi. \]

Using the fact that \((-x)^\beta D^\beta h_\rho(\xi)\) is the inverse Fourier transform of \((-x)^\beta F_\rho(x)\), one sees that

\[ F_\rho(x) = O \left( \frac{1}{|x|^N} \right) \text{ as } |x| \to \infty \]

for any \( N \geq 1 \). It follows that

\[
\eta \left( \frac{|(\mathbf{n} + \mathbf{k}) A(\mathbf{n} + \mathbf{k})^T|^2 / \rho^2}{(\mathbf{n} + \mathbf{k}) A(\mathbf{n} + \mathbf{k})^T} \right) = \eta \left( \frac{|\varphi(\mathbf{n} + \delta \mathbf{a} + \mathbf{b}, \rho)|^2 / \rho^2}{\varphi(\mathbf{n} + \delta \mathbf{a} + \mathbf{b}, \rho)} \right) = h_\rho(\mathbf{n} + \delta \mathbf{a} + \mathbf{b})
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \mathbf{n} + \delta \mathbf{a} + \mathbf{b}, x \rangle} F_\rho(x) dx
\]

\[
= \frac{1}{(2\pi)^d} \sum_{m \in \mathbb{Z}^d} \int_{\Omega} e^{-i\langle \mathbf{n} + \delta \mathbf{a} + \mathbf{b}, x + 2\pi m \rangle} F_\rho(x + 2\pi m) dx
\]

\[
= \frac{1}{(2\pi)^d} \int_{\Omega} e^{-i\langle \mathbf{n}, x \rangle} \sum_{m \in \mathbb{Z}^d} e^{-i\langle \delta \mathbf{a} + \mathbf{b}, x + 2\pi m \rangle} F_\rho(x + 2\pi m) dx.
\]

In view of (2.9), this implies that

\[ G_\rho(x) = \frac{1}{(2\pi)^d} \sum_{m \in \mathbb{Z}^d} e^{-i\langle \delta \mathbf{a} + \mathbf{b}, x + 2\pi m \rangle} F_\rho(x + 2\pi m), \]

which is a form of Poisson summation formula [20]. In particular,

\[ |G_\rho(x)| \leq \frac{1}{(2\pi)^d} \sum_{n \in \mathbb{Z}^d} |F_\rho(x + 2\pi n)|. \]

To estimate the function \( F_\rho(x) \), we first note that

\[ \varphi(\xi, \rho) = \rho^2 \varphi(\xi / \rho, 1) \]
and
\[ h_\rho(\xi) = \frac{\eta \left( \rho^2 |\varphi(\xi/\rho, 1)|^2 \right)}{\rho^2 \varphi(\xi/\rho, 1)}. \]

It follows that
\[ F_\rho(x) = (h_\rho)^\vee(x) = \rho^{d-2} \left( \frac{\eta \left( \rho^2 |\varphi(\cdot, 1)|^2 \right)}{\varphi(\cdot, 1)} \right)^\vee(x). \]

Let
\[ f_\rho(x) = \left( \frac{\eta \left( \rho^2 |\varphi(\cdot, 1)|^2 \right)}{\varphi(\cdot, 1)} \right)^\vee(x). \]

Then,
\[ F_\rho(x) = \rho^{d-2} f_\rho(\rho x). \]

Note that
\[ \varphi(\xi, 1) = \sum_{j,k} a_{jk} \xi_j \xi_k - s_0 a_1 + 2i s_0 \xi_1, \]
\[ |\varphi(\xi, 1)|^2 = \left| \sum_{j,k} a_{jk} \xi_j \xi_k - s_0 a_1 \right|^2 + 4s_0^2 \xi_1^2. \]

A direct computation yields the estimates
\[ \left| \frac{\partial^{\ell}}{\partial \xi_j^\ell} \left\{ \frac{1}{\varphi(\xi, 1)} \right\} \right| \leq C_\ell (1 + |\xi|)^\ell |\varphi(\xi, 1)|^{\ell+1}, \]
\[ \left| \frac{\partial^{\ell}}{\partial \xi_j^\ell} \left\{ \eta \left( \rho^2 |\varphi(\cdot, 1)|^2 \right) \right\} \right| \leq C_\ell \rho^\ell \]
for \( \ell \geq 0 \) and \( j = 1, \ldots, d \).

**Lemma 2.20.** Let \( f_\rho(x) \) be defined by (2.15). Then, for any \( x \in \mathbb{R}^d \),
\[ |f_\rho(x)| \leq \frac{C \rho^3}{|x|^4}, \quad \text{if } d = 3, \]
\[ |f_\rho(x)| \leq \frac{C \rho^4}{|x|^4}, \quad \text{if } d = 2. \]

**Proof.** Since \( x_j^4 f_\rho(x) \) is the inverse Fourier transform of
\[ \frac{\partial^4}{\partial \xi_j^4} \left\{ \eta \left( \rho^2 |\varphi(\xi, 1)|^2 \right) \right\}/\varphi(\xi, 1), \]
we have

\[
|x|^4 f_\rho(x) | \leq \int_{\mathbb{R}^d} \left| \frac{\partial^4}{\partial \xi_j^4} \left\{ \frac{\eta (\rho^2 |\varphi(\xi, 1)|^2)}{\varphi(\xi, 1)} \right\} \right| d\xi
\]

\[
\leq C \int_{\mathbb{R}^d} \sum_{\ell=0}^4 \left| \frac{\partial^\ell}{\partial \xi_j^\ell} \right\{ \frac{1}{\varphi(\xi, 1)} \right\} \cdot \left| \frac{\partial^{4-\ell}}{\partial \xi_j^{4-\ell}} \right\{ \eta (\rho^2 |\varphi(\xi, 1)|^2) \right\} \right| d\xi
\]

\[
\leq C \int_{|\varphi(\xi, 1)| \geq c/\rho} \frac{(1 + |\xi|)^4}{|\varphi(\xi, 1)|^5} d\xi
\]

\[
+ C \int_{|\varphi(\xi, 1)| \sim 1/\rho} \sum_{\ell=0}^3 \frac{(1 + |\xi|)^\ell}{|\varphi(\xi, 1)|^{\ell+1}} \cdot \rho^{4-\ell} d\xi
\]

\[
\leq C \int_{|\varphi(\xi, 1)| \geq c/\rho} \frac{(1 + |\xi|)^4}{|\varphi(\xi, 1)|^5} d\xi
\]

\[
\leq C \int_{\mathbb{R}^d} \left\{ |\varphi(\xi, 1)| + 1/\rho \right\} d\xi.
\]

Note that

\[
(2.23) \quad |\varphi(\xi, 1)| \sim |\xi_1| + |\xi A \xi^T - aAa^T|
\]

\[
= |\xi_1| + \{ |\xi B| + |aB| \} \{ |\xi B| - |aB| \},
\]

where \( B = \sqrt{A} \geq 0 \). Using this it is not hard to see that

\[
I_1 := \int_{|\xi B| \geq 2|aB|} \frac{(1 + |\xi|)^4}{\left\{ |\varphi(\xi, 1)| + 1/\rho \right\}^5} d\xi \leq C \int_{|\xi| \geq c \|\xi\|^5} \frac{|\xi|^4}{|\xi|^5} d\xi \leq C.
\]

Also,

\[
I_2 := \int_{|\xi B| \leq 2|aB|} \frac{(1 + |\xi|)^4}{\left\{ |\varphi(\xi, 1)| + 1/\rho \right\}^5} d\xi
\]

\[
\leq C \int_{|\xi B| \leq 2|aB|} \frac{d\xi}{\left\{ |\xi_1| + \|B - aB\| + 1/\rho \right\}^5}
\]

\[
\leq C \int_{|\xi B - aB| \leq 2|aB|} \frac{d\xi}{\left\{ |(\xi B - aB)_1| + \|\xi - aB\| + 1/\rho \right\}^5}
\]

\[
\leq C \int_{|\xi| \leq 2|aB|} \frac{d\xi}{\left\{ |\xi_1| + \|\xi - aB\| + 1/\rho \right\}^5},
\]

where the last inequality follows by a rotation.
Now suppose $d = 3$. Then, using spherical coordinates with $\xi_1 = r \cos \theta$, we have

\[
I_2 \leq C \int_0^{2|aB|} r^2 \left\{ \int_0^{\pi/2} \frac{\sin \theta d\theta}{\{r \cos \theta + |r - |aB|| + 1/\rho\}^5} \right\} dr
\]
\[
\leq C \int_0^{2|aB|} dr \frac{\{|r - |aB|| + 1/\rho\}^4}{\{r + 1/\rho\}^5}
\]
\[
\leq C \rho^3.
\]

Similarly, if $d = 2$,

\[
I_2 \leq C \int_0^{2|aB|} r \left\{ \int_0^{\pi/2} \frac{d\theta}{\{r \cos \theta + |r - |aB|| + 1/\rho\}^5} \right\} dr
\]
\[
\leq C \int_0^{|aB|} dr \frac{\{|r - |aB|| + 1/\rho\}^4}{\{r + 1/\rho\}^5}
\]
\[
\leq C \rho^4.
\]

Thus we have proved that, for $j = 1, \cdots, d$,

\[
|x_j^4 f_\rho(x)| \leq C \{I_1 + I_2\} \leq \begin{cases} 
C \rho^3, & \text{if } d = 3, \\
C \ln \frac{1}{|x|}, & \text{if } d = 2.
\end{cases}
\]

The estimates (2.21) and (2.22) then follow.

It follows from (2.16) and Lemma 2.20 that, for any $x \in \mathbb{R}^d$,

(2.24) \quad |F_\rho(x)| \leq \frac{C}{|x|^4} \quad \text{for } d = 2 \text{ or } 3.

This will be used to estimate the terms on the right hand side of (2.14), where $|x + 2\pi n| \geq 1/2$.

3. Pointwise estimate of the kernel function $G_\rho(x)$

In this section we will show that, if $|x| \leq 1/2$, then

(3.1) \quad |F_\rho(x)| \leq \begin{cases} 
C \frac{1}{|x|^4}, & \text{if } d = 3, \\
C \ln \frac{1}{|x|}, & \text{if } d = 2.
\end{cases}
Together with (2.24) and (2.14), this implies that
\begin{equation}
|G_\rho(x)| \leq \begin{cases} 
C \left\{ 1 + \sum_{|x + 2\pi n| \leq 1/2} \frac{1}{|x + 2\pi n|} \right\}, & \text{if } d = 3, \\
C \left\{ 1 + \sum_{|x + 2\pi n| \leq 1/2} \ln \frac{1}{|x + 2\pi n|} \right\}, & \text{if } d = 2.
\end{cases}
\end{equation}

To prove (3.1), we recall that \( F_\rho(x) = \rho^{d-2} f_\rho(\rho x) \) and write
\begin{equation}
f_\rho(x) = \left\{ \frac{1}{\varphi(\cdot, 1)} \right\} (x) + \left\{ \frac{\eta (\rho^2 |\varphi(\cdot, 1)|^2 - 1)}{\varphi(\cdot, 1)} \right\} (x).
\end{equation}

**Lemma 3.4.** We have
\[
\left| \int_{\mathbb{R}^d} \left| \frac{\eta (\rho^2 |\varphi(\xi, 1)|^2 - 1)}{\varphi(\xi, 1)} \right| d\xi \right| \leq \frac{C}{\rho}.
\]

*Proof.* Recall that \( \eta(r) = 1 \) for \( r \geq s_0^2 \). Thus, as in the proof of Lemma 2.20, we have
\[
\int_{\mathbb{R}^d} \left| \frac{\eta (\rho^2 |\varphi(\xi, 1)|^2 - 1)}{\varphi(\xi, 1)} \right| d\xi \leq C \int |\varphi(\xi, 1)| \frac{d\xi}{|\varphi(\xi, 1)|}
\leq C \int_{|\xi_1| + |\xi'| - |aB| \leq c/\rho} \frac{d\xi}{|\xi_1| + |\xi'| - |aB|}
\leq C \int_{|\xi_1| \leq c/\rho} \frac{d\xi_1}{|\xi_1| + |r - aB|}
\leq C \int_{0 < r < c} \frac{d\xi_1}{|\xi_1| + r}
\leq \frac{C}{\rho}.
\]

It follows from Lemma 3.4 that
\begin{equation}
\left| \left\{ \frac{\eta (\rho^2 |\varphi(\cdot, 1)|^2 - 1)}{\varphi(\cdot, 1)} \right\} (x) \right| \leq \frac{C}{\rho}.
\end{equation}

To estimate the first term on the right hand side of (3.3), we first note that, by several changes of variables, we have
\begin{equation}
\left\{ \frac{1}{\varphi(\cdot, 1)} \right\} (x) = \frac{|aB|^{d-2}}{\det(B)} \left\{ \frac{1}{|\xi|^2 - 1 + 2\xi_1} \right\} (|aB|xB^{-1}O^{-1}),
\end{equation}
where $O$ is a $d \times d$ orthogonal matrix such that $aBO^{-1} = (|aB|, 0, \cdots, 0)$.

**Lemma 3.7.** Let $u(x)$ denote the inverse Fourier transform of $\{ |\xi|^2 - 1 + 2i\xi_1 \}^{-1}$ in $\mathbb{R}^d$, $d = 2$ or $d = 3$. Let $x = (x_1, x') \in \mathbb{R}^d$. Then,

$$u(x) = 2\pi \int_0^\infty J_0(|x'| r) v(r, x_1) r \, dr, \quad \text{if } d = 3,$$

$$u(x) = 2 \int_0^\infty \cos(|x_2| r) v(r, x_1) \, dr, \quad \text{if } d = 2,$$

where

$$v(r, x_1) = \int_{\mathbb{R}} e^{ix_1 \xi_1} \frac{e^{i\xi_1} - 1 + 2i\xi_1}{\xi_1^2 + 1 + 2i\xi_1} \, d\xi_1,
$$

and

$$J_0(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{it\cos \omega} d\omega$$

is the Bessel function of the first kind of order 0.

**Proof.** One may verify that, for $R > 0$,

$$\{ |\xi|^2 - 1 + 2i\xi_1 \}^{-1} \chi_{\{\xi \in \mathbb{R}^d : |\xi| \leq R\}} \in L^1(\mathbb{R}^d),$$

where $\xi = (\xi_1, \xi')$. Since $\{ |\xi|^2 - 1 + 2i\xi_1 \}^{-1} \in L^p(\mathbb{R}^d)$ for $3/2 < p < 2$, we have, by the Hausdorff–Young inequality [20],

$$u(x) = \lim_{R \to \infty} \int_{\mathbb{R}^d} \frac{e^{i(x, \xi)}}{|\xi|^2 - 1 + 2i\xi_1} \, d\xi,$$

where the limit is taken in the $L^p'$-space. From this, (3.8) and (3.9) follow by using Fubini’s theorem and polar coordinates. We omit the details. \hfill \Box

**Lemma 3.12.** Let $v(r, x_1)$ be the function defined by (3.10). Then,

$$v(r, x_1) = \begin{cases} \frac{\pi}{r} e^{x_1 - r|x_1|}, & \text{if } r > 1, \\ \frac{\pi}{r} \left\{ e^{x_1 - r|x_1|} - e^{(1-r)x_1} \right\}, & \text{if } 0 < r < 1. \end{cases}$$

**Proof.** First we write

$$v(r, x_1) = e^{x_1} \int_{\mathbb{R}} \frac{e^{ix_1(\xi_1 + i)}}{r^2 + (\xi_1 + i)^2} \, d\xi_1.$$

Applying Cauchy’s integral theorem to the function

$$w(z) = \frac{e^{ix_1 z}}{r^2 + z^2} = \frac{e^{ix_1 z}}{(z + ri)(z - ri)},$$
we obtain

\begin{equation}
(v(r, x_1) = \begin{cases}
  e^{x_1} \int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2 + y^2} \, dy, & \text{if } r > 1, \\
  e^{x_1} \int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2 + y^2} \, dy - \frac{\pi}{r} e^{(1-r)x_1}, & \text{if } 0 < r < 1.
\end{cases}
\end{equation}

By a routine application of the residue theorem, one may show that

\begin{equation}
\int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2 + y^2} \, dy = \frac{\pi}{r} e^{-r|x_1|};
\end{equation}

see, e.g., [14, pp. 389–390]. This, together with (3.13), yields the lemma.

**Lemma 3.15.** Let \( v(r, x_1) \) be the function defined by (3.10). Then,

\[ |v(r, x_1)| \leq \begin{cases} 
  \frac{\pi}{r} e^{(1-r)|x_1|}, & \text{if } r > 1, \\
  C \left\{ e^{(r-1)|x_1|} + |x_1| e^{-|x_1|/2} \right\}, & \text{if } 0 < r < 1,
\end{cases} \]

and

\[ \left| \frac{\partial v}{\partial r} (r, x_1) \right| \leq \begin{cases} 
  \frac{C}{r^2} (1 + r|x_1|) e^{(1-r)|x_1|}, & \text{if } r > 1, \\
  C \left\{ |x_1|^2 e^{-|x_1|/2} + (1 + |x_1|) e^{(r-1)|x_1|} \right\}, & \text{if } 0 < r < 1.
\end{cases} \]

**Proof.** We will only prove the second estimate, using Lemma 3.12. The proof of the first estimate is easier.

If \( r > 1 \),

\[ \frac{\partial v}{\partial r} = -\frac{\pi}{r^2} e^{x_1-r|x_1|} + \frac{\pi}{r} e^{x_1-r|x_1|} (-|x_1|). \]

Hence,

\[ \left| \frac{\partial v}{\partial r} \right| = \frac{\pi}{r^2} (1 + r|x_1|) e^{x_1-r|x_1|} \leq \frac{\pi}{r^2} (1 + r|x_1|) e^{(1-r)|x_1|}. \]

Next suppose \( 0 < r < 1 \). We may assume \( x_1 < 0 \) since \( v(r, x_1) = 0 \) if \( 0 < r < 1 \) and \( x_1 \geq 0 \). Note that, in this case, we have

\[ \frac{\partial v}{\partial r} = -\frac{\pi}{r^2} \left\{ e^{(1+r)x_1} - e^{(1-r)x_1} \right\} + \frac{\pi x_1}{r} \left\{ e^{(1-r)x_1} + e^{(1-r)x_1} \right\}. \]

If \( 1/2 \leq r < 1 \), it is easy to see that

\[ \left| \frac{\partial v}{\partial r} \right| \leq C \left\{ e^{(1+r)x_1} - e^{(1-r)x_1} \right\} + C |x_1| \left\{ e^{(1-r)x_1} + e^{(1-r)x_1} \right\} \]
\[ \leq C \left\{ 1 + |x_1| \right\} e^{(r-1)|x_1|}. \]
Also, if \( 0 < r < 1/2 \) and \(|rx_1| \geq 1\), then \(|x_1| \geq 1/r\). It follows that
\[
\left| \frac{\partial v}{\partial r} \right| \leq C |x_1|^2 \left\{ e^{(1+r)x_1} + e^{(1-r)x_1} \right\} \\
\leq C |x_1|^2 e^{-|x_1|/2}.
\]

Finally, if \( 0 < r < 1/2 \) and \(|rx_1| < 1\), we use \( e^t = 1 + t + O(t^2) \) for \(|t| < 1\) to obtain
\[
\frac{\partial v}{\partial r} = -\frac{\pi}{r^2} e^{x_1} \{ 2rx_1 + O((rx_1)^2) \} + \frac{\pi x_1}{r} e^{x_1} \{ 2 + O((rx_1)^2) \} \\
= \frac{\pi}{r^2} e^{x_1} O((rx_1)^2) + \frac{\pi x_1}{r} e^{x_1} O((rx_1)^2).
\]

It follows that
\[
\left| \frac{\partial v}{\partial r} \right| \leq C \left\{ |x_1|^2 e^{x_1} + r |x_1|^3 e^{x_1} \right\} \leq C |x_1|^2 e^{-|x_1|}.
\]

The proof is now complete. \(\square\)

**Lemma 3.16.** Let \( u(x) \) be the inverse Fourier transform of \( |\xi|^2 - 1 + 2i\xi_1 \)^{-1} in \( \mathbb{R}^d \). Then, if \( d = 3 \),
\[
|u(x)| \leq \frac{C}{|x|},
\]
and, if \( d = 2 \),
\[
|u(x)| \leq \begin{cases} 
C \ln \frac{1}{|x|}, & \text{if } |x| \leq \frac{1}{2}; \\
\frac{C}{|x|}, & \text{if } |x| > \frac{1}{2}. 
\end{cases}
\]

**Proof.** We first consider the case \( d = 3 \). It follows from (3.8) that
\[
u(x) = 2\pi \int_0^{1/|x'|} J_0(|x'| r) v(r, x_1) r dr + 2\pi \int_{1/|x'|}^{\infty} J_0(|x'| r) v(r, x_1) r dr \\
= I_1 + I_2.
\]

By Lemma 3.15, \(|v(r, x_1)| r \leq C\). This, together with the observation \(|J_0(t)| \leq 1\), gives
\[
|I_1| \leq 2\pi \int_0^{1/|x'|} |v(r, x_1)| r dr \leq \frac{C}{|x'|}.
\]

To estimate \( I_2 \) we first assume that \(|x'| \leq 1\). Since
\[
rJ_0(r) = \frac{d}{dr} \{ rJ_1(r) \},
\]

(3.17)
where $J_\nu(r)$ denotes the Bessel function of the first kind of order $\nu$ (see [11]), we may use integration by parts to obtain
\[
I_2 = \frac{2\pi}{|x'|^3} \int_1^\infty r J_0(r) v \left( \frac{r}{|x'|}, x_1 \right) dr 
\]
\[
= -\frac{2\pi}{|x'|^3} J_1(1) v \left( \frac{1}{|x'|}, x_1 \right) - \frac{2\pi}{|x'|^3} \int_1^\infty r J_1(r) \frac{\partial v}{\partial r} \left( \frac{r}{|x'|}, x_1 \right) dr.
\]
It then follows from the estimate (see [11])
\[
|J_1(r)| \leq C \frac{r^{-1/2}}{r}, \quad \text{for } r \geq 1
\]
and Lemma 3.15 that
\[
|I_2| \leq \frac{C}{|x'|} + \frac{C}{|x'|^3} \int_1^\infty r^{1/2} \left| \frac{\partial v}{\partial r} \left( \frac{r}{|x'|}, x_1 \right) \right| dr 
\]
\[
\leq \frac{C}{|x'|} + \frac{C}{|x'|^{3/2}} \int_1^\infty r^{1/2} \left| \frac{\partial v}{\partial r}(r, x_1) \right| dr 
\]
\[
\leq \frac{C}{|x'|} + \frac{C}{|x'|^{3/2}} \int_1^{1/|x'|} r^{1/2} \cdot \frac{1}{r^{1/2}} \cdot (1 + r |x_1|) e^{(r-1)|x_1|} dr 
\]
\[
\leq \frac{C}{|x'|}.
\]
If $|x'| \geq 1$, we write
\[
I_2 = \frac{2\pi}{|x'|^3} \int_1^{|x'|} r J_0(r) v \left( \frac{r}{|x'|}, x_1 \right) dr + \frac{2\pi}{|x'|^3} \int_{|x'|}^\infty r J_0(r) v \left( \frac{r}{|x'|}, x_1 \right) dr
\]
\[
= I_{21} + I_{22}.
\]
Note that, using (3.17), integration by parts, and (3.18), we have
\[
|I_{21}| \leq \frac{C}{|x'|^{3/2}} + \frac{C}{|x'|^{3/2}} \int_1^{|x'|} r^{1/2} \left| \frac{\partial v}{\partial r} \left( \frac{r}{|x'|}, x_1 \right) \right| dr 
\]
\[
\leq \frac{C}{|x'|^{3/2}} + \frac{C}{|x'|^{3/2}} \int_1^{1/|x'|} r^{1/2} \left| \frac{\partial v}{\partial r}(r, x_1) \right| dr 
\]
\[
\leq \frac{C}{|x'|^{3/2}} + \frac{C}{|x'|^{3/2}} \int_0^{|x_1|} |x_1| e^{(r-1)|x_1|} dr 
\]
\[
\leq \frac{C}{|x'|^{3/2}} \leq \frac{C}{|x'|}.
\]
Similarly,

\[ |I_{22}| = \frac{2\pi}{|x'|^2} \left| \int_0^\infty \frac{d}{dr} \{ r J_1(r) \} v \left( \frac{r}{|x'|}, x_1 \right) dr \right| \]

\[ \leq \frac{C}{|x'|^{3/2}} + \frac{C}{|x'|^3} \int_0^\infty r^{1/2} \left| \frac{\partial v}{\partial r} \left( \frac{r}{|x'|}, x_1 \right) \right| dr \]

\[ = \frac{C}{|x'|^{3/2}} + \frac{C}{|x'|^3} \int_1^\infty r^{1/2} \left| \frac{\partial v}{\partial r}(r, x_1) \right| dr \]

\[ \leq \frac{C}{|x'|^{3/2}} \leq C. \]

Thus we have proved that, for any \( x \in \mathbb{R}^3 \),

\[ (3.19) \quad |u(x)| \leq \frac{C}{|x'|}. \]

To finish the case \( d = 3 \), we still need to show that

\[ (3.20) \quad |u(x)| \leq \frac{C}{|x_1|}, \quad \text{for any } x \in \mathbb{R}^3. \]

Clearly, (3.19) and (3.20) imply that \( |u(x)| \leq C/|x| \) for any \( x \in \mathbb{R}^3 \).

To see (3.20), we use Lemma 3.15 to obtain

\[ |u(x)| \leq 2\pi \int_0^\infty |v(r, x_1)| r \, dr \]

\[ \leq C \int_0^1 \left\{ e^{(r-1)|x_1|} + |x_1| e^{-|x_1|/2} \right\} dr + C \int_1^\infty e^{(1-r)|x_1|} dr \]

\[ \leq \frac{C}{|x_1|}. \]

We now consider the case \( d = 2 \). By Lemmas 3.7 and 3.5,

\[ |u(x)| = 2 \left| \int_0^\infty \cos(|x_2| r) v(r, x_1) \, dr \right| \]

\[ \leq 2 \int_0^\infty |v(r, x_1)| \, dr \]

\[ \leq C \int_0^1 \left\{ e^{(r-1)|x_1|} + |x_1| e^{-|x_1|/2} \right\} dr + C \int_1^\infty e^{(1-r)|x_1|} \frac{dr}{r} \]

\[ \leq C |x_1| e^{-|x_1|/2} + C \int_0^1 e^{-r|x_1|} \, dr + C \int_0^\infty e^{-r|x_1|} \frac{dr}{r+1}. \]
From this it is not hard to see that

\[
|u(x)| \leq \begin{cases} 
C \ln \frac{1}{|x_1|}, & \text{if } |x_1| \leq \frac{1}{2}, \\
\frac{C}{|x_1|}, & \text{if } |x_1| > \frac{1}{2}.
\end{cases}
\]  

(3.21)

Finally, we will show that

\[
|u(x)| \leq \begin{cases} 
C \ln \frac{1}{|x_2|}, & \text{if } |x_2| \leq \frac{1}{2}, \\
\frac{C}{|x_2|}, & \text{if } |x_2| > \frac{1}{2}.
\end{cases}
\]  

(3.22)

The desired estimate for $|u(x)|$ follows easily from (3.21) and (3.22).

To see (3.22) we write

\[
u(x) = 2 \int_0^{1/|x_2|} \cos(|x_2|r)v(r, x_1) \, dr + 2 \int_{1/|x_2|}^{\infty} \cos(|x_2|r)v(r, x_1) \, dr
\]

\[= I_3 + I_4,
\]
as in the case of $d = 3$. If $|x_2| > 1/2$, by Lemma 3.15, we have

\[|I_3| \leq C \int_0^{1/|x_2|} |v(r, x_1)| \, dr \leq C \int_0^{1/|x_2|} \, dr \leq \frac{C}{|x_2|}.
\]

Similarly, if $|x_2| \leq 1/2$,

\[|I_3| \leq 2 \int_0^1 |v(r, x_1)| \, dr + 2 \int_1^{1/|x_2|} |v(r, x_1)| \, dr \leq C + C \int_1^{1/|x_2|} \frac{dr}{r} \leq C \ln \frac{1}{|x_2|}.
\]

To estimate $I_4$ we use integration by parts. Suppose $|x_2| \leq 1/2$. Then

\[|I_4| = \frac{2}{|x_2|} \left| \int_0^\infty \frac{\partial}{\partial r} \left\{ \sin(|x_2| r) \right\} v(r, x_1) \, dr \right|
\]

\[\leq C + \frac{C}{|x_2|} \int_{1/|x_2|}^{\infty} \left| \frac{\partial v}{\partial r}(r, x_1) \right| \, dr
\]

\[\leq C + \frac{C}{|x_2|} \int_{1/|x_2|}^{\infty} \frac{1}{r^2} \left( 1 + r |x_1| \right) e^{(1-r)|x_1|} \, dr
\]

\[\leq C \leq C \ln \frac{1}{|x_2|}.
\]
If $|x_2| > 1/2$, then
\[
|I_4| \leq \frac{2}{|x_2|} \left| \int_{1/|x_2|}^{1} \frac{\partial}{\partial r} \{\sin(|x_2| r)\} \cdot v(r, x_1) \, dr \right| + \frac{2}{|x_2|} \left| \int_{1}^{\infty} \frac{\partial}{\partial r} \{\sin(|x_2| r)\} \cdot v(r, x_1) \, dr \right|
\]
\[
\leq \frac{C}{|x_2|} + \frac{C}{|x_2|} \int_{1/|x_2|}^{1} \left| \frac{\partial v}{\partial r}(r, x_1) \right| \, dr + \frac{C}{|x_2|} \int_{1}^{\infty} \left| \frac{\partial v}{\partial r}(r, x_1) \right| \, dr
\]
\[
\leq \frac{C}{|x_2|}.
\]
This proves (3.22) and completes the proof of Lemma 3.16.

It follows from Lemma 3.16 and (3.6) that
\[
\left\{ \frac{1}{\varphi(\cdot, 1)} \right\}^\vee (x) \leq \begin{cases} 
C \frac{1}{|x|}, & \text{if } d = 3, \\
C \ln \left(1 + \frac{1}{|x|}\right), & \text{if } d = 2.
\end{cases}
\]
This, together with (3.3) and (3.5), implies that
\[
|f_\rho(x)| \leq \begin{cases} 
C \left\{ \frac{1}{\rho} + \frac{1}{|x|} \right\}, & \text{if } d = 3, \\
C \left\{ \frac{1}{\rho} + \ln \left(1 + \frac{1}{|x|}\right) \right\}, & \text{if } d = 2.
\end{cases}
\]
Thus, by (2.16), for any $x \in \mathbb{R}^d$,
\[
|F_\rho(x)| = \rho^{d-2} |f_\rho(\rho x)| \leq \begin{cases} 
C \left\{ \frac{1+1}{|x|} \right\}, & \text{if } d = 3, \\
C \left\{ \frac{1}{\rho} + \ln \left(1 + \frac{1}{\rho|x|}\right) \right\}, & \text{if } d = 2.
\end{cases}
\]
The estimate (3.1) now follows from (3.25), and the proof of (3.2) is complete.

4. Proof of the Main Theorem

Suppose $V \in K_d$. It is well known that, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon, V} > 0$ such that
\[
\int_{\mathbb{R}^d} |g|^2 |V| \, dx \leq \varepsilon \int_{\mathbb{R}^d} |\nabla g|^2 d x + C_{\varepsilon, V} \int_{\mathbb{R}^d} |g|^2 d x
\]
for any $g \in H^1(\mathbb{R}^d)$; see [7], [18]. It follows from (4.1) that the quadratic form associated with $DAD^T + V$ generates a unique self-adjoint operator on $L^2(\mathbb{R}^d)$, which we also denote by $DAD^T + V$. 
Let \( \psi \in H^1(T^d) \), where

\[
H^1(T^d) = \left\{ \phi \in L^2(T^d) : \phi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{i(x,n)} \text{ and } \sum_{n \in \mathbb{Z}^d} |n|^2 |a_n|^2 < \infty \right\}.
\]

Extending \( \psi \) by periodicity to \( \mathbb{R}^d \) and then applying (4.1) to \( \tilde{\psi}_\eta \), where \( \tilde{\eta} \) is a \( C^\infty \) cut-off function such that \( \tilde{\eta} = 1 \) on \( \Omega \), we obtain

\[
\left( \int_\Omega |\psi|^2 |V| \, dx \right) \leq \varepsilon \int_\Omega |\nabla \psi|^2 \, dx + \tilde{C}_{\varepsilon,V} \int_\Omega |\psi|^2 \, dx
\]

for any \( \varepsilon > 0 \). This implies that, for any \( k \in \mathbb{C}^d \), the quadratic form associated with \( (D + k)A(D + k)^T + V \) on \( T^d \) defines a unique closed operator on \( L^2(T^d) \), which we denote by \( \mathbb{H}_V(k) \). Moreover,

\[
\text{Domain } (\mathbb{H}_V(k)) = \left\{ \psi \in H^1(T^d) : \mathbb{H}_V(0)\psi = (DAD^T + V)\psi \in L^2(T^d) \right\}.
\]

Let \( a \in \mathbb{R}^d \) be a vector satisfying (2.1) and

\[
L = \left\{ b \in \mathbb{R}^d : \langle b, a \rangle = 0 \text{ and } |b| \leq \sqrt{d} \right\}.
\]

**Proposition 4.5.** If, for every \( b \in L \), the family of operators \( \{ \mathbb{H}_V(za + b) : z \in \mathbb{C} \} \) has no common eigenvalue, then the spectrum of the operator \( DAD^T + V \) on \( L^2(\mathbb{R}^d) \) is purely absolutely continuous.

**Proof.** See [13] and [16]. \( \square \)

Fix \( b \in L \) and let

\[
\delta = \frac{1}{a_1} \left( \frac{1}{2} - b_1 \right),
\]

as in (2.2). We will show that the family of operators \( \{ \mathbb{H}_V((\delta + i\rho)a + b) : \rho \geq 1 \} \) has no common eigenvalue under the assumption of our main theorem.

We need the following estimate on the norm of \( \{ \mathbb{H}_0((\delta + i\rho)a + b) \}^{-1} \) on \( L^1(T^d) \).

**Theorem 4.6.** There exists a constant \( C > 0 \) such that

\[
\left\| \{ \mathbb{H}_0((\delta + i\rho)a + b) \}^{-1} \right\|_{L^1(T^d) \rightarrow L^1(T^d)} \leq \begin{cases} \frac{C \ln(\rho + 1)}{\rho^{1/2}}, & \text{if } d = 3, \\ \frac{C}{\rho^{1/2}}, & \text{if } d = 2. \end{cases}
\]

**Proof.** In view of (2.8), it suffices to show that

\[
\int_\Omega |G_\rho(x)| \, dx \leq \begin{cases} \frac{C \ln(\rho + 1)}{\rho^{1/2}}, & \text{if } d = 3, \\ \frac{C}{\rho^{1/2}}, & \text{if } d = 2. \end{cases}
\]
To this end, note that, by Hölder’s inequality, (2.9), and the Plancherel theorem, we have
\[
\int_{\Omega} |G_\rho(x)| \, dx \leq |\Omega|^{1/2} \left\{ \int_{\Omega} |G_\rho(x)|^2 \, dx \right\}^{1/2} \\
= C \left\{ \sum_{n \in \mathbb{Z}^d} \frac{1}{|n + k| A(n + k)^T|^2} \right\}^{1/2} \\
\leq C \left\{ \sum_{n \in \mathbb{Z}^d} \frac{1}{\{(n + b)A(n + b)^T - \rho^2 a_1 s_0| + \rho |n_1 + \frac{1}{2}|^2 \}^2} \right\}^{1/2}.
\]
The desired estimate (4.7) follows from the proof of Lemma 3.2 in [16] (see the estimate (3.11) in [16]). We omit the details. □

The next theorem is a consequence of the pointwise estimate (3.2) of the kernel function $G_\rho$.

**Theorem 4.8.** There exists a constant $C > 0$ such that
\[
\left\| V \{ \mathbb{H}_0((\delta + i\rho)a + b) \}^{-1} \right\|_{L^1(\mathbb{T}^d) \rightarrow L^1(\mathbb{T}^d)} \leq \begin{cases} 
C \sup_{x \in \Omega} \int_\Omega \frac{|V(y)|}{|y - x|} \, dy, & \text{if } d = 3, \\
C \sup_{x \in \Omega} \int_\Omega |V(y)| \{1 + |\ln|y - x||\} \, dy, & \text{if } d = 2.
\end{cases}
\]

**Proof.** Recall that, if $\psi \in C^\infty(\mathbb{T}^d)$, then
\[
\{ \mathbb{H}_0((\delta + i\rho)a + b) \}^{-1} \psi(x) = \int_{\Omega} G_\rho(x - y)\psi(y) \, dy.
\]
It follows that
\[
\left\| V \{ \mathbb{H}_0((\delta + i\rho)a + b) \}^{-1} \psi \right\|_1 \leq \int_{\Omega} |V(x)| \left\{ \int_{\Omega} |G_\rho(x - y)| \, |\psi(y)| \, dy \right\} \, dx \\
\leq \sup_{y \in \Omega} \int_{\Omega} |V(x)| \, |G_\rho(x - y)| \, dx \|\psi\|_1.
\]
The desired estimate now follows easily from (3.2). □

**Proof of Main Theorem.** We give the proof for the case $d = 3$. The case $d = 2$ can be handled in the same manner.

To show that $\{ \mathbb{H}_V((\delta + i\rho)a + b) : \rho \geq 1 \}$ has no common eigenvalue, we argue by contradiction. Suppose that there exists $E \in \mathbb{R}$ such that, for every $\rho \geq 1$, there exists $\psi_\rho \in \text{Domain}(\mathbb{H}_V((\delta + i\rho)a + b))$ such that $\|\psi_\rho\|_2 = 1$ and
\[
\mathbb{H}_V((\delta + i\rho)a + b)\psi_\rho = E\psi_\rho.
\]
Since \( \psi_\rho \in H^1(\mathbb{T}^d) \), by the Cauchy inequality and (4.2), we have
\[
\int_{\Omega} |\psi_\rho| \, |V| \, dx \leq \left\{ \int_{\Omega} |V| \, dx \right\}^{1/2} \left\{ \int_{\Omega} |\psi_\rho|^2 \, |V| \, dx \right\}^{1/2} < \infty.
\]
It follows that \((D + k)A(D + k)^T \psi_\rho = E \psi_\rho - V \psi_\rho \in L^1(\mathbb{T}^d)\).

Let
\[
V_N(x) = \begin{cases} 
V(x), & \text{if } |V(x)| > N, \\
0, & \text{if } |V(x)| \leq N.
\end{cases}
\]
Then,
\[
(4.9) \quad \| (D + k)A(D + k)^T \psi_\rho \|_1 \leq \{|E| + N\} \| \psi_\rho \|_1 + \| V_N \psi_\rho \|_1.
\]
By Theorem 4.8,
\[
(4.10) \quad \| V_N \psi_\rho \|_1 \leq C \sup_{x \in \Omega} \int_{\Omega} \frac{|V_N(y)|}{|y - x|} \, dy \cdot \| (D + k)A(D + k)^T \psi_\rho \|_1.
\]
Note that
\[
\sup_{x \in \Omega} \int_{\Omega} \frac{|V_N(y)|}{|y - x|} \, dy \leq \sup_{x \in \Omega} \int_{|y - x| \leq r} \frac{|V(y)|}{|y - x|} \, dy + \frac{1}{r} \int_{\Omega} |V_N(y)| \, dy.
\]
It follows that
\[
\lim_{N \to \infty} \sup_{x \in \Omega} \int_{\Omega} \frac{|V_N(y)|}{|y - x|} \, dy \leq \lim_{r \to 0} \sup_{x \in \Omega} \int_{|y - x| \leq r} \frac{|V(y)|}{|y - x|} \, dy = 0.
\]
This implies that, if \( N \) is sufficiently large,
\[
(4.11) \quad \| V_N \psi_\rho \|_1 \leq \frac{1}{2} \| (D + k)A(D + k)^T \psi_\rho \|_1.
\]
In view of (4.9) and (4.11), we obtain
\[
\| (D + k)A(D + k)^T \psi_\rho \|_1 \leq 2(|E| + N) \| \psi_\rho \|_1.
\]
This, together with Theorem 4.6, gives
\[
\frac{C \rho^{1/2}}{\ln(\rho + 1)} \| \psi_\rho \|_1 \leq 2(|E| + N) \| \psi_\rho \|_1
\]
or
\[
\frac{C \rho^{1/2}}{\ln(\rho + 1)} \leq 2(|E| + N),
\]
for any \( \rho \geq 1 \). This is impossible if we let \( \rho \to \infty \). \( \square \)
References


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