MA 137
Calculus I with Life Science Applications
THIRD MIDTERM

Fall 2020
11/10/2020

Name: Answer Key

Sect. #: ______________

Length of exam: Unless you have a DRC accommodation letter, you will have until 7:30 PM on November 10, 2020 to upload a PDF with your answers for the exam in the same group assignment on Canvas where you downloaded the exam. The exam is written so that it should take you at most 2 hours for the exam, allowing 30 minutes to scan and upload the exam as a single PDF on Canvas. Budget your time appropriately as NO extensions will be given.

Students with a DRC accommodation letter will take the exam with the students in sections 001 and 002. Thus they should use the Zoom number for sections 001 and 002. Their exam will end at 8:30 PM if they are allowed 50% extra time or at 9:30 PM if they are allowed 100% extra time.

Submitting your exam: You can annotate the PDF on an e-device, for example on a university issued iPad. Alternatively, you could print the test and write all your solutions on the printed exam. If it is too time consuming and/or impossible to print the test, just write on blank sheets of paper your work for the multiple choice questions and for the open response questions.

Please make sure to write your name and list the correct section number on the front page of your exam. In case you have forgotten your section number, consult the table below.

Please make sure to write your answers for the multiple choice questions either on the second page of the exam or on a single sheet of paper. You should include any supporting work that you deem appropriate for the multiple choice questions. The answers must be in the same order as the multiple choice questions (namely, question 1/answer, question 2/answer, etc,). Similarly, please write your answers to the open response questions on either the exam pages or on separate sheets of paper, making sure your answer pages are scanned in sequential order (answer to problem 13, then answer to problem 14, etc.). *You will be penalized 10 points if you provide the answers in a scrambled order.*

Questions during exam: You will be proctored for the entire exam time by your TA at the following Zoom link from 5 pm to 7:30 pm. You are required to have your camera on during the entire exam. If you need any clarification during the exam please ask a private question in the Zoom chat.

Section	Time/Recitation Location	TA	Zoom number
001	TR 08:00-08:50 AM, CB 240	J. Garagnani	https://uky.zoom.us/j/88435600496 passcode: MA137
002	TR 09:00-9:50 AM, CB 240		
003	TR 10:00-10:50 AM, CB 242	J. Britt	https://uky.zoom.us/j/88473566767 passcode: 137
004	TR 11:00-11:50 AM, CB 242		
005	TR 12:00-12:50 PM, CB 246	W. Rizer	https://uky.zoom.us/j/83960932487 passcode: MA137
006	TR 01:00-01:50 PM, CB 246		
007	TR 12:00-12:50 PM, CB 244	R. Righi	https://uky.zoom.us/j/83990722986 passcode: MA137
008	TR 01:00-01:50 PM, CB 244		
009	TR 02:00-02:50 PM, CB 246	M. McCarver	https://uky.zoom.us/j/85264041638 passcode: MA137
010	TR 03:00-03:50 PM, CB 246		

Restrictions on books, notes, calculators and cell phones: You will return the whole exam with your answers or the sheets that you want us to grade. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. No books or notes may be used. Absolutely no cell phone use during the exam is allowed, except for scanning your exam pages. Make sure to work in a quiet environment.

The **first part of the exam** consists of 12 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

a b c d e

It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of four open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

Cheating (Senate Rule 6.3.2): Cheating is a serious offense and will not be tolerated. It will be thoroughly investigated, and might lead to failure in the course or even to expulsion from the university. Cheating is defined by its general usage. It includes, but is not limited to, wrongfully giving, taking, or presenting any information or material by a student with the intent of aiding themselves or another on any academic work which is considered in any way in the determination of the final grade. The fact that a student could not have benefited from an action is not by itself proof that the action does not constitute cheating. Any question of definition shall be referred to the University Appeals Board.

1. a ■ c d e

2. a b ■ d e

3. ■ b c d e

4. a b ■ d e

5. a b c d ■

6. a b c d ■

7. a b c ■ e

8. ■ b c d e

9. a ■ c d e

10. a b c d ■

11. a b ■ d e

12. a ■ c d e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		60 pts
13.		10 pts
14.		10 pts
15.		10 pts
16.		10 pts
Bonus.		10 pts
TOTAL		100 pts

1. Suppose $f(x) = \ln(\cos x)$. Find $f'(x)$.

$$f'(x) = \underbrace{\frac{1}{\cos(x)} \cdot (-\sin(x))}_{\text{chain rule}}$$

$$= -\frac{\sin(x)}{\cos(x)} = \boxed{-\tan x}$$

Possibilities:

(a) $f'(x) = (\ln x)(-\sin x) + (\cos x)(\ln x)$

(b) $f'(x) = -\tan x$

(c) $f'(x) = \cot x$

(d) $f'(x) = \sec x$

(e) $f'(x) = \dfrac{1}{\ln(\cos x)}$

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2. Suppose $f(x) = x^{4x}$. Use logarithmic differentiation to find $f'(x)$.

Set $y = x^{4x}$. Take "ln" of both sides

$\ln y = \ln x^{4x} = 4x \cdot \ln x$. Take $\frac{d}{dx}$ of $\boxed{\ln y = 4x \cdot \ln x}$. We get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 4 \cdot \ln x + 4x \cdot \frac{1}{x}$$

$$= 4((\ln x) + 1)$$

Possibilities:

(a) $f'(x) = 4x^{4x-1}$

(b) $f'(x) = (4 + \ln x)x^{4x}$

(c) $f'(x) = 4(1 + \ln x)x^{4x}$

(d) $f'(x) = 4(1 + \ln x)$

(e) $f'(x) = (1 + \ln x)x^{4x}$

or $$\frac{dy}{dx} = y \cdot 4((\ln x) + 1)$$

or $$= \boxed{4x^{4x} \cdot ((\ln x) + 1)}$$

3. The linear approximation to $f(x) = \sqrt{x^2+3}$ at $x = -1$ is:

$$f(-1) = \sqrt{(-1)^2+3} = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2}(x^2+3)^{-1/2} \cdot (2x) = \frac{x}{\sqrt{x^2+3}}$$

$$f'(-1) = \frac{-1}{\sqrt{(-1)^2+3}} = -1/2$$

Possibilities:

(a) $\boxed{L(x) = -\frac{1}{2}x + \frac{3}{2}}$

(b) $L(x) = -\frac{1}{2}x + \frac{1}{2}$

(c) $L(x) = -x + 2$

(d) $L(x) = -\frac{1}{4}x + \frac{1}{2}$

(e) $L(x) = -x + \frac{1}{2}$

Thus

$$L(x) = 2 - \frac{1}{2}(x - (-1))$$
$$= 2 - 1/2\, x - 1/2$$
$$\boxed{= -1/2\, x + 3/2}$$

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4. Find the absolute maximum and absolute minimum values of $f(x) = \frac{x^2-4}{x^2+4}$ on the interval $[-4, 4]$.

Notice that the function is continuous for all values of $x \in \mathbb{R}$, as the denominator is never 0 and $f(x)$ is the quotient of 2 continuous function. By the EVT it has both a global max and minimum as the interval $[-4,4]$ is closed. The critical #s are

$$f'(x) = \frac{2x(x^2+4) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2} \stackrel{\downarrow}{=} 0$$

for $\boxed{x=0}$

Possibilities:

(a) absolute maximum $3/5$; absolute minimum $-3/5$

(b) absolute maximum $-3/5$; absolute minimum -1

(c) $\boxed{\text{absolute maximum } 3/5\text{; absolute minimum } -1}$

(d) absolute maximum 4; absolute minimum -4

(e) absolute maximum 4; absolute minimum 0

x	± 4	0
$f(x)$	$\frac{12}{20} = 3/5$	-1

5. The function $f(x) = x^{2/3}$ on $[-8, 8]$ does not satisfy the conditions of the Mean Value Theorem because

$$f'(x) = \frac{2}{3} \cdot x^{\frac{2}{3}-1} = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

$\therefore$ f' is not defined at $x=0$

Possibilities:

(a) $f(0)$ does not exist

(b) f is not continuous on $[-8, 8]$

(c) $f(1)$ does not exist

(d) f is not defined for $x < 0$

$\boxed{\text{(e) } f'(0) \text{ does not exist}}$

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6. Denote the biomass at time t by $B(t)$, and assume that $B(t)$ is continuous on the interval $[1, 5]$ and differentiable on the interval $(1, 5)$ with $B(1) = 100$ and $-2 \leq dB/dt \leq 3$ for all $t \in (1, 5)$. What can you say about $B(5)$?

By the Mean Value Thereom there exists $c \in (1,5)$ such that

$$\frac{B(5)-B(1)}{5-1} = B'(c)$$

As $-2 \leq B'(t) \leq 3$ for all $t \in (1,5)$ we have

$$-2 \leq \frac{B(5)-B(1)}{4} \leq 3$$

$$-8 \leq B(5)-B(1) \leq 12$$

$$\underbrace{B(1)-8}_{92} \leq B(5) \leq \underbrace{B(1)+12}_{112}$$

Possibilities:

(a) $108 \leq B(5) \leq 112$

(b) $B(5) = 103$

(c) $92 \leq B(5) \leq 108$

(d) $B(5) = 98$

$\boxed{\text{(e) } 92 \leq B(5) \leq 112}$

7. Let $f(x) = x^3 - 2x^2$. Find the largest open interval on which f is decreasing.

$$f'(x) = 3x^2 - 4x = x(3x-4)$$

x: $- - - - \; \overset{0}{|} \; + + + + + + + + + +$

$3x-4$: $- - - - - - - \; | \; + + + + +$ $\quad 4/3$

Possibilities: $f'(x)$: $+ + + \; | \; - - - \; | \; + + + + +$ sign of derivative

$\qquad 0 \qquad 4/3$

$f \nearrow \qquad f \searrow \qquad f \nearrow$

(a) $(-2, 0)$
(b) $(0, 2)$
(c) $(-\infty, 2/3)$
(d) $\boxed{(0, 4/3)}$
(e) $(2/3, \infty)$

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8. If $f(x) = \dfrac{1}{x^2+5}$, we have that $f(x)$ is concave up when which of the following is true?

$$f'(x) = \frac{-2x}{(x^2+5)^2} \qquad f'' = \frac{-2(x^2+5)^2 - (-2x)2(x^2+5)\cdot 2x}{((x^2+5)^2)^2}$$

$$f'' = \frac{2\cancel{(x^2+5)}\left[-(x^2+5) + 4x^2\right]}{(x^2+5)^{\cancel{4}\,3}}$$

Possibilities:

(a) $\boxed{6x^2 - 10 \text{ is positive}}$
(b) $6x^2 - 10$ is negative
(c) $2x$ is positive
(d) $2x$ is negative
(e) None of the above

$$f''(x) = \frac{2(3x^2-5)}{(x^2+5)^3}$$

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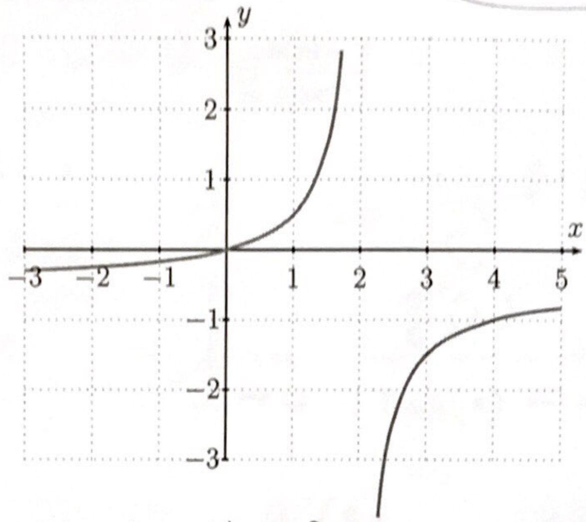
need $6x^2 - 10$ to be positive

9. Let $f(x)$ be a function defined on the interval $[-3, 5]$. The graph of the **derivative** of $f(x)$ is shown below:



Which of the following statements must be true?

true

I. The graph of $f(x)$ is decreasing on the intervals $(-3, 0)$ and $(2, 5)$.

II. The function $f(x)$ has a local minimum at $x = 0$.

III. The graph of $f(x)$ is concave down on the interval $(2, 5)$.

false

Possibilities:

(a) I only

(b) I and II only

(c) III only

(d) I and III only

(e) II and III only

sign of $f'(x)$

$-3 \quad - - - - \quad 0 \quad + + + + \quad 2 \quad - - - -$

$f \searrow \quad f \nearrow \quad \searrow f$

$\therefore f$ has a local min at $x=0$

f' is increasing on $(2,5)$ so f is concave up

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10. Find two positive numbers whose product is 144 and whose sum is a minimum.

x, y positive numbers $\quad x \cdot y = 144$

so $\quad y = \frac{144}{x}$

need to minimize $\boxed{S = x + \frac{144}{x}}$

on $0 < x < +\infty$

Possibilities:

$x=12$
$y=12$

(a) 2, 72

(b) 3, 48

(c) 4, 36

(d) 6, 24

(e) None of the above

$S=24$

$S'(x) = 1 - \frac{144}{x^2} = 0 \iff x^2 = 144$

so $x = \pm 12$

$0 \quad - - - \quad 12 \quad + + + +$ sign of $S'(x)$

$S \searrow \quad S \nearrow$

11. Let $g(x)$ be a differentiable function such that $\lim_{x\to 0} g(x) = 2$ and $\lim_{x\to 0} g'(x) = 4$.
Use l'Hôpital's rule to compute the limit

$$\lim_{x\to 0} \frac{g(x)-2}{x\cos(x)}.$$

$\lim_{x\to 0} \frac{g(x)-2}{x\cos x} = \frac{0}{0}$ so by l'Hôpital's Rule

we have $= \lim_{x\to 0} \frac{g'(x)}{\cos(x) - x\sin x} =$

$= \frac{g'(0)}{\cos(0)} = \frac{4}{1} = \boxed{4}$

Possibilities:

(a) 2

(b) 3

$\boxed{\text{(c) } 4}$

(d) 5

(e) ∞

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12. Suppose a is any positive number. It is given that the first order recursion (= difference equation)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = f(x_n) \text{ with } f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

has two fixed points $\widehat{x} = \pm\sqrt{a}$. What does the Stability Criterion say about the positive fixed point?

$f(x)$ = updating function $= \frac{1}{2}\left(x + \frac{a}{x}\right)$

$f'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$ so $f'(\sqrt{a}) = \frac{1}{2}\left(1 - \frac{a}{(\sqrt{a})^2}\right)$

$= \frac{1}{2}(1-1)$

$= 0$

Possibilities:

(a) $f'(\sqrt{a}) = 0$ hence $\widehat{x} = \sqrt{a}$ is unstable

$\boxed{\text{(b) } f'(\sqrt{a}) = 0 \text{ hence } \widehat{x} = \sqrt{a} \text{ is locally stable}}$

(c) $f'(\sqrt{a}) = 1$ hence $\widehat{x} = \sqrt{a}$ is unstable

(d) $f'(\sqrt{a}) = 1$ hence $\widehat{x} = \sqrt{a}$ is locally stable

(e) $f'(\sqrt{a}) = 1$ hence we cannot conclude anything about the stability of $\widehat{x} = \sqrt{a}$

13. Recall that the **Mean Value Theorem** states that if f is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there exists at least one number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

A particle moves in a straight line. At time t (measured in seconds) its position (measured in meters) is

$$s(t) = \frac{1}{100}t^3 \qquad 0 \leq t \leq 5$$

(a) Find its average velocity between $t = 0$ and $t = 5$.

$$\frac{s(5) - s(0)}{5 - 0} = \frac{\frac{5^3}{100} - \frac{0^3}{100}}{5} = \underline{\underline{0.25}}$$

(b) Find its (instantaneous) velocity at time t.

$$v(t) = s'(t) = \underline{\frac{3}{100}t^2}$$

(c) At what time is the average velocity of the particle between $t = 0$ and $t = 5$ equal to the instantaneous velocity?

Want to find $t_0 \in (0, 5)$ such that

$v(t_0) = 0.25$ so $\frac{3}{100} t_0^2 = 0.25$

$\rightsquigarrow t_0^2 = \frac{25}{3}$ $\quad t_0 = \pm \frac{5}{\sqrt{3}} = \pm 2.88$

But t_0 must be positive so $\boxed{t_0 = 2.88}$ pts: /10

14. Let $g(x) = \dfrac{x^2}{1+x}$ $\quad x \neq -1$. You are given that

$$g'(x) = \frac{2x + x^2}{(1+x)^2} \qquad g''(x) = \frac{2}{(1+x)^3}.$$

(a) Find the intervals over which $g(x)$ is increasing, and find the intervals over which $g(x)$ is decreasing. Find all local minima and all local maxima.

$g'(x) = 0 \iff 2x + x^2 = 0 \iff x(2+x) = 0$

so that $\boxed{x = 0,\ x = -2}$ are the 2 critical #s

		-2		-1		0	
$x(2+x)$	+ + +		− − −		− − −		+ + + +
$(1+x)^2$	+ + + +		+ +	⊗	+ +		+ + + + +
sign of $g'(x)$	+ + +		− −		− −		+ + + + +
	g ↗	-2	g ↘	-1	g ↘	0	↗ g

local max at $\boxed{x = -2}$

local min at $\boxed{x = 0}$

g decreasing on $(-2,-1) \cup (-1,0)$; increasing otherwise

(b) Find the intervals over which $g(x)$ is concave up, and find the intervals over which $g(x)$ is concave down. Find all inflection points.

$g''(x) = \dfrac{2}{(1+x)^3}$ is never zero

it is not defined at $x = -1$

		-1	
sign of g''	− − −	⊗	+ + + + +
	g concave down		g concave up

There are no inflection points

pts: /10

15. (*a*) Nina has one more gift to wrap for her family. Her gift is rather small, but it has a very recognizable shape so she wants to keep her family guessing by using the largest possible box. She doesn't have any boxes left. It is late on Christmas Eve and all the stores are closed. She decides to make a box (with no lid) out of a 100 cm by 100 cm square sheet of cardboard. She cuts a square from each corner of the sheet so that she can fold up the sides to make her box.

Write the expression of the volume $V(x)$ of the box as a function of the size x of the square she cuts from each corner.

$$\text{Volume} = \text{square base} \cdot \text{height} = (100-2x)^2 \cdot x$$

$$= 4(50-x)^2 \cdot x = 4(x^2 - 100x + 2500)\cdot x$$

$$\boxed{= 4(x^3 - 100x^2 + 2500x)} \qquad 0 \le x \le 50$$

(*b*) What is the volume of the largest possible box that Nina can make?

We can use the Extreme Value Theorem as $V(x)$ is a continuous function on a closed interval $[0, 50]$.

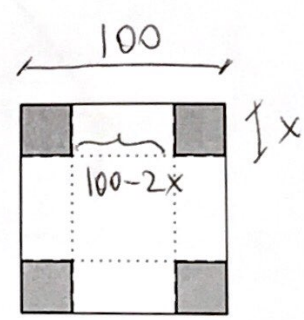
$$V'(x) = 4(3x^2 - 200x + 2500) = 0$$

$$x = \frac{200 \pm \sqrt{200^2 - 4\cdot 3\cdot 2500}}{6} = \frac{200 \pm 100}{6} = \begin{cases} 50 \\ \frac{50}{3} = 16.66 \end{cases}$$

	x	$V(x)$
end points	0	0
	50	0
critical #s	16.66	74,074.06

(*c*) What is the length of the side of the square she should cut from each corner to obtain the maximum volume?

$x = \frac{50}{3} = 16.66$ is the size of the cut that gives the largest volume



Volume = 74,074.06

pts: /10

16. (*a*) Find all the fixed (equilibrium) points for the recursive sequence $x_{t+1} = x_t\, e^{\frac{1}{2}(1-x_t^3)}$.

We need to solve $x = f(x)$ to find the fixed pts.

$$x = x \cdot e^{\frac{1}{2}(1-x^3)} \iff \boxed{\hat{x} = 0} \quad \text{or} \quad 1 = e^{\frac{1}{2}(1-x^3)} \iff$$

$$\ln 1 = \ln e^{\frac{1}{2}(1-x^3)} \iff 0 = \frac{1}{2}(1-x^3) \iff \boxed{\hat{x} = 1}$$

(*b*) What does the Stability Criterion say about the fixed (equilibrium) points found in part (*a*)?

[You can use the fact that if $f(x) = x\, e^{\frac{1}{2}(1-x^3)}$, then $f'(x) = (1 - \frac{3}{2}x^3)\, e^{\frac{1}{2}(1-x^3)}$]

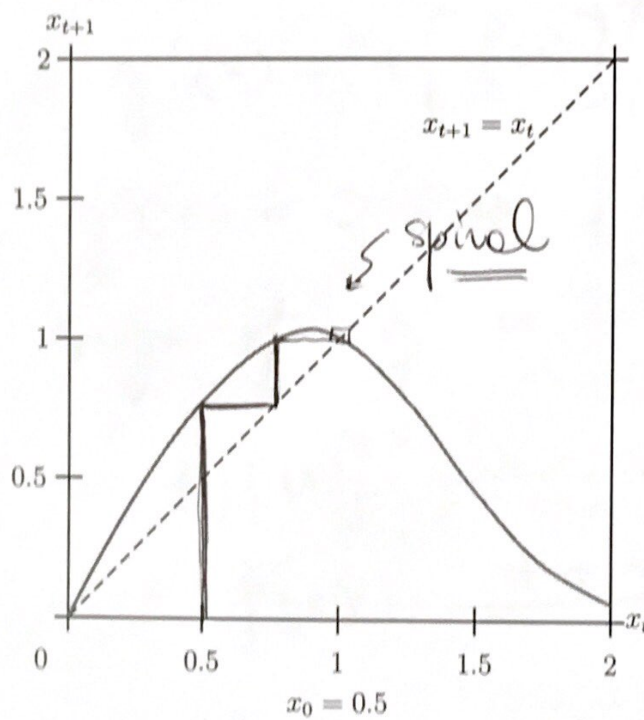
$$f'(0) = (1 - \tfrac{3}{2}0^3) \cdot e^{\frac{1}{2}(1-0^3)} = 1 \cdot e^{1/2} \approx 1.6487$$

$$f'(1) = (1 - \tfrac{3}{2}1^3) \cdot e^{\frac{1}{2}(1-1^3)} = -1/2$$

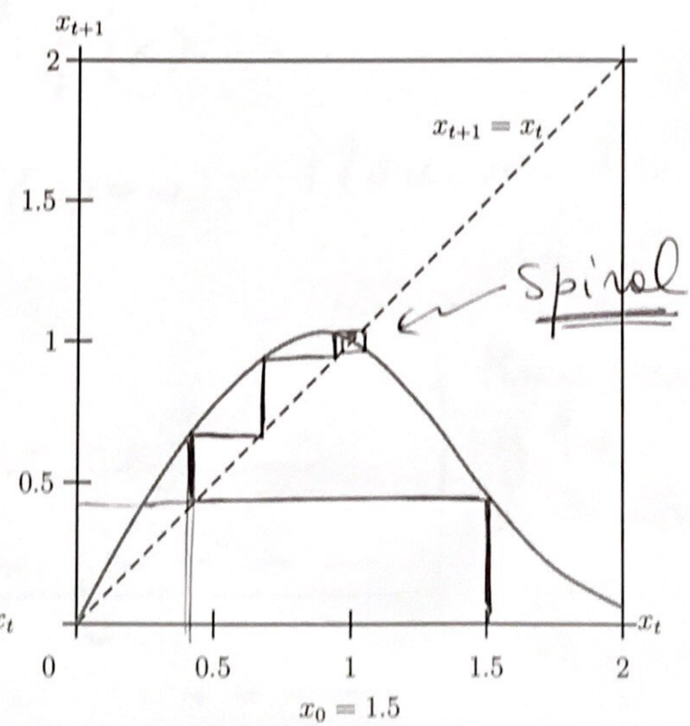
$\hat{x} = 0$ is unstable

$\hat{x} = 1$ is locally stable

(*c*) Sketch a cobweb graph starting at $x_0 = 0.5$ and $x_0 = 1.5$, respectively, on each of the figures below. Use it to determine $\lim_{t \to \infty} x_t$ in each case.



$x_0 = 0.5$



$x_0 = 1.5$

pts: /10

Bonus. Suppose that $f(x)$ is a differentiable function with $f(x) < 0$ for all $x \in \mathbb{R}$ and suppose that $f(x)$ has a local maximum at $x = c$.

Show that the function $g(x) = [f(x)]^4$ has a local minimum at $x = c$.

[**Hint:** apply the first derivative test; in other words, compare the sign of the derivative of f and g nearby the value $x = c$.]

Since $f(x)$ has a local maximum at $x = c$ we know that the sign of $f'(x)$ near c is:

sign of $f'(x)$: $+ + + + + \mid - - - - -$ (at c) $\qquad f'(c) = 0$

Given that $g(x) = [f(x)]^4$ we have that

$$g'(x) = 4[f(x)]^3 \cdot f'(x)$$

$g'(c) = 0$ as $f'(c) = 0$. Moreover the sign of g' is

$4[f(x)]^3$: $- - - - - \mid - - - - - -$

$f'(x)$: $+ + + \mid - - - -$ (at c)

sign of $g'(x)$: $- - - - \mid + + + + +$ (at c)

$g \searrow \quad c \quad \nearrow g$

therefore g has a local min at $x = c$

pts: /10