Jordan Canonical Form

Recall the following definition:

Definition

1. We say that two square matrices ***A*** and ***B*** are ***similar*** provided there exists an invertible matrix ***P*** so that $B = P^{-1} A P$.
2. We say a matrix ***A*** is ***diagonalizable*** if it is similar to a diagonal matrix.

We noted in an earlier unit that not all square matrices are diagonalizable. The following theorem yields necessary and sufficient conditions for a square matrix to be diagonalizable.

Theorem

An $n \times n$ matrix ***A*** is diagonalizable if and only if ***A*** has *n* linearly independent eigenvectors.

Example

1. Let $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. Then

$$\begin{aligned} Det\left(\boldsymbol{A} - \lambda\, \boldsymbol{I_n}\right) &= Det\begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix}. \\ &= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 \\ &= -(\lambda - 1)^2 (\lambda - 10) \end{aligned}$$

We see that $\lambda = 1$ & $\lambda = 10$ are the associated eigenvalues for ***A***. We seek the corresponding eigenspaces:

The null space for

$$\begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} \underset{\lambda=1}{=} \begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is $\left\langle \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

and the null space for

$$\begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} \underset{\lambda=10}{=} \begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

is $\left\langle \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$. Since

$$Det\left(\begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \right) = -9 \neq 0,$$

the three vectors $\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$ are linearly independent and so by the

above theorem the given matrix is diagonalizable. In particular,

$$\begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

We note that the columns for $\begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$ consist of the set of three linearly independent eigenvectors for $\boldsymbol{A}$. (Hold that thought!)

2. Let $A = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Then

$$\begin{vmatrix} 2-\lambda & 0 & 1 & -3 \\ 0 & 2-\lambda & 10 & 4 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} = \lambda^4 - 9\lambda^3 + 30\lambda^2 - 44\lambda + 24$$

$$= (\lambda - 2)^3 (\lambda - 3).$$

The eigenspace corresponding to $\lambda = 2$ is given by $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ and the eigenspace corresponding to $\lambda = 3$ is given by $\left\langle \begin{pmatrix} -3 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\rangle$. Since we have only three linearly independent eigenvectors and the given matrix $A = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is of

size 4×4, the above theorem tells us that A is *not* diagonalizable.

While the matrix $A = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is not diagonalizable, it is similar to a matrix that is

"nearly" diagonal:

$$\begin{pmatrix} 0 & 1 & 0 & -3 \\ 1 & 10 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & -3 \\ 1 & 10 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

The matrix on the right-hand side of the above is an example of a matrix in ***Jordan Canonical Form***. Here we note that

$$\left(A - 2I_4\right)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0_{4\times1}, \ \left(A - 2I_4\right)\begin{pmatrix} 1 \\ 10 \\ 0 \\ 0 \end{pmatrix} = 0_{4\times1}, \text{ and } \left(A - 3I_4\right)\begin{pmatrix} -3 \\ 4 \\ 0 \\ 1 \end{pmatrix} = 0_{4\times1}$$

but

$$\left(A - 2I_4\right)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \neq 0_{4\times 1}.$$

Hence, three of the four columns of

$$\begin{pmatrix} 0 & 1 & 0 & -3 \\ 1 & 10 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

consist of (linearly independent) eigenvectors of $\boldsymbol{A}$. (Hold that thought!)

We illustrate the notion of a ***Jordan matrix*** via two sets of examples.

Example

1. The following are Jordan matrices:

$$\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix} \qquad \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -9 \end{pmatrix} \qquad \begin{pmatrix} 12 & 1 & 0 \\ 0 & 12 & 1 \\ 0 & 0 & 12 \end{pmatrix}$$

$$\begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} i & 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} i & 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 6 & 1 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

2. The following are *not* Jordan matrices:

$$\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} \qquad \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 0 & 0 \\ 1 & 12 & 1 \\ 0 & 0 & 12 \end{pmatrix} \qquad \begin{pmatrix} 5 & 10 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -9 \end{pmatrix} \qquad \begin{pmatrix} 12 & 1 & 0 \\ 0 & 12 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 1 & 0 & 0 & -3 \\ 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Example

1. The only 2×2 Jordan matrices are $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ (the $\lambda' s$ may or may not be distinct) and

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

2. The only 3×3 Jordan matrices are

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

where the λ_1, λ_2, & λ_3 are not necessarily distinct.

3. The only 4×4 Jordan matrices are

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \quad \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

where λ_1, λ_2, λ_3 & λ_4 are not necessarily distinct and the “blocks” may be permuted.

An $m \times m$ matrix of the form

$$
\begin{pmatrix}
\lambda & 1 & 0 & 0 & 0 & \ldots 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & \ldots 0 & 0 \\
0 & 0 & \lambda & 1 & 0 & \ldots 0 & 0 \\
0 & 0 & 0 & \lambda & 1 & \ldots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots 0 & \lambda
\end{pmatrix}
$$

is called a ***Jordan block***. An $n \times n$ matrix J is said to be in ***Jordan canonical form*** if it is a matrix of the form

$$
\begin{pmatrix}
J_1 & & & 0 \\
 & J_2 & & \\
 & & \ldots. & \\
0 & & & J_s
\end{pmatrix}
$$

where each J_k is either a diagonal matrix or a Jordan block matrix. *That is, a Jordan matrix is a matrix with Jordan blocks down the diagonal and zeros everywhere else.*

Theorem

Every $n \times n$ matrix is similar to a matrix in Jordan canonical form. That is, for every matrix A there exists an invertible matrix M so that $J = M^{-1} A M$ where J is in Jordan canonical form.

The "trick" to producing the Jordan matrix J is to find the invertible matrix M having the desired properties. As this process is similar to diagonalizing a matrix, we will see that the matrix M consists of columns of eigenvectors or "generalized" eigenvectors.

Recall that an eigenvector v associated with the eigenvalue λ for A satisfies the equation $\left(A - \lambda I_n\right) \boldsymbol{v} = \mathbf{0}$.

Definition

A nonzero n-vector $\boldsymbol{v}$ is called a ***generalized eigenvector of rank r*** associated with the eigenvalue λ if and only if $\left(A - \lambda I_n\right)^r \boldsymbol{v} = \mathbf{0}$ and $\left(A - \lambda I_n\right)^{r-1} \boldsymbol{v} \neq \mathbf{0}$.

We note that a generalized eigenvector of rank 1 is an ordinary eigenvector associated with λ.

Example

Earlier we observed that

$$\begin{pmatrix} 0 & 1 & 0 & -3 \\ 1 & 10 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & -3 \\ 1 & 10 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and that $\left(A - 2 I_4\right) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \neq 0_{4\times 1}$. It can be shown that $\left(A - 2 I_4\right)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0_{4\times 1}$ and so the

vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector of rank 2 for $\lambda = 2$. So, the matrix

$\begin{pmatrix} 0 & 1 & 0 & -3 \\ 1 & 10 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is now seen to have columns consisting of either eigenvectors or generalized eigenvectors.

Theorem

If λ is an eigenvalue of algebraic multiplicity m of the matrix $\boldsymbol{A}$, then there are m linearly independent generalized eigenvectors associated with λ.

Example

For the eigenvalue of $\lambda = 2$ for the matrix $\begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ we have three linearly independent generalized eigenvectors $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 10 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ (with ranks of 1, 1, and 2, respectively).

Example (*Focus on the flow, not the details!*)

Let $A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 7 & -16 & -48 & 46 \\ -6 & 16 & 43 & -38 \\ -3 & 9 & 23 & -19 \end{pmatrix}$. Then

$$Det[\boldsymbol{A} - \lambda I_4] = \lambda^4 - 12\lambda^3 + 54\lambda^2 - 108\lambda + 81 = (\lambda - 3)^4.$$

Direct, nontrivial computations show that for $\lambda = 3$ we have that

1. $A - \lambda I_4 = \begin{pmatrix} 1 & -4 & -11 & 11 \\ 7 & -19 & -48 & 46 \\ -6 & 16 & 40 & -38 \\ -3 & 9 & 23 & -22 \end{pmatrix}$ and the associated null space has a basis

consisting of $\left\{ \begin{pmatrix} -1 \\ -3 \\ 2 \\ 1 \end{pmatrix} \right\}$. Hence, all rank 1 generalized eigenvectors are in this null

space. We note here that $\lambda = 3$ has geometric multiplicity of 1.

2. $(A - \lambda I_4)^2 = \begin{pmatrix} 6 & -5 & -6 & 3 \\ 24 & -21 & -27 & 15 \\ -20 & 18 & 24 & -14 \\ -12 & 11 & 15 & -9 \end{pmatrix}$ and the associated null space has a basis

consisting of $\left\{ \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ 2 \\ 0 \end{pmatrix} \right\}$. Hence, all rank 2 generalized eigenvectors are in this

null space but since $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 1 \end{pmatrix}$ this subspace also contains the rank 1

generalized eigenvectors too.

3. $\left(A - \lambda\, i_4\right)^3 = \begin{pmatrix} -2 & 2 & 3 & -2 \\ -6 & 6 & 9 & -6 \\ 4 & -4 & -6 & 4 \\ 2 & -2 & -3 & 2 \end{pmatrix}$ and the associated null space has a basis

consisting of $\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. Hence, all rank 3 generalized eigenvectors are

in this null space (as well as the rank 2 and rank 1 generalized eigenvectors).

4. $\left(A - \lambda\, I_4\right)^4 = 0_{4\times4}$ and the associated null space has a standard basis consisting of

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Hence, all rank 4 generalized eigenvectors are in this

null space (as well as the rank 3, rank 2 and rank 1 generalized eigenvectors). One can show that

$$\left\{ \begin{pmatrix} -1 \\ -3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is a linearly independent set of generalized eigenvectors (one of rank 1, one of rank 2, one of rank 3, and one of rank 4).

To construct the matrix M so that $J = M^{-1} A M$ is in ***Jordan canonical form*** we are in general not interested in any set of linearly independent generalized eigenvectors but in a set of linearly independent generalized eigenvectors related in a particular manner.

Example - Continued

Let $A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 7 & -16 & -48 & 46 \\ -6 & 16 & 43 & -38 \\ -3 & 9 & 23 & -19 \end{pmatrix}$. Then the eigenvalue of $\lambda = 3$ has algebraic multiplicity 4 and geometric multiplicity 1. The vector $x_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a generalized eigenvector of rank 4. We now construct a ***chain*** using x_4 as a seed:

$$x_3 = \left(A - 3I_4\right) x_4 = \begin{pmatrix} 1 \\ 7 \\ -6 \\ -3 \end{pmatrix}$$

$$x_2 = \left(A - 3I_4\right) x_3 = \left(A - 3I_4\right)^2 x_4 = \begin{pmatrix} 6 \\ 24 \\ -20 \\ -12 \end{pmatrix}$$

$$x_1 = \left(A - 3I_4\right) x_2 = \left(A - 3I_4\right)^3 x_4 = \begin{pmatrix} -2 \\ -6 \\ 4 \\ 2 \end{pmatrix} \quad (= \text{eigenvector since } x_1 = 2 \begin{pmatrix} -1 \\ -3 \\ 2 \\ 1 \end{pmatrix}).$$

The set $\left\{ x_1, x_2, x_3, x_4 \right\}$ is a linearly independent set of generalized eigenvectors for $\lambda = 3$ (one of rank 1, one of rank 2, one of rank 3, and one of rank 4). Define a matrix $\boldsymbol{P}$ by

$$P = \left(x_1 \ x_2 \ x_3 \ x_4 \right) = \begin{pmatrix} -2 & 6 & 1 & 1 \\ -6 & 24 & 7 & 0 \\ 4 & -20 & -6 & 0 \\ 2 & -12 & -3 & 0 \end{pmatrix}.$$

Then

$$P^{-1} A P = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

We conclude that $A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 7 & -16 & -48 & 46 \\ -6 & 16 & 43 & -38 \\ -3 & 9 & 23 & -19 \end{pmatrix}$ is similar to matrix in Jordan canonical form.

Example (algebraic multiplicity 3, geometric multiplicity 1)

Let $A = \begin{pmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{pmatrix}$. Then $Det[A - \lambda I_3] = -\left(\lambda + 2\right)^3$ and the null space for

$A + 2I_3 = \begin{pmatrix} -3 & -18 & -7 \\ 1 & -15 & -4 \\ -1 & 25 & 6 \end{pmatrix}$ is given by $\left\langle \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix} \right\rangle$. Thus, $\lambda = -2$ has algebraic

multiplicity of 3 and geometric multiplicity of 1. We find that the null space for

$$(A + 2I_3)^2 = \begin{pmatrix} -10 & 5 & -5 \\ -6 & 3 & -3 \\ 14 & -7 & 7 \end{pmatrix}$$

is given by $\left\langle \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\rangle$ and the null space for

$$(A + 2I_3)^3 = 0_{3\times 3}$$

has the standard basis of $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$. So, set

$$x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$x_2 = (A + 2I_3)\, x_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and}$$

$$x_1 = (A + 2I_3)\, x_2 = \begin{pmatrix} -10 \\ -6 \\ 14 \end{pmatrix} \text{ (= eigenvector since } x_1 = 2\begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix}\text{).}$$

Define

$$P = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} -10 & 1 & 1 \\ -6 & 1 & 0 \\ 14 & -1 & 0 \end{pmatrix}.$$

We then see that

$$P^{-1} A P = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

Hence, ***A*** is similar to a matrix in Jordan canonical form.

Example

Let $A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{pmatrix}$. Then $Det[A - \lambda I_4] = (\lambda - 3)^4$ and the null space

of $A - 3 I_4$ is given by $\left\langle \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\rangle$. Thus, $\lambda = 3$ has algebraic multiplicity 4 and

geometric multiplicity 2. Here

$$N\left((A - 3I_4)^2 \right) = \left\langle \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

and

$$N\left(\left(A - 3I_4\right)^3 \right) = N\left(0_{4\times 4} \right) = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle.$$

Observe here that we cannot arbitrarily choose x_3 from among $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

as $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has rank 2 rather than rank three (why?). Set

$$x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$x_2 = \left(A - 3I_3\right) x_3 = \begin{pmatrix} 11 \\ 42 \\ -34 \\ -20 \end{pmatrix}, \text{ and}$$

$$x_1 = \left(A + 3I_4\right) x_2 = \begin{pmatrix} -3 \\ -9 \\ 6 \\ 3 \end{pmatrix} \quad (= \text{eigenvector as } x_1 = 3 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix} + 6 \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}).$$

Unfortunately, at this point we only have three linearly independent generalized eigenvectors for $\lambda = 3$. *So, we seek another chain of generalized eigenvectors of length one.* That is, we seek an eigenvector (why?) that is linearly independent from x_1, x_2, & x_3. Set $y_1 = \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$. Now,

since $Det[\ (x_1\ x_2\ x_3\ y_1)\ ] = -27 \neq 0$, the matrix ***P*** given by

$$P = \left(x_1\ x_2\ x_3\ y_1 \right) = \begin{pmatrix} -3 & 11 & 0 & -1 \\ -9 & 42 & 0 & -3 \\ 6 & -34 & 0 & 1 \\ 3 & -20 & 1 & 0 \end{pmatrix}$$

is invertible and

$$P^{-1}\ A\ P = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Again, we see that A is similar to a matrix in Jordan canonical form.

Example

Let $A = \begin{pmatrix} 4 & -1 & -2 & 2 \\ 7 & -4 & -12 & 10 \\ -6 & 6 & 13 & -8 \\ -3 & 3 & 5 & -1 \end{pmatrix}$. Then $Det[A - \lambda I_4] = (\lambda - 3)^4$ and the null space of

$A - 3I_4$ is given by $\left\langle \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$. Thus, $\lambda = 3$ has algebraic multiplicity 4 and

geometric multiplicity 2. Here

$$N\left((A - 3I_4)^2 \right) = N\left(0_{4\times 4} \right) = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle.$$

So, we set

$$x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$x_1 = (A - 3I_4)\, x_2 = \begin{pmatrix} 1 \\ 7 \\ -6 \\ -3 \end{pmatrix}.$$

As before $x_1 = (-3)\begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and x_1 is an eigenvector associated with $\lambda = 3$.

Our set of generalized eigenvectors has cardinality two and so we seek either a single chain of *length two or two chains of length one to bring the total number of generalized eigenvectors up to four.* We try

$$y_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We note that $y_2 \neq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ for in this case $x_1 + y_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ which is an eigenvector. Why is this problematic?

Set

$$y_1 = \left(A - 3I_4\right) y_2 = \begin{pmatrix} 2 \\ 10 \\ -8 \\ -4 \end{pmatrix}$$

and observe that

$$y_1 = (-4) \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix} + 10 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; (= \text{eigenvector}).$$

Here we set

$$P = \left(y_1 \; y_2 \; x_1 \; x_2 \right) = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 10 & 0 & 7 & 0 \\ -8 & 0 & -6 & 0 \\ -4 & 1 & -3 & 0 \end{pmatrix}$$

and see that

$$P^{-1} A P = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Yet again we find that A is similar to a matrix in Jordan canonical form.

<u>Example</u> - Characteristic Polynomial of $Det[A - \lambda I_4] = (\lambda - 3)^4$

1. The matrix $A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 7 & -16 & -48 & 46 \\ -6 & 16 & 43 & -38 \\ -3 & 9 & 23 & -19 \end{pmatrix}$ has characteristic polynomial of

$Det[A - \lambda I_4] = (\lambda - 3)^4$. The eigenvalue $\lambda = 3$ has algebraic multiplicity 4 and geometric multiplicity 1. The matrix A has Jordan canonical form of $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

2. The matrix $A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{pmatrix}$ has characteristic polynomial of

$Det[A - \lambda I_4] = (\lambda - 3)^4$. The eigenvalue $\lambda = 3$ has algebraic multiplicity 4 and geometric multiplicity 2. The matrix $\boldsymbol{A}$ has Jordan canonical form of $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

3. The matrix $A = \begin{pmatrix} 4 & -1 & -2 & 2 \\ 7 & -4 & -12 & 10 \\ -6 & 6 & 13 & -8 \\ -3 & 3 & 5 & -1 \end{pmatrix}$ has characteristic polynomial of

$Det[A - \lambda I_4] = (\lambda - 3)^4$. The eigenvalue $\lambda = 3$ has algebraic multiplicity 4 and

geometric multiplicity 2. The matrix $\boldsymbol{A}$ has Jordan canonical form of $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

<u>Example</u> - Multiple eigenvalues. (Details missing!!)

1. Let $A = \begin{pmatrix} 2 & 5 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$. Since A is upper triangular, we see that the

eigenvalues are -1 & 2. For $\lambda = 2$ we find that the null space for $A - 2 I_5$ is given by $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$, the null space for $(A - 2I_5)^2$ is given by $\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ and the null space for $(A - 2I_5)^3 \neq 0_5$ is also given by $\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$. This sequence has stalled and nothing new will be added. (Why? Btw, what is the algebraic multiplicity of $\lambda = 2$?) We take $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (do we really have a choice here?) and then

$$x_2 = (A - 2I_5)\,x_2 = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = -1$ we find that the null space for $A + I_5$ is given by

$$\left\langle \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$. Since the algebra multiplicity equals the geometric

multiplicity, we are done with respect to $\lambda = -1$. (Why?)

Set

$$P = \begin{pmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 5 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & \\ 0 & 2 & 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 & \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$. As A is upper triangular we see that

$\lambda = 2$ is an eigenvalue with algebraic multiplicity 6 and $\lambda = 3$ is an eigenvalue with algebraic multiplicity 2.

Case $\lambda = 2$:

$$x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (rank 4)}, \quad x_3 = \left(A - 2 I_8\right) x_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (rank 3)},$$

$$x_2 = \left(A - 2 I_8\right) x_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (rank 2), } x_1 = \left(A - 2 I_8\right) x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (rank 1)}$$

$$\text{Set } y_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ (rank 2), } y_1 = \left(A - 2 I_8\right) y_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (rank 1)}$$

Case $\lambda = 3$:

$$\text{Set } z_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ (rank 2), } z_1 = \left(A - 2 I_8\right) z_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ (rank 1)}$$

Now, if $P = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & z_1 & z_2 \end{pmatrix}$, then

$$P^{-1} A P = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$