Marie Meyer

(1) Let $G$ be a finite group and let $P$ be a normal $p$-subgroup of $G$. Show that $P$ is contained in every Sylow $p$-subgroup of $G$.

(2) Determine all groups of order 21 up to isomorphism.

(3) Let $P$ be a Sylow $p$-subgroup of $G$ and let $H$ be any subgroup of $G$. Prove that $P \cap H$ is the unique Sylow $p$-subgroup of $H$.

(4) Let $G$ be a finite group of composite order $n$ with the property that $G$ has a subgroup of order $k$ for each positive integer $k$ dividing $n$. Prove that $G$ is not simple.

Fouché F. Smith

(1) Scavenger Hunt 1 : Algebra Prelim June 2004
Let $(G, \cdot)$ be a group with identity element $e$. Suppose that $a \neq e$ is an element of $G$ such that $a^6 = a^{10} = e$. Determine the order of $a$.

(2) Scavenger Hunt 2 : J. Fraleigh Section 6 Exercise 44
Let $G$ be a cyclic group with generator $a$, and let $G'$ be a group isomorphic to $G$. If $\phi : G \to G$ is an isomorphism, show that, for every $x \in G$, $\phi(x)$ is completely determined by the value $\phi(a)$. That is, if $\phi : G \to G'$ and $\psi : G \to G'$ are two isomorphisms such that $\phi(a) = \psi(a)$, then $\phi(x) = \psi(x)$ for all $x \in G$.

(3) Scavenger Hunt : D. Dummit Section 1.6 Exercise 22
Let $A$ be an abelian group and fix some $k \in \mathbb{Z}$. Prove that the map $a \mapsto a^k$ is a homomorphism from $A$ to itself. If $k = -1$ prove that this homomorphism is an isomorphism (i.e., is an automorphism of $A$.)

(4) Scavenger Hunt : D. Dummit Section 3.2 Exercise 31
Let $N \leq G$ and $N$ is a normal subgroup of $H$, then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of $G$ in which $N$ is normal (i.e., is the join of all subgroups $H$ for which $N \triangleleft H$)

Sarah Orchard

(1) 1. Let $G$ be a finite group and let $H$ be a normal Sylow $p$-subgroup of $G$. Show that $\alpha(H) = H$ for all automorphisms $\alpha$ of $G$. 

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(2) 2. Suppose that $G$ is a group of order $p^n$, where $p$ is prime, and $G$ has exactly one subgroup for each divisor of $p^n$. Show that $G$ is cyclic.

(3) 3. Let $H$ be a Sylow $p$-subgroup of $G$. Prove that $H$ is the only Sylow $p$-subgroup of $G$ contained in $N(H)$.

(4) 4. Show that if $G$ is a group of order 168 that has a normal subgroup of order 4, then $G$ has a normal subgroup of order 28.

**Florian Kohl**

(1) Prove that there are 45 elements of order 2 in $A_6$.

(2) Let $G$ be an abelian group, $K$ a group and $f: G \to K$ a group homomorphism. Prove that $f(G) \subset K$ is an abelian subgroup of $K$.

(3) Prove that $G$ is abelian if and only if the map $f: G \to G$ by $f(g) = g^2$ is a group homomorphism.

(4) Prove that $(\mathbb{Q} \setminus 0, \cdot)$ is not a cyclic group.

**George Lytle**

(1) Let $K$ be a Sylow $p$-subgroup of $G$ and $N$ a normal subgroup of $G$. Prove that $K \cap N$ is a Sylow $p$-subgroup of $N$.

(2) Prove that there are no simple subgroups of order 30.

(3) Let $K$ be a $p$-Sylow subgroup of $G$ and $N$ a normal subgroup of $G$. If $K$ is a normal subgroup of $N$, prove that $K$ is normal in $G$.

(4) If $K$ is a $p$-Sylow subgroup of $G$ and $H$ is a subgroup that contains $N(K)$, prove that $[G : H] \equiv 1 \mod p$.  

**Lola Davidson**

(1) How many elements of order 5 does a non-cyclic group of order 55 have?

(2) If $P$ is a Sylow $p$-subgroup of $G$, prove that $P$ is the only Sylow $p$-subgroup of $N(P)$.

(3) Let $G$ be a group of order 105. Show that $G$ has a subgroup of order 35.

(4) If $|G| = pqr$ with $p \leq q \leq r$ primes, prove that $G$ is not simple.

**Ian Barnett**

(1) Prove that every non-abelian group of order 6 has a non-normal subgroup of order 2, and in fact there is only one such group.

(2) Prove that there are 28 homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $D_4$.

(3) Prove that for every integer $1 \leq n \leq 59$ there are no non-abelian simple groups of order $n$.

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1All problems are from *Abstract Algebra: an Introduction* Second Edition by Thomas Hungerford
(4) An abelian group has 8192 has elements of the following orders:

<table>
<thead>
<tr>
<th>order</th>
<th># of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>31</td>
</tr>
<tr>
<td>4</td>
<td>224</td>
</tr>
<tr>
<td>8</td>
<td>1792</td>
</tr>
<tr>
<td>16</td>
<td>2048</td>
</tr>
<tr>
<td>32</td>
<td>4096</td>
</tr>
</tbody>
</table>

Determine the isomorphism type of the group.

Robert Cass

1.: Show that the multiplicative group \((\mathbb{Z}/2^l\mathbb{Z})^*\) for \(l \geq 3\) is a direct product of a cyclic group of order 2 and another cyclic group of order 2\(l-2\).

To do this, it will help to show that \(\{(-1)^a5^b | a = 0, 1 \text{ and } 0 \leq b < 2^{l-2}\}\) is a reduced residue system mod 2\(l\). You may also use the fact that the order of 5 modulo 2\(l\) is 2\(l^2\). 


2.: Let \(H\) be a proper subgroup of a finite group \(G\). Prove the group \(G\) is not the union of the conjugate subgroups of \(H\).


3.: Prove that any group of order 1365 is not simple.

Source: Jim Brown, homework problem from MA 851 Fall 2010 at Clemson University

4.: Show that there are two isomorphism classes of groups of order 6, the class of the cyclic group with six elements and the class of the symmetric group \(S_3\).


Cyrus Hettle

1. Count and give a combinatorial interpretation of the number of abelian groups of order 2\(n\) for \(n \in \mathbb{N}\). Give a geometric interpretation of the abelian groups of order 8.

2. Suppose a group \(G\) has elements \(u\) and \(v\) such that \(u^m = e, uvu^{-1} = v^k\), where \(k > 1, m > 0\). Prove that \(|v|\) is finite.

3. Let \(G\) be a group, and let \(f : G \to G\) be defined by \(f(g) = g^2\). Give necessary and sufficient conditions for \(f\) to be an automorphism.

4. Let \(G\) be a finite group and let \(P\) be a normal \(p\)-subgroup of \(G\). Show that \(P\) is contained in every Sylow \(p\)-subgroup of \(G\).

Eric Kaper

1. Show that \(A_5\) has no subgroup of order 15.

2. Show that \(A_5\) has no subgroup of order 30. (One possible approach to this is showing that every group of order 30 has a subgroup of order 15).
(3) Show that the number of conjugacy classes in $S_n$ is $p(n)$ where $p(n)$ is the number of ways, neglecting the order of the summands, that $n$ can be expressed as a sum of positive integers. The number $p(n)$ is the number of partitions of $n$.

(4) Show that the number of conjugacy classes in $S_n$ is also the number of different abelian groups (up to isomorphism) of order $p^n$, where $p$ is a prime number.

(5) Let $H$ be a normal subgroup of order $p^k$ of a finite group $G$. Show that $H$ is contained in every $p$-Sylow subgroup of $G$.

(6) Let $G$ be a finite group with the property that for each positive integer $n$, the equation $x^n = 1$ has at most $n$ solutions in the group. Prove that $G$ is cyclic.

(7) Show that any finite $p$-group $G$ is isomorphic to a group of upper triangular matrices with ones on the diagonal (unitriangular matrices) over $\mathbb{F}_p$.

A possible approach to this problem follows:

- Take $n \in \mathbb{N}$ to be given. Use a counting argument to show that the unitriangular group (group of all $n \times n$ unitriangular matrices) is a $p$-Sylow subgroup of the general linear group (group of all invertible $n \times n$ matrices) over $\mathbb{F}_p$.
- Note that the symmetric group embeds in the general linear group using permutation matrices.
- Note that $G$ is isomorphic to a subgroup of a symmetric group.
- Apply the fact that any two $p$-Sylow subgroups are conjugate.

Chase Russell

(1) Let $G$ be a group, and let $\text{Aut}(G)$ be the group of all automorphisms of $G$ together with the operation of function composition. Suppose that $G$ is non-Abelian. Show that $\text{Aut}(G)$ is not cyclic.

(2) Let $G$ be a group and $p$ be a prime. Suppose that $H = \{g^p | g \in G\}$. Show that $H$ is a normal subgroup of $G$ and that every nonidentity element of $G/H$ has order $p$.

(3) Let $G$ be an Abelian group. Determine all homomorphisms from $S_3$ to $G$.

(4) Let $G$ be an Abelian group and let $n$ be a positive integer. Let $G_n = \{g \in G | g^n = e\}$ and $G^n = \{g^n | g \in G\}$. Prove that $G/G_n$ is isomorphic to $G^n$.

Olsen McCabe
(1) How many elements of order 5 does a non-cyclic group of order 55 have?
(2) Prove that there are no simple groups of order 120.
(3) Show that every group of order 56 has a proper normal subgroup.
(4) If $|G| = pqr$, with $p < q < r$ primes, the $G$ is not simple.
(5) If $G/Z(G)$ is cyclic, prove that $G$ is abelian.
(6) Prove that a non-cyclic group of order 21 must have 14 elements of order 3.