MA 137
Calculus I with Life Science Applications
SECOND MIDTERM

Fall 2021
10/19/2021

Name: Answer Key

Sect. #: ____________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 12 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ b c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of four open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. a b c ■ e

2. a b c ■ e

3. a ■ c d e

4. a ■ c d e

5. a b ■ d e

6. a b ■ d e

7. a ■ c d e

8. ■ b c d e

9. a b c d ■

10. a ■ c d e

11. a b c ■ e

12. ■ b c d e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		60 pts
13.		10 pts
14.		10 pts
15.		10 pts
16.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Time/Lecture Location	Lecturer
001-010	MWF 10:00 am - 10:50 am, CB 106	Alberto Corso

Section #	Time/ Recitation Location	TA
001	TR 08:00-08:50 AM, CB 339	Nicholas Arsenault
002	TR 09:00-9:50 AM, CB 339	Nicholas Arsenault
003	TR 10:00-10:50 AM, CB 339	Katherine (Kat) Henneberger
004	TR 11:00-11:50 AM, CB 339	Katherine (Kat) Henneberger
005	TR 12:00-12:50 PM, CB 339	Faith Hensley
006	TR 01:00-01:50 PM, CB 339	Faith Hensley
007	TR 12:00-12:50 PM, CB 341	Michael Morrow
008	TR 01:00-01:50 PM, CB 341	Michael Morrow
009	TR 02:00-02:50 PM, CB 339	Karen Reed
010	TR 03:00-03:50 PM, CB 339	Karen Reed

1. Suppose $\quad 3x+1 \le f(x) \le x^2+7x+5 \quad$ for all x.

Use this information and the Sandwich Theorem to compute $\lim_{x\to -2} f(x)$

$$\lim_{x\to -2}(3x+1) = 3\left(\lim_{x\to -2} x\right)+1 = 3(-2)+1 = -5$$

$$\lim_{x\to -2}(x^2+7x+5) = \left(\lim_{x\to -2} x\right)^2 + 7\left(\lim_{x\to -2} x\right)+5$$

$$= (-2)^2 + 7(-2)+5 = 9-14=-5$$

Hence $\boxed{\lim_{x\to -2} f(x) = -5}$

Possibilities:

(a) $\lim_{x\to -2} f(x) = 7$

(b) $\lim_{x\to -2} f(x) = 5$

(c) $\lim_{x\to -2} f(x) = 23$

(d) $\lim_{x\to -2} f(x) = -5$

(e) $\lim_{x\to -2} f(x) = -7$

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2. Suppose $\lim_{x\to 0} f(x) = 4$ and $\lim_{x\to 0} g(x) = 7$. Find

$$\lim_{x\to 0} \frac{f(x)\sqrt{2+g(x)}}{[g(x)]^2 - f(x)}$$

Using the limit properties !

$$= \frac{\lim_{x\to 0} f(x)\cdot\sqrt{2+\lim_{x\to 0} g(x)}}{\left[\lim_{x\to 0} g(x)\right]^2 - \lim_{x\to 0} f(x)} = \frac{4\cdot\sqrt{2+7}}{7^2-4}$$

$$= \frac{4\cdot \cancel{3}}{\cancel{45}_{15}} = \boxed{\frac{4}{15}}$$

Possibilities:

(a) $4/3$

(b) 4

(c) $\sqrt{2}$

(d) $4/15$

(e) None of the above

3. Find the values of c that make the following piecewise-defined function continuous everywhere

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ (c^2-c)x-8 & \text{if } x \geq 2. \end{cases}$$

We need to guarantee the continuity at $x = 2$.

For $x \neq 2$ $f(x)$ is already continuous

Thus $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ OR

$$\lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^+} (c^2-c)x-8$$

$$\Leftrightarrow \lim_{x \to 2^-} (x+2) = \lim_{x \to 2^+} (c^2-c)x-8$$

$$4 = (c^2-c) \cdot 2 - 8$$

$$\therefore \boxed{c^2 - c - 6 = 0}$$

$$(c-3)(c+2) = 0$$

$$\therefore c = 3 \text{ or } c = -2$$

Possibilities:

(a) 2 and 4

(b) -2 and 3

(c) 0 and 8

(d) -4 and -8

(e) None of the above

4. Find the value of the limit $\lim_{x \to 0} \dfrac{x}{\tan(2x)}$.

$$\lim_{x \to 0} \frac{x}{\tan(2x)} = \frac{0}{0} \text{ by direct substitution}$$

$$= \lim_{x \to 0} \frac{x}{\dfrac{\sin(2x)}{\cos(2x)}} = \lim_{x \to 0} \frac{\cos(2x)}{\dfrac{\sin(2x)}{x}} \cdot \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\cos(2x)}{\dfrac{\sin(2x)}{2x}} \quad \cos(2x) \to 1, \quad \frac{\sin(2x)}{2x} \to 1$$

$$= \boxed{\frac{1}{2}}$$

Possibilities:

(a) 2

(b) $\dfrac{1}{2}$

(c) 0

(d) $\dfrac{1}{\tan(2)}$

(e) The limit does not exist

5. For which of the examples below can the Intermediate Value Theorem be used to conclude that the equation has a solution lying in the given interval?

I. $x^3 - x - 3 = 0, \quad [0, 2]$

II. $\cos(x) - x = 0, \quad [0, 1]$

III. $2^x - 3x - 2 = 0, \quad [0, 3]$

$\boxed{\text{I.}}$ Yes $g(x) = x^3 - x - 3$ is continuous on $[0,2]$. $g(0) = -3$ and $g(2) = 3$. Hence by IVT there exists $c \in (0,2)$ such that $g(c) = 0$

$\boxed{\text{II.}}$ Yes $f(x) = \cos(x) - x$ is continuous on $[0,1]$ $f(0) = 1$ and $f(1) \cong -0.46$ Hence by IVT there exist $c \in (0,1)$ such that $f(c) = 0$

$\boxed{\text{III.}}$ No $h(x) = 2^x - 3x - 2$ is continuous on $[0,3]$. $h(0) = -1$ and $h(3) = -3$ hence the IVT does not tell us anything

Possibilities:

(a) I. only

(b) II. only

$\boxed{\text{(c) I. and II. only}}$

(d) I. and III. only

(e) II. and III. only

6. The average rate of change of a particle over the time interval $[5, 5+h]$ is given by

$$\frac{\frac{1}{5+h} - \frac{1}{5}}{h}.$$

The instantaneous rate of change of the particle at $t = 5$ equals

Observe that the instantaneous rate of change is the derivative of $\frac{1}{t}$ at $t = 5$.

$f'(t) = -\frac{1}{t^2}$ at $t = 5$

$f'(5) = -1/25$ OR by direct Computation

$$\frac{\frac{1}{5+h} - \frac{1}{5}}{h} = \frac{5-(5+h)}{h \cdot 5(5+h)} = \frac{-h}{h \cdot 5 \cdot (5+h)} = \frac{-1}{5(5+h)} \xrightarrow{h \to 0} -1/25$$

Possibilities:

(a) 0

(b) -1

$\boxed{\text{(c) } -\frac{1}{25}}$

(d) $-\frac{1}{5}$

(e) None of the above

7. Assume that $f(x)$ is everywhere continuous and it is given to you that

$$\lim_{x \to 6} \frac{f(x)+10}{x-6} = 6.$$

It follows that

$f(6) = -10 \qquad f'(6) = 6$

so the equation of the tg line is at $\underline{P(6,-10)}$

$y-(-10) = 6(x-6)$

Possibilities: OR $\boxed{y = 6x - 46}$

(a) $y = 6x - 26$ is the equation of the tangent line to $y = f(x)$ at $P(6, 10)$

$\boxed{\text{(b)} \quad y = 6x - 46 \text{ is the equation of the tangent line to } y = f(x) \text{ at } P(6, -10)}$

(c) $y = 10x - 54$ is the equation of the tangent line to $y = f(x)$ at $P(6, -10)$

(d) $f(6) = 10$

(e) $y = 6x - 46$ is the equation of the tangent line to $y = f(x)$ at $P(6, 10)$

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8. Suppose $F(x) = \dfrac{4 - x^2}{h(x)}$ and $h(-1) = -3$ and $h'(-1) = 4$. Find $F'(-1)$.

$$F'(x) = \frac{-2x \cdot h(x) - (4-x^2) \cdot h'(x)}{[h(x)]^2}$$

$$F'(-1) = \frac{(-2)(-1)\, h(-1) - (4-(-1)^2) \cdot h'(-1)}{[h(-1)]^2}$$

Possibilities:

$\boxed{\text{(a)} \quad -2}$

(b) $-\frac{9}{8}$

(c) 0

(d) $\frac{9}{8}$

(e) 2

$$= \frac{2(-3) - (3) \cdot 4}{(-3)^2} = \frac{-6-12}{9}$$

$$= \frac{-18}{9} = -2$$

9. Let $g(t) = \frac{1}{9}(2t+1)^3$. Find an equation of the tangent line to the graph of g at $t = 1$.

$g(1) = \frac{1}{9}(2\cdot 1+1)^3 = \frac{27}{9} = 3$

$g'(t) = \frac{1}{9}\cdot 3(2t+1)^2 \cdot 2 = \frac{2}{3}(2t+1)^2$

$g'(1) = \frac{2}{3}(2\cdot 1+1)^2 = 6$

$\therefore$ equation of tg line

$y - 3 = 6(t-1)$

OR $\boxed{y = 6t - 3}$

Possibilities:

(a) $y = 3$

(b) $y = 3t$

(c) $y = 6$

(d) $y = 6t - 6$

(e) $y = 6t - 3$

10. Let $F(x) = f(f(x))$ and $G(x) = (F(x))^2$ and suppose that

$$f(5) = 3 \qquad f(7) = 5 \qquad f'(5) = 8 \qquad f'(7) = 13.$$

Find $F'(7)$ and $G'(7)$.

$\boxed{F(7) = f(\underbrace{f(7)}_{5}) = f(5) = 3}$ $\quad F'(x) = f'(f(x))\cdot f'(x)$

So $\boxed{F'(7) = f'(f(7))\cdot f'(7) = f'(5)\cdot f'(7) = 8\cdot 13 = 104}$

$G'(x) = 2(F(x))\cdot F'(x)$ $\quad$ so $\boxed{G'(7) = 2F(7)\cdot F'(7)}$

$= 2(3)\cdot 104$

$= \boxed{624}$

Possibilities:

(a) $F'(7) = 40$ and $G'(7) = 1040$

(b) $F'(7) = 104$ and $G'(7) = 624$

(c) $F'(7) = 104$ and $G'(7) = 1040$

(d) $F'(7) = 3$ and $G'(7) = 169$

(e) $F'(7) = 40$ and $G'(7) = 624$

11. Suppose a function $y = y(x)$ is implicitly defined by the equation

$$x^2 + xy + y^2 = 1.$$

Find a formula for dy/dx in terms of x and y.

$$\frac{d}{dx}\left[x^2 + xy + y^2\right] = \frac{d}{dx}[1]$$

$$2x + 1\cdot y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(x+2y)\frac{dy}{dx} = -2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{-2x-y}{x+2y}}$$

Possibilities:

(a) $\frac{dy}{dx} = \frac{-(2x+y)}{x+y}$

(b) $\frac{dy}{dx} = \frac{-2x}{x+2y}$

(c) $\frac{dy}{dx} = \frac{2x+y}{x+2y}$

(d) $\boxed{\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}}$

(e) None of the above

12. A spherical water balloon is connected to a high-pressure water hose which pumps 2 cubic feet of water per second. How fast is the radius of the water balloon increasing when its radius is 1 foot? **[Recall:** the volume of a sphere in terms of its radius is $V(r) = \frac{4}{3}\pi r^3$.]

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dt}(r^3)$$

so $$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi\,\cancel{3}r^2\cdot\frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}}$$

with $\frac{dV}{dt} = 2\ \text{ft}^3/\text{sec}$ $\qquad r = 1$

$$\frac{dr}{dt} = \frac{1}{2\pi}\ \text{ft/sec}$$

Possibilities:

(a) $\boxed{1/(2\pi)\text{ feet per second}}$

(b) $1/\pi$ feet per second

(c) $1/(2\pi)$ feet per minute

(d) $1/\pi$ feet per minute

(e) $1/(4\pi)$ feet per minute

13. (*a*) (5 pts) The growth of fish can be described by the von Bertalanffy growth function

$$L(t) = L_\infty - (L_\infty - L_0)e^{-rt}$$

where t denotes the age of the fish and r, L_0 and L_∞ are positive constants.

Find:

$$\lim_{t\to 0^+} L(t) \qquad \lim_{t\to\infty} L(t).$$

$$\lim_{t\to 0^+}\left(L_\infty - (L_\infty - L_0)e^{-rt}\right) = L_\infty - (L_\infty - L_0)\cdot \underbrace{\lim_{t\to 0^+} e^{-rt}}_{1}$$

$$= \cancel{L_\infty} - \cancel{L_\infty} + L_0 = L_0$$

$$\lim_{t\to\infty}\left(L_\infty - (L_\infty - L_0)e^{-rt}\right) = L_\infty - (L_\infty - L_0)\cdot \underbrace{\lim_{t\to\infty} e^{-rt}}_{0}$$

$$= L_\infty$$

(*b*) (5 pts) Evaluate the following limit $\displaystyle\lim_{x\to\infty}\frac{\sqrt{4x^2+3x}}{1-8x}$

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+3x}}{1-8x} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{\sqrt{4x^2+3x}\cdot\frac{1}{x}}{(1-8x)\cdot\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{\frac{4x^2+3x}{x^2}}}{\frac{1}{x}-8} = \lim_{x\to\infty}\frac{\sqrt{4+\frac{3}{x}}}{\frac{1}{x}-8}$$

$$= \frac{\sqrt{4+\lim_{x\to\infty}\frac{3}{x}}}{\left(\lim_{x\to\infty}\frac{1}{x}\right)-8} = \frac{\sqrt{4}}{-8} = \boxed{-\frac{1}{4}}$$

pts: /10

14. (*a*) (5 pts) Differentiate

$$g(N) = \frac{bN^2 + N}{K + b}$$

with respect to N. Assume that b and K are positive constants.

$$g(N) = \frac{b}{K+b} N^2 + \frac{1}{K+b} \cdot N$$

$$g'(N) = \frac{dg}{dN} = \frac{b}{K+b} \cdot 2N + \frac{1}{K+b} \cdot 1 =$$

$$= \boxed{\frac{2bN+1}{K+b}}$$

(*b*) (5 pts) Find the derivative of the function

$$f(x) = \frac{3}{2-x}$$

using the <u>definition</u> of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{2-(x+h)} - \frac{3}{2-x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3(2-x) - 3(2-(x+h))}{(2-x-h)(2-x)}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{6} \cancel{-3x} \cancel{-6} \cancel{+3x} + 3h}{h(2-x-h)(2-x)} = \lim_{h\to 0} \frac{3}{(2-x-h)(2-x)}$$

$$= \boxed{\frac{3}{(2-x)^2}}$$

pts: /10

15. (*a*) (5 pts) Find the derivative of $g(x) = x^2(2+3x)^4$.

$$g'(x) = \underbrace{2x(2+3x)^4 + x^2 \cdot \left[(2+3x)^4\right]'}_{\text{product rule}}$$

$$= 2x(2+3x)^4 + x^2 \cdot \left(\underbrace{4(2+3x)^3 \cdot (3)}_{\text{power chain rule}}\right)$$

$$= 2x(2+3x)^4 + 12x^2(2+3x)^3$$

$$= 2x(2+3x)^3\left[2+3x + 6x\right] = \boxed{2x(2+3x)^3(2+9x)}$$

(*b*) (5 pts) Suppose that $f(x) = (x^2-5)^{-3/2}$. Find $f'(3)$.

$$f'(x) = -\tfrac{3}{2}(x^2-5)^{-3/2-1} \cdot (2x)$$

$$= -3x(x^2-5)^{-5/2} = \frac{-3x}{(x^2-5)^{5/2}}$$

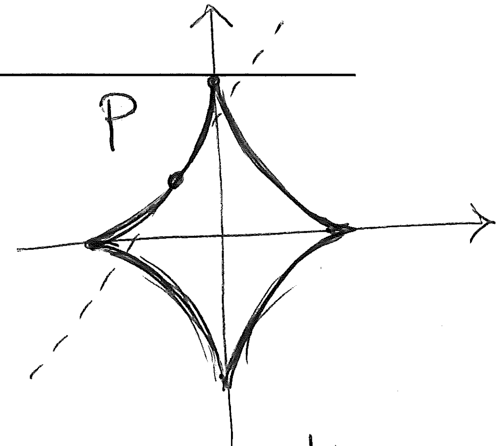
$$f'(3) = \frac{-3(3)}{(3^2-5)^{5/2}} = \frac{-9}{(4)^{5/2}} = \boxed{-\frac{9}{32}}$$

pts: /10

16. (*a*) (5 pts) Suppose x and y satisfy the equation

$$x^{2/3} + y^{2/3} = 4.$$

Use implicit differentiation to compute $\frac{dy}{dx}$ at $(-1, 3\sqrt{3})$.



$$\frac{d}{dx}\left[x^{2/3} + y^{2/3}\right] = \frac{d}{dx}[4]$$

$$\frac{2}{3}x^{2/3-1} + \underbrace{\frac{2}{3}y^{2/3-1} \cdot \frac{dy}{dx}}_{\text{chain rule}} = 0$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\cancel{\tfrac{2}{3}}y^{-1/3}\frac{dy}{dx} = -\cancel{\tfrac{2}{3}}x^{-1/3}$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

OR $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

at $(-1, 3\sqrt{3})$

$$\frac{dy}{dx} = -\left(\frac{3\sqrt{3}}{-1}\right)^{1/3} = \boxed{\sqrt{3}}$$

$= 1.732$

(*b*) (5 pts) Boyles Law states that when a sample gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = c$, where c is a constant. Suppose that a gas is being compressed and at a certain instant the volume is 100 cubic centimeters, the pressure is 4 kPa, and the pressure is increasing at a rate of 2 kPa/min. At what rate is the volume decreasing at this instant?

$PV = c$

$V = 100$ $\quad \frac{dP}{dt} = 2$

$P = 4$

$\frac{dV}{dt} = ?$

$$\frac{d}{dt}(PV) = \frac{d}{dt}(c)$$

$$\frac{dP}{dt} \cdot V + P\frac{dV}{dt} = 0$$

$$P\frac{dV}{dt} = -\frac{dP}{dt} \cdot V$$

OR $\boxed{\frac{dV}{dt} = -\frac{V}{P} \cdot \frac{dP}{dt}}$

Thus with our data

$$\frac{dV}{dt} = -\frac{100}{4} \cdot 2 = -50 \text{ cm}^3/\text{min}$$

pts: /10

Bonus. (*a*) (5 pts) Hill's function models how the amount of oxygen bound to hemoglobin in the blood depends on oxygen concentration, P, in the surrounding tissues. In its most general form Hill's function models the fraction of hemoglobin molecules in blood that are bound to oxygen by

$$f(P) = \frac{P^n}{k^n + P^n}$$

where k is a positive constant, and n is a positive integer.
Find $f'(P)$. Simplify the expression as much as you can.

$$f'(P) = \frac{nP^{n-1}(k^n + P^n) - P^n(nP^{n-1})}{(k^n + P^n)^2}$$

$$= \frac{nP^{n-1} \cdot k^n + \cancel{nP^{2n-1}} - \cancel{nP^{2n-1}}}{(k^n + P^n)^2}$$

$$= \boxed{\frac{nP^{n-1} \cdot k^n}{(k^n + P^n)^2}}$$

notice that $\underline{f'(P) > 0}$

this means that increasing the oxygen concentration always increases the fraction of hemoglobin molecules that are bound to oxygen

(*b*) (5 pts) The following limit represents the derivative $f'(x_0)$ of a function f at a point x_0. State $f(x)$ and x_0.

$$\lim_{h \to 0} \frac{-(2+h)^3 + 8}{h}$$

$$\boxed{\begin{array}{l} f(x) = -x^3 \\ x_0 = 2 \end{array}}$$

pts: /10