Worksheet # 26: The Fundamental Theorems of Calculus and the Net Change Theorem

1. (a) State both parts of the Fundamental Theorem of Calculus using complete sentences.
   (b) Consider the function \( f(x) \) on \([1, \infty)\) defined by \( f(x) = \int_{1}^{x} \sqrt{t^5 - 1} \, dt \). Argue that \( f \) is increasing.
   (c) Find the derivative of the function \( g(x) = \int_{1}^{x} \sqrt{t^5 - 1} \, dt \) on \((1, \infty)\).

2. Use Part II of the Fundamental Theorem of Calculus to find the derivative of the following functions:
   (a) \( g(x) = \int_{1}^{x} (2 + t^4)^5 \, dt \)
   (b) \( F(x) = \int_{x}^{4} \cos(t^5) \, dt \)
   (c) \( h(x) = \int_{0}^{x^2} \sqrt{1 + r^3} \, dr \)
   (d) \( y(x) = \int_{1}^{x^3} \sin^3(t) \, dt \)
   (e) \( G(x) = \int_{\sqrt{x}}^{x^2} \sqrt{t} \sin(t) \, dt \)

3. Below is pictured the graph of the function \( f(x) \), its derivative \( f'(x) \), and an antiderivative \( \int f(x) \, dx \). Identify \( f(x), f'(x) \) and \( \int f(x) \, dx \).

4. A population of rabbits at time \( t \) increases at a rate of \( 40 - 12t + 3t^2 \) rabbits per year. Find the population after 8 years if there are 10 rabbits at \( t = 0 \).

5. Suppose the velocity of a particle traveling along the \( x \)-axis is given by \( v(t) = 3t^2 + 8t + 15 \) m/s and the particle is initially located 5 meters left of the origin. How far does the particle travel from \( t = 2 \) seconds to \( t = 3 \) seconds? After 3 seconds, where is the particle with respect to the origin?

6. Suppose an object traveling in a straight line has a velocity function given by \( v(t) = t^2 - 8t + 15 \) km/hr. Find the displacement and distance traveled by the object from \( t = 2 \) to \( t = 4 \) hours.

7. An oil storage tank ruptures and oil leaks from the tank at a rate of \( r(t) = 100e^{-0.01t} \) liters per minute. How much oil leaks out during the first hour?

8. Recognize each of the sums as a Riemann sum, express the limit as an integral and use the Fundamental Theorem to evaluate the limit.
   (a) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\sqrt{3 + \frac{i}{n}}}{n} \)
   (b) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2(2 + \frac{2i}{n})^2}{n} \)
1. Consider the functions \( h(t) = \frac{1}{t} \) and \( j(t) = -5t^4 + \frac{1}{\sqrt{t}}. \)

   (a) Evaluate \( \int_c^e h(t) \, dt. \)

   (b) Calculate \( \int_0^x j(t) \, dt. \)

2. Let \( f \) be a continuous function such that

   \[
   \int_0^x f(t) \, dt = xe^{2x} + \int_0^x e^{-t} f(t) \, dt.
   \]

   Find an explicit formula for \( f(x) \). \textit{Hint: It may be useful to differentiate to find an equation for } \( f. \)

3. What is wrong with the following calculation? Explain.

   \[
   \int_{-1}^3 \frac{1}{x^2} \, dx = x^{-1}\bigg|_{-1}^{3} = \frac{-1}{3} - 1 = \frac{-4}{3}
   \]