14 Simplifying Trigonometric Expressions and Proving Trigonometric Identities

Concepts:

- Expressions vs. Identities
- Simplifying Trigonometric Expressions
- Proving Trigonometric Identities
- Disproving Trigonometric Identities

(Section 7.1)

14.1 Simplifying Trigonometric Expressions

**Definition 14.1**

A trigonometric expression is an expression that contains trigonometric functions. Like all mathematical expressions, trigonometric expressions do not contain an equal sign (=).

In your previous mathematics classes, you have undoubtedly simplified many algebraic expressions. Why did your mathematics teachers ask you to do so much simplification? It is true that simplifying expressions can help you learn the rules. But why do you need to learn the rules? For one thing, these rules are the first step toward solving harder problems. For example, in Calculus you will need to rationalize a numerator in order to find the derivative of the square root function. For many students, Calculus is the first mathematics course you take where you can really start to see all of the pieces coming together. Today we add another piece to the puzzle by learning to simplify trigonometric expressions.

When you are simplifying a trigonometric expression you need to:

1. Know the rules.
2. Follow the rules.
3. Recognize that you can only multiply by ________.
4. Recognize that you can only add ________.

The rules come from definitions or identities that we have already proven. So far, we have defined the trigonometric functions, and we have discussed the periodicity identities, the
Pythagorean identities, and the negative angle identities. These are the only rules that are available to you. As we proceed through the course, more rules will become available to you.

The Procedure for Simplifying Trigonometric Expressions:

When you are simplifying, at each step you can:

1. apply a known rule,
2. add 0, or
3. multiply by 1.

Your final answer should have the form:

\[
\begin{align*}
\text{the original expression} & = \text{simplification} \\
& = \text{simplification} \\
& = \text{simplification} \\
& \vdots \\
& = \text{the final simplified expression}
\end{align*}
\]

In all of the examples that follow, assume that the denominators do not evaluate to zero.

Example 14.2
This example was taken from


Simplify the expression.

\[
\frac{\sin^4(x) - \cos^4(x)}{\sin^2(x) - \cos^2(x)}
\]
Example 14.3
This example was taken from
Simplify the expression. \[
\frac{\sec(x) \sin^2(x)}{1 + \sec(x)}
\]

Example 14.4
This example was taken from
Simplify the expression. \[
\sin^2(x) - \cos^2(x) \sin^2(x)
\]

Example 14.5
This example was taken from
Simplify the expression. \[
\tan^4(x) + 2 \tan^2(x) + 1
\]
14.2 Proving a Trigonometric Identity

**Definition 14.6**
An identity is an equation that is true for every possible value of the variable. 
A trigonometric identity is an identity that involves trigonometric functions.

There are a few things to note about identities. First, identities have to be true for all possible values of the variable. This does not include values of the variable that would cause one side of the equation to be undefined. If an equation is only true for some of the possible values of the variable, then it is not an identity. This is why you cannot prove an identity with an example. On the other hand, you can disprove an identity by finding one value of the variable that does not yield a true equation.

Second, you cannot prove an identity with your calculator because calculators only produce approximations. However, you can use your calculator to determine if an equation has any possibility of being an identity. For example, the picture below shows the graphs of $Y_1 = (x + 3)^2$ and $Y_2 = x^2 + 6x + 9$ in the $[-10, 10] \times [-10, 10]$ viewing window.

![Graphs of $(x + 3)^2 = x^2 + 6x + 9$](image)

These two graphs are indistinguishable, so there is hope that 
\[(x + 3)^2 = x^2 + 6x + 9\]
is an identity. In fact, we know from Algebra that this is an identity because 
\[
(x + 3)^2 = (x + 3)(x + 3) \quad \text{(Definition of Exponentiation)}
= (x + 3)x + (x + 3)3 \quad \text{(Distributive Law)}
= x^2 + 3x + 3x + 9 \quad \text{(Distributive Law)}
= x^2 + 6x + 9 \quad \text{(Combining Like Terms by the Distributive Law)}
\]
On the other hand, the picture below shows the graphs of $Y_1 = (x + 3)^2$ and $Y_2 = x^2 + 9$ in the $[-10, 10] \times [-5, 15]$ viewing window.

Clearly, these graphs are different. In fact, they only intersect when $x = 0$. We can show that $(x + 3)^2 = x^2 + 9$ is NOT an identity by selecting any value of $x$ that is not equal to zero and showing that the equation is not true for that value of $x$. For example, when $x = 2$

$$(x + 3)^2 = (2 + 3)^2 = 5^2 = 25$$

but

$$x^2 + 9 = 2^2 + 9 = 4 + 9 = 13$$

Since 25 $\neq$ 13, then $(x + 3)^2 \neq x^2 + 9$ when $x = 2$. Thus, $(x + 3)^2 = x^2 + 9$ is NOT an identity.

In this section, we are going to prove or disprove trigonometric identities. Proving trigonometric identities is nothing more than simplifying an expression until you reach a target expression. The expression you are simplifying is one side of the equation and the target expression is the other side of the equation. It is very important that you start with only one side of the identity. Disproving an identity involves finding a value for the variable that makes the equation false.
The Procedure for Proving Trigonometric Identities:

When you are simplifying, at each step you can:

1. apply a known rule,
2. add 0, or
3. multiply by 1.

Your final answer should have the form:

\[
\text{one side of the identity} = \text{simplification} \\
= \text{simplification} \\
= \text{simplification} \\
\vdots \\
= \text{the other side of the identity}
\]

When you prove a trigonometric identity, it becomes a new rule that you can use to simplify other trigonometric expressions.

In all of the examples that follow, you should assume that the denominators do not evaluate to zero. In other words, you are proving the identity for all values of the variable for which both sides of the equation are defined.

Example 14.7
This example was taken from

http://www.intmath.com/Analytic-trigonometry/1_Trigonometric-identities.php

Prove or disprove that \(\sin(x)\cos(x)\tan(x) = 1 - \cos^2(x)\).
Example 14.8
This example was taken from

http://www.intmath.com/Analytic-trigonometry/1_Trigonometric-identities.php

Prove or disprove that \( \tan(x) + \cot(x) = \sec(x) \csc(x) \).

Example 14.9
Prove or disprove that \( \sin(2x) = 2 \sin(x) \).

What rules are you using each time to simplify during your proofs? This is your toolbox of methods to try if you don’t know where to begin.
Example 14.10
Prove or disprove that \( \sec(x)(\sec(x) - \cos(x)) = \tan^2(x) \).