MA 137
Calculus I with Life Science Applications
THIRD MIDTERM

Fall 2021
11/16/2021

Name: Answer Key

Sect. #: ____________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 12 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

a b c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of four open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. a b **c** d e

2. a b c **d** e

3. **a** b c d e

4. a **b** c d e

5. a b **c** d e

6. a b **c** d e

7. a b c d **e**

8. a b c **d** e

9. a b c d **e**

10. a b c d **e**

11. **a** b c d e

12. a b c **d** e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		60 pts
13.		10 pts
14.		10 pts
15.		10 pts
16.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Time/Lecture Location	Lecturer
001-010	MWF 10:00 am - 10:50 am, CB 106	Alberto Corso
Section #	**Time/ Recitation Location**	**TA**
001	TR 08:00-08:50 AM, CB 339	Nicholas Arsenault
002	TR 09:00-9:50 AM, CB 339	
003	TR 10:00-10:50 AM, CB 339	Katherine (Kat) Henneberger
004	TR 11:00-11:50 AM, CB 339	
005	TR 12:00-12:50 PM, CB 339	Faith Hensley
006	TR 01:00-01:50 PM, CB 339	
007	TR 12:00-12:50 PM, CB 341	Michael Morrow
008	TR 01:00-01:50 PM, CB 341	
009	TR 02:00-02:50 PM, CB 339	Karen Reed
010	TR 03:00-03:50 PM, CB 339	

1. Suppose that $f(x) = x\cos x$. Find $f''(x)$.

$$f'(x) = 1 \cdot \cos(x) + x(-\sin(x))$$
$$= \cos(x) - x\sin(x)$$

Hence $f''(x) = (f')'(x) = -\sin(x) - [1 \cdot \sin(x) + x\cos(x)]$

$$= -2\sin(x) - x\cos(x)$$

Possibilities:

(a) $2\sin x + x\cos x$

(b) $-2\sin x + x\cos x$

(c) $\boxed{-2\sin x - x\cos x}$

(d) $2\sin x - x\cos x$

(e) $2\cos x + x\sin x$

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2. Suppose that $h(t) = \ln(5t^3 + 2t^2 + t + 2)$. Then $h'(0)$ is

$$h'(t) = \frac{1}{5t^3+2t^2+t+2} \cdot (15t^2 + 4t + 1)$$

chain rule

$$= \frac{15t^2+4t+1}{5t^3+2t^2+t+2}$$

Hence $h'(0) = \frac{1}{2}$

Possibilities:

(a) 0

(b) 1

(c) $1/\ln 2$

(d) $\boxed{1/2}$

(e) None of the above

3. Suppose $f(x) = x\,e^{2x}$. Find all the values of x where the third derivative of f is equal to zero, i.e., $f^{(3)}(x) = 0$.

$$f'(x) = 1 \cdot e^{2x} + x \cdot \left(e^{2x} \cdot 2\right) = e^{2x}(1+2x)$$

$$f''(x) = \left(e^{2x} \cdot 2\right) \cdot (1+2x) + e^{2x} \cdot (2)$$
$$= e^{2x}(2+4x+2) = 4e^{2x}(1+x)$$

$$f'''(x) = 4(e^{2x} \cdot 2)(1+x) + 4e^{2x}(1)$$
$$= 4e^{2x}(2+2x+1)$$
$$= 4e^{2x}(3+2x)$$

$f'''(x) = 0$ if and only if $\boxed{x = -3/2}$

Possibilities:

(a) $-3/2$

(b) $-2/3$

(c) $3/2$

(d) -1

(e) 1

4. Find the second derivative of $f(x) = 3^{30}$.

$f(x)$ is a constant

hence $f'(x) = 0$ so that

$$\boxed{f''(x) = 0}$$

Possibilities:

(a) $30 \cdot 29 \cdot 3^{28}$

(b) 0

(c) $[\ln(3)]^2 \cdot 3^{30}$

(d) 3

(e) None of the above

5. Suppose $f(x) = x^x$. Use logarithmic differentiation to find $f'(x)$

$y = x^x$. Take ln of both sides

$\ln y = x \ln x$ Now use implicit diff.

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$ $\underline{\text{OR}}$ $\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$

Possibilities:

(a) $x \cdot x^{x-1}$

(b) $x^x \cdot \ln x$

(c) $\boxed{x^x \cdot (1 + \ln x)}$

(d) $1 + x^x \ln x$

(e) None of the above

$\therefore \frac{dy}{dx} = y((\ln x) + 1)$

$= \boxed{x^x (1 + \ln x)}$

6. The linear approximation to $f(x) = \sqrt{7x + 25}$ at $x_0 = 0$ is:

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$f(0) = \sqrt{7 \cdot 0 + 25} = 5$ $f'(x) = \frac{1}{2\sqrt{7x+25}} \cdot 7$

$f'(0) = \frac{7}{10}$ Hence

Possibilities:

(a) $L(x) = \frac{7}{10}x - 5$

(b) $L(x) = \frac{7}{5}x - 5$

(c) $\boxed{L(x) = \frac{7}{10}x + 5}$

(d) $L(x) = \frac{7}{5}x + 5$

(e) $L(x) = -x + \frac{1}{2}$

$L(x) = 5 + \frac{7}{10} \cdot (x - 0)$

$= \boxed{5 + \frac{7}{10} x}$

7. The absolute minimum and absolute maximum of the function $f(x) = 2x^3 - 9x^2 + 11$ on the interval $[-2, 1]$ is:

The Extreme Value Theorem applies as $f(x)$ is continuous on a closed interval. The extrema can occur at the end points or at the points in the interval where $f' = 0$ OR f' DNE. $f'(x) = 6x^2 - 18x = 0$

$6x(x-3) = 0 \iff x = 0$, $\cancel{x = 3}$

Possibilities:

(a) absolute minimum is -16 and absolute maximum is 11

(b) absolute minimum is 4 and absolute maximum is 11

(c) absolute minimum is -41 and absolute maximum is 4

(d) absolute minimum is -16 and absolute maximum is 4

(e) absolute minimum is -41 and absolute maximum is 11

x	$f(x)$	
-2	-41	$\leftarrow$ min
1	4	
0	11	$\leftarrow$ max

8. Suppose that $f(x)$ is a differentiable function with $6 \le f'(x) \le 10$ for all x in the open interval $(1, 7)$. If $f(1) = 3$, then the Mean Value Theorem for $f(x)$ on the interval $[1, 7]$ implies that the largest possible value of $f(7)$ is

We know that for some $c \in (1,7)$

$$\frac{f(7) - f(1)}{7 - 1} = f'(c)$$

But $f'(x)$ for all $x \in (1,7)$ is: $6 \le f'(x) \le 10$

Possibilities:

Hence $6 \le \dfrac{f(7) - f(1)}{6} \le 10$

(a) 53

(b) 57

(c) 60

(d) 63

(e) 67

So $36 \le f(7) - f(1) \le 60$

OR $\underbrace{f(1) + 36}_{39} \le f(7) \le 60 + \underbrace{f(1)}_{3} = 63$

9. The function f is given by $f(x) = x^4 + 4x^3$. On which of the following intervals is f decreasing?

We need $\underline{\underline{f'(x) < 0}}$

$f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$

We know $f'(x) = 0$ for $x = 0$ OR $x = -3$

test -4 test -1 test 2

Sign of $f'(x)$: $- - - - -$ -3 $+ + +$ 0 $+ + +$

Possibilities:

(a) $(-3, 0)$
(b) $(0, \infty)$
(c) $(-3, \infty)$
(d) $(-\infty, 0)$
$\boxed{\text{(e)} \quad (-\infty, -3)}$

f decreasing on $\underline{\underline{(-\infty, -3)}}$

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10. The value of c that satisfies the conclusions of the Mean Value Theorem on the interval $[0, 5]$ for the function $f(x) = x^3 - 6x$ is:

We need to find $c \in (0, 5)$ such that

$$\frac{f(5) - f(0)}{5 - 0} = f'(c) \quad ; \quad f'(x) = 3x^2 - 6$$

Hence we need to solve

$$\frac{95 - 0}{5 - 0} = 3c^2 - 6$$

Possibilities:

(a) $-\frac{5}{\sqrt{3}}$
(b) 0
(c) 1
(d) $\frac{5}{3}$
$\boxed{\text{(e)} \quad \frac{5}{\sqrt{3}}}$

So that

$$19 = 3c^2 - 6$$

$$3c^2 = 25 \quad \therefore \quad c = \pm \frac{5}{\sqrt{3}}$$

But c is in $(0, 5)$

11. Let f be a function defined for all real numbers x. You are given that

$$f'(x) = \frac{2x}{x^2+1} \quad \text{and} \quad f''(x) = \frac{2-2x^2}{(x^2+1)^2}.$$

On what interval(s) is f concave downward?

We need $f''(x) < 0$

test the pts -2, 0, 2

sign f'': $- - - \;|_{-1}\; + + + + \;|_{1}\; - - -$

f conc down f conc down

Possibilities:

- $\boxed{\text{(a)} \quad f \text{ is concave downward on } (-\infty,-1)\cup(1,+\infty)}$
- (b) f is concave downward on $(0,+\infty)$
- (c) f is concave downward on $(-\infty,-1)$
- (d) f is concave downward on $(-\infty,0)$
- (e) f is concave downward on $(-1,1)$

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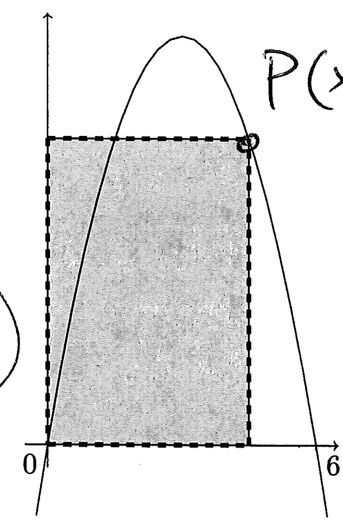
12. Consider a rectangle that lies in the first quadrant with one vertex at the origin and two of the sides along the coordinates axis. The fourth vertex lies on the parabola $y = 6x - x^2$. Find the area of the largest such rectangle.



$P(x, 6x-x^2)$

$$A(x) = \underbrace{x}_{\text{base}}\ \underbrace{(6x-x^2)}_{\text{height}}$$

$$\boxed{\begin{array}{l} 0 \le x \le 6 \\ \text{maximize} \\ A(x) = 6x^2 - x^3 \end{array}}$$

We use the EVT

The critical #s are: $A'(x) = 0$ so

$$A'(x) = 12x - 3x^2 = 0$$

$$\iff 3x(4-x) = 0 \qquad x = 0, 4$$

x	0	6	4
$A(x)$	0	0	32

Possibilities:

- (a) 0
- (b) 3
- (c) 27
- $\boxed{\text{(d)} \quad 32}$
- (e) 40

13. Find the derivative with respect to x of the following functions:

(a) $f(x) = \sin^2 x + \cos^2 x$

notice that $f(x) = 1$

So $f'(x) = 0$

(b) $g(x) = x \ln x$

$$g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= (\ln x) + 1$$

pts: /10

14. (*a*) Suppose that the specific growth rate of a plant is 3%; that is, if $B(t)$ denotes the **biomass** at time t then

$$\frac{1}{B(t)}\frac{dB}{dt} = 0.03.$$

Suppose that the biomass at time $t = 2$ is equal to 800 grams.
Use a linear approximation to compute the biomass at time $t = 2.3$.

$B(2) = 800$ and $B'(2) = 0.03 \cdot B(2)$

$= 0.03 \cdot 800 = \underline{24}$

Thus $L(t)$ at $t=2$ is $:= B(2) + B'(2)(t-2)$

$= \underline{800 + 24(t-2)}$

$B(2.3) \approx L(2.3) = 800 + 24(\underbrace{2.3-2}_{0.3}) = \boxed{807.2}$

(*b*) Explain why the function $f(x) = x^{2/3}$ on $[-8, 8]$ does not satisfy the conditions of the Mean Value Theorem.

$f'(x) = 2/3 \cdot x^{2/3-1} = \frac{2}{3} \cdot x^{-1/3} = \frac{2}{3 \cdot \sqrt[3]{x}}$

Hence f is not differentiable at $x=0$

So the function f is not differentiable on $(-8, 8)$.

That is the condition that fails in the MVT.

pts: /10

15. Assume that the **derivative** of a function $f(x)$ satisfies $f'(x) = xe^{-x}$.

(a) Find the intervals over which f is increasing, the intervals where f is decreasing, and find where all the local minima and maxima of f occur.

$$f'(x) = x e^{-x}$$

sign of $f'(x)$ $\quad$ $- - - - - \overset{0}{|} + + + + +$

as $e^{-x} > 0$ for all x

f is increasing on $(0, +\infty)$

f is decreasing on $(-\infty, 0)$

thus there is a local (global) min at $x=0$

(b) Find the intervals over which f is concave down, the intervals over which f is concave up, and find where all points of inflection of f are.

$$f''(x) = 1 \cdot e^{-x} + x \cdot e^{-x}(-1) = e^{-x}(1-x)$$

sign of f'' $\quad$ $+ + + + + \underset{1}{|} - - - - -$

as $e^{-x} > 0$ for all x

f is concave up on $(-\infty, 1)$

f is concave down on $(1, +\infty)$

There is an inflection pt at $x=1$

pts: /10

16. Use an optimization technique to find two positive numbers whose product is 121 and whose sum is a minimum. What is the sum?

We are seeking x, y (positive) such that

$$xy = 121$$

and we want to minimize the sum $x + y$.

We know $y = \frac{121}{x}$

So we need to minimize

$$\boxed{S(x) = x + \frac{121}{x} \quad \text{on} \quad 0 < x < +\infty}$$

$$S'(x) = 1 - \frac{121}{x^2} = \frac{x^2 - 121}{x^2}$$

$S'(x) = 0$ for $x^2 - 121 = 0$ or $x = \pm 11$

0 test at 1   11   test at 20

sign $S'(x)$ $- - - -$ $+ + + +$

$S \searrow$ $S \nearrow$

Hence we have a minimum for

$x = y = 121$ $\boxed{S(11) = 22}$

pts: /10

Bonus. (5 pts each) Use l'Hôpital's rule to evaluate the following limits:

(a) $\lim_{x \longrightarrow 2} \frac{8x - 12 - x^2}{x^2 - 4}$ using the substitution then

$= \frac{8 \cdot 2 - 12 - 2^2}{2^2 - 4} = \frac{0}{0}$

Hence we can use l'Hôpital's Rule

$= \lim_{x \to 2} \frac{8 - 2x}{2x} = \frac{8 - 2(2)}{2(2)} = \frac{4}{4} = \boxed{1}$

(b) $\lim_{x \longrightarrow 0+} x^2 \cdot \ln x$ $= 0^2 \cdot \ln(0) = 0 \cdot (-\infty)$

this is one of the form of l'Hôpital's rule

$= \lim_{x \to 0^+} \frac{\ln(x)}{1/x^2} = \frac{0}{0}$

Apply l'Hôpital

$= \lim_{x \to 0^+} \frac{1/x}{-\frac{2}{x^3}} = \lim_{x \to 0^+} \frac{1}{x} \left(-\frac{x^3}{2}\right)$

$= \lim_{x \to 0^+} -\frac{x^2}{2} = \boxed{0}$

pts: /10