MA 137
Calculus I with Life Science Applications
SECOND MIDTERM

Fall 2020
10/13/2020

Name: Answer Key

Sect. #: ____________

Length of exam: Unless you have a DRC accommodation letter, you will have until 7:30 PM on October 13, 2020 to upload a PDF with your answers for the exam in the same group assignment on Canvas where you downloaded the exam. The exam is written so that it should take you at most 2 hours for the exam, allowing 30 minutes to scan and upload the exam as a single PDF on Canvas. Budget your time appropriately as NO extensions will be given.

Students with a DRC accommodation letter will take the exam with the students in sections 001 and 002. Thus they should use the Zoom number for sections 001 and 002. Their exam will end at 8:30 PM if they are allowed 50% extra time or at 9:30 PM if they are allowed 100% extra time.

Submitting your exam: You can annotate the PDF on an e-device, for example on a university issued iPad. Alternatively, you could print the test and write all your solutions on the printed exam. If it is too time consuming and/or impossible to print the test, just write on blank sheets of paper your work for the multiple choice questions and for the open response questions.

Please make sure to write your name and list the correct section number on the front page of your exam. In case you have forgotten your section number, consult the table below.

Please make sure to write your answers for the multiple choice questions either on the second page of the exam or on a single sheet of paper. You should include any supporting work that you deem appropriate for the multiple choice questions. The answers must be in the same order as the multiple choice questions (namely, question 1/answer, question 2/answer, etc,). Similarly, please write your answers to the open response questions on either the exam pages or on separate sheets of paper, making sure your answer pages are scanned in sequential order (answer to problem 13, then answer to problem 14, etc.). *You will be penalized 10 points if you provide the answers in a scrambled order.*

Questions during exam: You will be proctored for the entire exam time by your TA at the following Zoom link from 5 pm to 7:30 pm. You are required to have your camera on during the entire exam. If you need any clarification during the exam please ask a private question in the Zoom chat.

Section	Time/Recitation Location	TA	Zoom number
001	TR 08:00-08:50 AM, CB 240	J. Garagnani	https://uky.zoom.us/j/88380813488 passcode: MA137
002	TR 09:00-9:50 AM, CB 240		
003	TR 10:00-10:50 AM, CB 242	J. Britt	https://uky.zoom.us/j/88627959903 passcode: 137
004	TR 11:00-11:50 AM, CB 242		
005	TR 12:00-12:50 PM, CB 246	W. Rizer	https://uky.zoom.us/j/83349322897 passcode: MA137
006	TR 01:00-01:50 PM, CB 246		
007	TR 12:00-12:50 PM, CB 244	R. Righi	https://uky.zoom.us/j/81126842645 passcode: MA137
008	TR 01:00-01:50 PM, CB 244		
009	TR 02:00-02:50 PM, CB 246	M. McCarver	https://uky.zoom.us/j/85264041638 passcode: MA137
010	TR 03:00-03:50 PM, CB 246		

Restrictions on books, notes, calculators and cell phones: You will return the whole exam with your answers or the sheets that you want us to grade. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. No books or notes may be used. Absolutely no cell phone use during the exam is allowed, except for scanning your exam pages. Make sure to work in a quiet environment.

The **first part of the exam** consists of 12 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

a b c d e

It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of four open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

Cheating (Senate Rule 6.3.2): Cheating is a serious offense and will not be tolerated. It will be thoroughly investigated, and might lead to failure in the course or even to expulsion from the university. Cheating is defined by its general usage. It includes, but is not limited to, wrongfully giving, taking, or presenting any information or material by a student with the intent of aiding themselves or another on any academic work which is considered in any way in the determination of the final grade. The fact that a student could not have benefited from an action is not by itself proof that the action does not constitute cheating. Any question of definition shall be referred to the University Appeals Board.

1. a b c d ■

2. ■ b c d e

3. a b c d ■

4. ■ b c d e

5. a b c d ■

6. a b c ■ e

7. a b ■ d e

8. a ■ c d e

9. a b c d ■

10. a b ■ d e

11. a b c ■ e

12. a ■ c d e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		60 pts
13.		10 pts
14.		10 pts
15.		10 pts
16.		10 pts
Bonus.		10 pts
TOTAL		100 pts

1. Suppose $\quad -8x - 22 \le f(x) \le x^2 - 2x - 13 \quad$ for all x.

Use this information and the Sandwich Theorem to compute $\lim_{x \to -3} f(x)$:

notice that $\lim_{x \to -3} (-8x - 22) \underset{\uparrow}{=} -8(-3) - 22 = \underline{2}$

Substitution THM

and $\lim_{x \to -3} (x^2 - 2x - 13) = (-3)^2 - 2(-3) - 13$

$= 9 + 6 - 13 = \underline{2}$

Possibilities:

(a) $\lim_{x \to -3} f(x) = -2$

(b) $\lim_{x \to -3} f(x) = -1$

(c) $\lim_{x \to -3} f(x) = 0$

(d) $\lim_{x \to -3} f(x) = 1$

(e) $\boxed{\lim_{x \to -3} f(x) = 2}$

By the Sandwich Theorem

$$\boxed{\lim_{x \to -3} f(x) = 2}$$

---

2. Find the value of the limit $\lim_{x \to 0} \dfrac{\sin(3x)}{x}$

$$= \lim_{x \to 0} 3 \cdot \left[\frac{\sin(3x)}{3x} \right] =$$

$$= 3 \lim_{x \to 0} \left[\underbrace{\frac{\sin(3x)}{3x}}_{\to 1} \right]$$

(also $3x \to 0$)

$$= 3 \cdot 1 = \boxed{3}$$

Possibilities:

(a) $\boxed{3}$

(b) $1/3$

(c) $\sin 3$

(d) 0

(e) The limit does not exist.

3. In Einstein's theory of relativity, the mass m of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens to the mass m of the particle as the speed v tends to the speed of light c? That is, evaluate:

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{\sqrt{1 - \lim_{v \to c^-} \frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1 - 1^-}} = \frac{m_0}{0^+} = +\infty$$

(or DNE)

Possibilities:

(a) The limiting value of the mass is 0

(b) The limiting value of the mass is $m_0/2$

(c) The limiting value of the mass is m_0

(d) The limiting value of the mass is $2m_0$

(e) The limiting value of the mass is ∞

---

4. Assume that $f(x)$ is everywhere continuous and it is given to you that

$$\boxed{\lim_{x \to 7} \frac{f(x) + 9}{x - 7} = 10.}$$

It follows that

From the equation we have $f(7) = -9$; $f'(7) = 10$

So the equation of the tg. line at $P(7, -9)$ is

$\boxed{y + 9 = 10(x - 7)}$ or $\boxed{y = 10x - 79}$

Possibilities:

(a) $y = 10x - 79$ is the equation of the tangent line to $y = f(x)$ at $P(7, -9)$

(b) $y = 10x - 61$ is the equation of the tangent line to $y = f(x)$ at $P(7, 9)$

(c) $y = 10x - 61$ is the equation of the tangent line to $y = f(x)$ at $P(7, -9)$

(d) $y = 10x - 79$ is the equation of the tangent line to $y = f(x)$ at $P(7, 9)$

(e) $f(7) = 9$

5. Suppose $F(x) = \dfrac{1-x^2}{h(x)}$ and $h(2) = -1$ and $h'(2) = -2$. Find $F'(2)$.

$$F'(x) = \frac{-2x \cdot h(x) - (1-x^2)\,h'(x)}{[h(x)]^2}$$

Possibilities:

(a) 2

(b) 1

(c) 0

(d) -1

(e) -2

$$F'(2) = \frac{-4\,h(2) - (1-4)\,h'(2)}{[h(2)]^2}$$

$$F'(2) = \frac{-4(-1) + 3(-2)}{(-1)^2} = 4 - 6 = \boxed{-2}$$

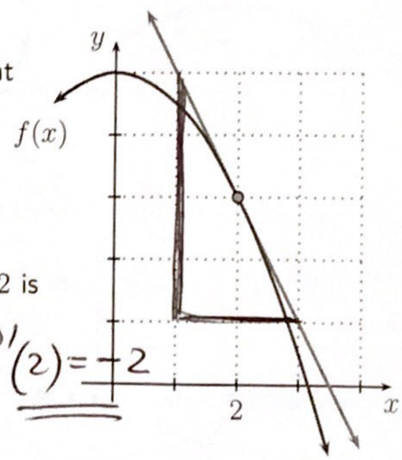
6.

A segment of the tangent line to the graph of $f(x)$ at the point $P(2,3)$ is shown in the picture.

Consider now the new function

$$g(x) = 5x + f(x).$$

The equation of the tangent line to the graph of $g(x)$ at $x = 2$ is given by the equation:



From the graph $f(2)=3$ and $f'(2)=-2$

Now $g(2) = 5\cdot 2 + f(2) = 10 + 3 = 13$

$g'(x) = 5 + f'(x)$ so $g'(2) = 5 - 2 = 3$

Possibilities:

(a) $y = -2x + 7$

(b) $y = 7x - 3$

(c) $y = 3x - 19$

(d) $y = 3x + 7$

(e) $y = 7x + 3$

Thus the tg line to $g(x)$ at $(2, 13)$ is

$y - 13 = 3(x-2)$ or

$$\boxed{y = 3x + 7}$$

7. Suppose that $f(x) = (x^2 - 5)^{-3/2}$. Find $f'(3)$.

power chain rule

$$f'(x) = -\tfrac{3}{2}(x^2-5)^{-3/2-1} \cdot 2x = -\frac{3}{\cancel{2}}(x^2-5)^{-5/2} \cdot \cancel{2}x$$

$$= \frac{-3 \cdot \cancel{2}x}{\cancel{2}\left(\sqrt{x^2-5}\right)^5}$$

Possibilities:

(a) 9/2

(b) 3/64

(c) $\boxed{-9/32}$

(d) $-3/64$

(e) $-9/2$

$$f'(3) = \frac{-3x}{\left(\sqrt{9-5}\right)^5}$$

$$= \frac{-3(3)}{2^5} = \frac{-9}{32}$$

---

8. Suppose that $f(x) = \sin^2(x^3 + 1)$. Find $f'(x)$.

$$f(x) = \left[\sin(x^3+1)\right]^2$$

use the chain rule (there are 3 nested functions):

$$f'(x) = 2\left[\sin(x^3+1)\right]^1 \cdot \cos(x^3+1) \cdot (3x^2)$$

Possibilities:

(a) $x^3 \sin(x^3 + 1)$

(b) $\boxed{6x^2 \sin(x^3 + 1) \cos(x^3 + 1)}$

(c) $2 \sin(x^3 + 1) \cos(x^3 + 1)$

(d) $3x^5 \sin(x^3 + 1)$

(e) $3x^2 \sin(x^3 + 1) \cos(3x^2)$

$$\boxed{f'(x) = 6x^2 \sin(x^3+1) \cdot \cos(x^3+1)}$$

9. Suppose a function $y = y(x)$ is implicitly defined by the equation

$$y^3x^4 - 10x + y = -3.$$

What is dy/dx at the point $(2, 1)$? Use implicit differentiation

$$\frac{d}{dx}\left(y^3x^4 - 10x + y\right) = \frac{d}{dx}(-3)$$

Possibilities:

(a) $-\frac{11}{24}$

(b) $\frac{15}{41}$

(c) $-\frac{16}{25}$

(d) $-\frac{21}{48}$

(e) $\boxed{-\frac{22}{49}}$

$$3y^2\frac{dy}{dx}\cdot x^4 + y^3(4x^3) - 10 + \frac{dy}{dx} = 0$$

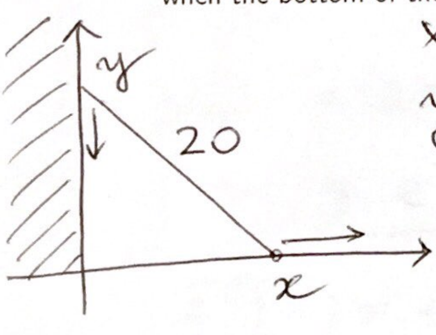
$$\frac{dy}{dx}\left(1 + 3y^2x^4\right) = 10 - 4x^3y^3$$

$$\boxed{\frac{dy}{dx} = \frac{10 - 4x^3y^3}{1 + 3y^2x^4}}$$

at $(2, 1)$

$$\boxed{\frac{dy}{dx} = -\frac{22}{49}}$$

10. A ladder 20 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 16 feet from the wall?



x = foot of the ladder from wall

y = top of the ladder from floor

$$\boxed{x^2 + y^2 = 20^2}$$

Differentiate both sides wrt t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Possibilities:

(a) -4/3 feet per second

(b) -6/5 feet per second

(c) $\boxed{\text{-8/3 feet per second}}$

(d) -10/3 feet per second

(e) -8/5 feet per second

Thus $\boxed{\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}}$

Now $\frac{dx}{dt} = 2$ ft/sec $\quad x = 16$

At that time $y = \sqrt{20^2 - 16^2} = 12$ so $\frac{dy}{dt} = -\frac{16}{12}\cdot 2$

$= -8/3$

11. Find the equation of the tangent line to the graph of $f(x) = x^2 e^x$ at $x = 1$.

when $x=1$ then $y = 1^2 \cdot e^1 = e$ so

$P(1, e)$. About the slope

$f'(x) = 2x \cdot e^x + x^2 \cdot e^x$ so $f'(1) = 2e + e$

$= \underline{\underline{3e}}$

Possibilities:

(a) $y = 3e \cdot x + 2e$

(b) $y = 2e \cdot x + 3e$

(c) $y = 2e \cdot x - 3e$

(d) $\boxed{y = 3e \cdot x - 2e}$

(e) $y = 2e \cdot x - e$

equation of tg. line

$y - e = 3e(x-1)$

$\boxed{y = 3ex - 2e}$

12. Suppose $f(x) = x^2 e^{-2x}$. Find $f''(1)$.

$f'(x) = 2x e^{-2x} + x^2 \left[e^{-2x}(-2) \right]$

$= 2x e^{-2x} - 2x^2 e^{-2x}$

$= \underline{e^{-2x}(2x - 2x^2)}$

$f''(x) = e^{-2x}(-2)(2x - 2x^2) + e^{-2x}(2 - 4x)$

$= e^{-2x}(-4x + 4x^2 + 2 - 4x]$

$f''(x) = e^{-2x}(2 - 8x + 4x^2)$

$f''(1) = e^{-2}(2 - 8 + 4) = \boxed{-2e^{-2}}$

Possibilities:

(a) $-e^{-2}$

(b) $\boxed{-2e^{-2}}$

(c) $-3e^{-2}$

(d) $-4e^{-2}$

(e) $-5e^{-2}$

13. (*a*) Let $Y(N)$ be the yield of an agricultural crop as a function of nitrogen level N in the soil. A model that is used for this relationship is

$$Y(N) = \frac{N}{1+N^2}$$

for $N \geq 0$. Find $\lim_{N\to\infty} Y(N)$. Why is your answer plausible?

$$\lim_{N\to\infty} \frac{N}{1+N^2} = \lim_{N\to\infty} \frac{N\left(\frac{1}{N^2}\right)}{(1+N^2)\frac{1}{N^2}} =$$

$$= \lim_{N\to\infty} \frac{1/N}{1/N^2+1} = \frac{\lim_{N\to\infty}\frac{1}{N}}{\left(\lim_{N\to\infty}\frac{1}{N^2}\right)+1} = \frac{0}{1} = \boxed{0}$$

This is plausible as if the nitrogen level in the soil is way too large we are likely to destroy all our crop. So the yield is close to nothing

(*b*) Explain how to use the Intermediate Value Theorem to conclude that the equation

$$\sqrt{x^2+2} = 2$$

has a solution inside the interval $[1, 2]$.

$\sqrt{x^2+2} = 2 \iff \sqrt{x^2+2} - 2 = 0$. Consider the function $\boxed{f(x) = \sqrt{x^2+2} - 2}$. It is a continuous function for every choice of x and in particular for $x \in [1,2]$. Notice that $f(1) = \sqrt{3} - 2 \cong \underline{-0.2679}$ and $f(2) = \sqrt{6} - 2 \cong 0.4494$. By the I.V.T. there exists $c \in (1,2)$ s/t $f(c) = 0$ i.e.

$$\sqrt{c^2+2} - 2 = 0 \quad \text{or} \quad \sqrt{c^2+2} = 2.$$

P.S. notice that $c = \sqrt{2}$!!

pts: /10

14. (*a*) The following limit represents the derivative $f'(x_0)$ of a function f at a point x_0:

$$\lim_{h \to 0} \frac{4(2+h)^3 - 32}{h}$$

Find f and x_0. What is the value of the limit?

$$f'(2) = \lim_{h \to 0} \frac{4(2+h)^3 - 32}{h} \quad \text{where}$$

$$\underline{\underline{f(x) = 4x^3}} \quad \text{and} \quad \underline{\underline{x_0 = 2}}$$

<u>Now</u> $f'(x) = 12x^2$ so

$$\boxed{f'(2) = 12(2)^2 = 48}$$

(*b*) Let $g(x) = 5x^3 - 2x + \frac{2}{x} + \pi^2$. Compute $g'(x)$.

$2x^{-1}$ constant

$$g'(x) = 5 \cdot (3x^2) - 2(1) + 2(-1x^{-2}) + 0$$

$$= \boxed{15x^2 - 2 - 2/x^2}$$

pts: /10

15. Use the chain rule to find the derivative of the following functions:

(a) $f(N) = (1 + 3N^2)^3$

$$\frac{df}{dN} = 3(1+3N^2)^2 \cdot (3 \cdot 2N)$$

$$= \boxed{18N(1+3N^2)^2}$$

(b) $g(s) = \sqrt[4]{5s-3}$ $= (5s-3)^{1/4}$

$$g'(s) = \frac{dg}{ds} = \frac{1}{4}(5s-3)^{1/4-1} \cdot (5)$$

$$= \frac{5}{4}(5s-3)^{-3/4}$$

$$= \boxed{\frac{5}{4\sqrt[4]{(5s-3)^3}}}$$

pts:	/10

16. Find the derivative with respect to x of the following functions:

(a) $f(x) = \cos(x^2) + \cos^2 x = \cos(x^2) + [\cos x]^2$

$$f'(x) = -\sin(x^2) \cdot 2x + 2(\cos x)(-\sin x)$$

$$\boxed{f'(x) = -2x\sin(x^2) - 2\cos x \sin x}$$

(b) $g(x) = e^{\frac{1}{x}} + \sin(\sqrt{x})$

$$g'(x) = e^{\frac{1}{x}} \cdot (x^{-1})' + \cos(\sqrt{x}) \cdot (\sqrt{x})'$$

chain rule

$$= e^{\frac{1}{x}} \cdot (-x^{-2}) + \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \boxed{-\frac{e^{\frac{1}{x}}}{x^2} + \frac{\cos(\sqrt{x})}{2\sqrt{x}}}$$

pts:	/10

Bonus. According to the **Michaelis-Menton equation** when a chemical reaction involving a substrate S is catalyzed by an enzyme, the rate of reaction R is given by the expression

$$R = \frac{as}{b+s},$$

where s denotes substrate concentration (for examples in moles per liter), and a and b are constants.

For this problem we will assume that $a = b = 1$, from which we have that:

$$R = \frac{s}{1+s}.$$

If substrate is added at a rate $\frac{ds}{dt} = 0.1$ mol/L per second find the rate at which R is changing when $s = 9$ mol/L.

$$R = \frac{s}{1+s}$$

$$\frac{dR}{dt} = \frac{d}{dt}\left(\frac{s}{1+s}\right)$$ ← chain rule

$$= \frac{d}{ds}\left(\frac{s}{1+s}\right) \cdot \frac{ds}{dt}$$

$$= \frac{1 \cdot (1+s) - s(1)}{(1+s)^2} \cdot \frac{ds}{dt}$$

$$= \boxed{\frac{1}{(1+s)^2} \cdot \frac{ds}{dt}}$$

with our data

$$dR/dt = \frac{1}{(1+9)^2} \cdot 0.1 = \frac{0.1}{100} = \boxed{0.001}$$ mol/L per sec

pts: /10