MA 137
Calculus I with Life Science Applications
FINAL EXAM

Fall 2020
12/1/2020

Name: Answer Key
Sect. #:

Length of exam: You will have until 11:00 PM on December 1, 2020 to upload a PDF with your answers for the exam in the same group assignment on Canvas where you downloaded the exam. The exam is written so that it should take you at most 2 hours for the exam (8:30 PM - 10:30 PM), allowing 30 minutes to scan and upload the exam as a single PDF on Canvas. Budget your time appropriately as NO extensions will be given.

Students with a DRC accommodation have been contacted separately to make arrangements for the test. If proctored on Tuesday evening, they will take the exam with the students in sections 001 and 002.

Submitting your exam: You can annotate the PDF on an e-device, for example on a university issued iPad. Alternatively, you could print the test and write all your solutions on the printed exam. If it is too time consuming and/or impossible to print the test, just write on blank sheets of paper your work for the multiple choice questions and for the open response questions.

Please make sure to write your name and list the correct section number on the front page of your exam. In case you have forgotten your section number, consult the table below.

Please make sure to write your answers for the multiple choice questions either on the second page of the exam or on a single sheet of paper. You should include any supporting work that you deem appropriate for the multiple choice questions. The answers must be in the same order as the multiple choice questions (namely, question 1/answer, question 2/answer, etc,). Similarly, please write your answers to the open response questions on either the exam pages or on separate sheets of paper, making sure your answer pages are scanned in sequential order (answer to problem 13, then answer to problem 14, etc.). *You will be penalized 10 points if you provide the answers in a scrambled order.*

Questions during exam: You will be proctored for the entire exam time by your TA at the following Zoom link from 8:30 PM to 11:00 PM. You are required to have your camera on during the entire exam. If you need any clarification during the exam please ask a private question in the Zoom chat.

Section	Time/Recitation Location	TA	Zoom number
001	TR 08:00-08:50 AM, CB 240	J. Garagnani	https://uky.zoom.us/j/84033570444
002	TR 09:00-9:50 AM, CB 240		passcode: MA137
003	TR 10:00-10:50 AM, CB 242	J. Britt	https://uky.zoom.us/j/87196985574
004	TR 11:00-11:50 AM, CB 242		passcode: 137
005	TR 12:00-12:50 PM, CB 246	W. Rizer	https://uky.zoom.us/j/3784942964
006	TR 01:00-01:50 PM, CB 246		passcode: MA137
007	TR 12:00-12:50 PM, CB 244	R. Righi	https://uky.zoom.us/j/88313430055
008	TR 01:00-01:50 PM, CB 244		passcode: MA137
009	TR 02:00-02:50 PM, CB 246	M. McCarver	https://uky.zoom.us/j/85264041638
010	TR 03:00-03:50 PM, CB 246		passcode: MA137

Restrictions on books, notes, calculators and cell phones: You will return the whole exam with your answers or the sheets that you want us to grade. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. No books or notes may be used. Absolutely no cell phone use during the exam is allowed, except for scanning your exam pages. Make sure to work in a quiet environment.

The **first part of the exam** consists of 12 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

a b c d e

It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of four open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

Cheating (Senate Rule 6.3.2): Cheating is a serious offense and will not be tolerated. It will be thoroughly investigated, and might lead to failure in the course or even to expulsion from the university. Cheating is defined by its general usage. It includes, but is not limited to, wrongfully giving, taking, or presenting any information or material by a student with the intent of aiding themselves or another on any academic work which is considered in any way in the determination of the final grade. The fact that a student could not have benefited from an action is not by itself proof that the action does not constitute cheating. Any question of definition shall be referred to the University Appeals Board.

1. ■ b c d e

2. ■ b c d e

3. a b c ■ e

4. ■ b c d e

5. a b ■ d e

6. a b c ■ e

7. a ■ c d e

8. a b c d ■

9. a b c ■ e

10. a b c d ■

11. a ■ c d e

12. a b c ■ e

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		60 pts
13.		10 pts
14.		10 pts
15.		10 pts
16.		10 pts
Bonus.		10 pts
TOTAL		100 pts

1. Suppose $f(x) = x^3 - 2x + 5x^{-1}$. Find an equation of the tangent line to the graph of $y = f(x)$ at the point $(1, 4)$.

$$f'(x) = 3x^2 - 2 - 5x^{-2}$$

$$f'(1) = 3(1)^2 - 2 - 5(1)^{-2} = 3 - 2 - 5 = \boxed{-4}$$

Possibilities:

(a) $y = -4x + 8$

(b) $y = 4x$

(c) $y = x + 3$

(d) $y = 4$

(e) $y = 3x + 1$

$f(1) = 4$ (given already)

so the tg. line is

$$y - 4 = -4(x - 1)$$

or $\boxed{y = -4x + 8}$

2. Let

$$f(x) = \begin{cases} -2x + c & x < -1 \\ 3x^2 + 2x + 5 & x \geq -1 \end{cases}$$

For what value of c is this function continuous?

The function f is continuous for all values $\neq -1$ as it is made up by polynomials. We need to worry about the continuity at $x = -1$. We need

$$\lim_{x \to -1^-} f(x) = f(1) = \lim_{x \to -1^+} f(x)$$

Possibilities:

(a) $c = 4$

(b) $c = 8$

(c) $c = 12$

(d) $c = 16$

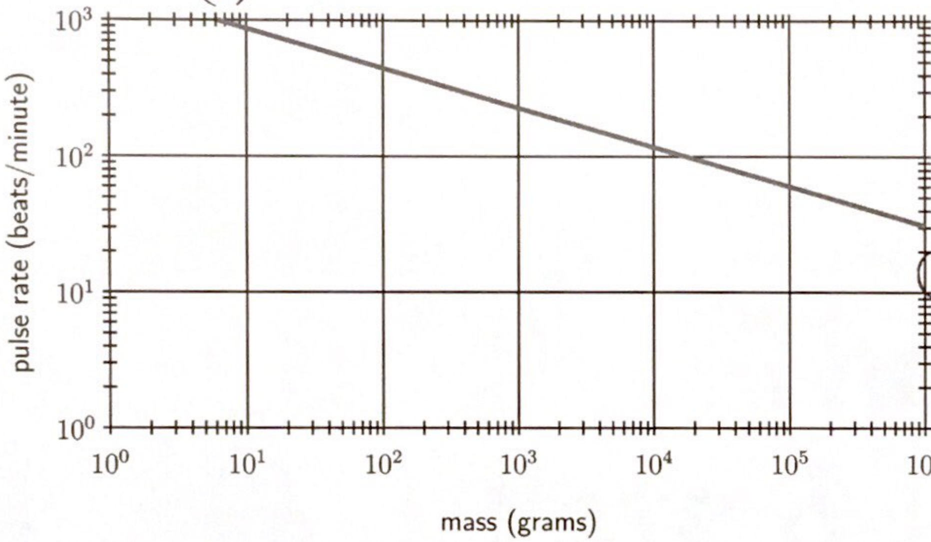
(e) $c = 20$

OR $-2(-1) + c = 3(-1)^2 + 2(-1) + 5$

$\hookrightarrow$ $c + 2 = 3 - 2 + 5$ $\boxed{c = 4}$

3. The following log-log plot gives resting heart rates for various animals.

(6,1000)



(1,000,000, 30)

Slope $= -0.29$

$= \dfrac{\log 30 - \log 1000}{\log 10^6 - \log 6}$

Find the functional relationship between the mass M of an animal and its pulse rate P best describing the above data.

[**Hint:** compute the slope of the line and observe this is a line in a log-log plot]

Possibilities:

(a) $P = 1,700 \cdot 10^{-0.29 \cdot M}$

(b) $P = 1,700 \cdot M^{0.29}$

(c) $P = -0.29M + 1,700$

$\boxed{\text{(d)} \quad P = 1,700 \cdot M^{-0.29}}$

(e) $P = 0.29M + 1,700$

Because we have a log-log plot we need to have a power function relation $P = A \cdot M^p$ with a negative slope p

4. The number of bacteria in a culture is modeled by the function

$$n(t) = 1,400e^{0.56t},$$

where the time t is measured in hours.

After how many hours will the number of bacteria reach 10,000?

Possibilities:

$\boxed{\text{(a)} \quad t \approx 3.51 \text{ hours}}$

(b) $t \approx 1/0.56 \cdot \log(10,000/1,400)$ hours

(c) $t \approx 12.75$ hours

(d) $t \approx \ln(2)/0.56$ hours

(e) None of the above

need to solve

$10,000 = 1,400\, e^{0.56t}$

$\dfrac{100}{14} = e^{0.56t}$

$\ln\left(\dfrac{100}{14}\right) = 0.56t$

$3.51 \approx t = \dfrac{\ln\left(\frac{100}{14}\right)}{0.56}$

5. Find all fixed points of the recursive sequence

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{64}{a_n^2}\right).$$

$f(x) = \frac{1}{2}\left(x + \frac{64}{x^2}\right)$

updating function

Use a table or the Stability Criterion to decide which fixed point, if any, is the limiting value for the initial condition $a_0 = -1$.

$$a = \frac{1}{2}\left(a + \frac{64}{a^2}\right) \iff 2a = a + \frac{64}{a^2}$$

$$\iff a = \frac{64}{a^2} \iff a^3 = 64 \quad \text{or} \quad \boxed{\hat{a} = 4 = \sqrt[3]{64}}$$

For the stability criterion we need $f'(x)$

Possibilities:

(a) There are no fixed points; $\lim_{n\to\infty} a_n$ does not exist

(b) One fixed point $\hat{a} = 4$; $\lim_{n\to\infty} a_n$ does not exist

$\boxed{\text{(c) One fixed point } \hat{a} = 4; \quad \lim_{n\to\infty} a_n = 4}$

(d) Two fixed points $\hat{a} = -4, 4$; $\lim_{n\to\infty} a_n = 4$

(e) Two fixed points $\hat{a} = -4, 4$; $\lim_{n\to\infty} a_n = -4$

$$f'(x) = \frac{1}{2}\left(1 - 2\cdot\frac{64}{x^3}\right)$$

$$f'(4) = \frac{1}{2}\left(1 - \frac{128}{4^3}\right) = \frac{1}{2}(1-2) = -\frac{1}{2}$$

$\therefore\ \hat{a} = 4$ is stable $\quad \boxed{\lim_{n\to\infty} a_n = 4}$

6. Suppose $f(x) = x^2[g(x)]^3$, $g(4) = 2$, and $g'(4) = 3$. Find $f'(4)$.

$$f'(x) = 2x[g(x)]^3 + x^2 \cdot 3[g(x)]^2 \cdot g'(x)$$

$$f'(4) = 2\cdot 4(g(4))^3 + 4^2\cdot 3[g(4)]^2 \cdot g'(4)$$

$$= 8\cdot 2^3 + 48(2)^2\cdot 3$$

$$= 64 + 576$$

$$= \boxed{640}$$

Possibilities:

(a) 256

(b) 288

(c) 425

$\boxed{\text{(d) } 640}$

(e) 696

7. Use the linear approximation of $f(x) = \sqrt{1+x}$ at $x_0 = 0$ to estimate $\sqrt{0.95}$.

$f(x) = \sqrt{1+x}$ $\qquad f'(x) = \frac{1}{2\sqrt{1+x}}$

at $x_0 = 0$ $\quad f(0) = 1 \quad f'(0) = \frac{1}{2}$

So the linearization is

$L(x) = f(0) + f'(0)\cdot(x-0) =$

$= 1 + \frac{1}{2}x$

$\sqrt{0.95} = f(-0.05) \approx L(-0.05) = 1 + \frac{1}{2}(-0.05)$

$= 0.9750$

Possibilities:

(a) 0.9502

$\boxed{\text{(b) } 0.9750}$

(c) 0.9747

(d) 0.9942

(e) none of the above

---

8. The volume of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. If the radius and the height both increase at a constant rate of 2 centimeters per second, at what rate, in cubic centimeters per second, is the volume increasing when the height is 9 centimeters and the radius is 6 centimeters?

[**Hint:** remember to use the product rule when differentiating with respect to time.]

$\frac{dr}{dt} = \frac{dh}{dt} = 2 \qquad \frac{dV}{dt} = ?$ when $r = 6$, $h = 9$

$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 \cdot h\right) = \frac{1}{3}\pi\, 2r\frac{dr}{dt}\cdot h$

$+ \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt}$

With our data

$\frac{dV}{dt} = \frac{1}{3}\pi \cdot 2(6)\,2\cdot 9 + \frac{1}{3} + \pi(6)^2\cdot 2$

$= 72\pi + 24\pi = 96\pi$

Possibilities:

(a) 2π

(b) 10π

(c) 24π

(d) 54π

$\boxed{\text{(e) } 96\pi}$

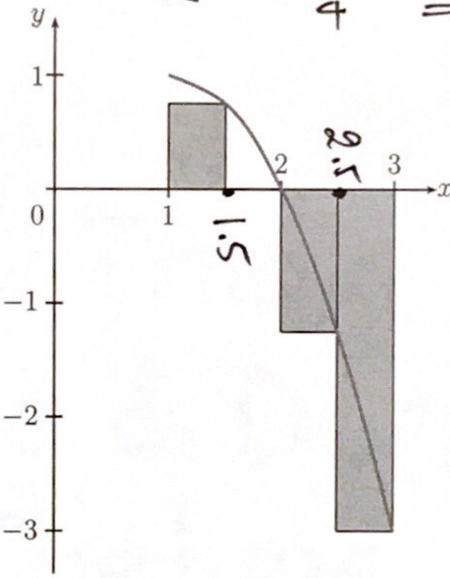
9. Consider the function $\boxed{f(x) = 2x - x^2}$ defined on the interval $[1, 3]$.

Estimate $\displaystyle\int_1^3 (2x - x^2)\,dx$ using right endpoints for $n = 4$ approximating rectangles all having bases of the same length, as shown in the picture.

$\Delta x = \frac{3-1}{4} = \underline{\underline{0.5}}$

x	$f(x)$
1.5	0.75
2	0
2.5	−1.25
3	−3

we need



Possibilities:

(a) −1.333

(b) 2.5

(c) 0

(d) −1.75

(e) 1.333

(signed) area of those rectangles

$= 0.5(0.75) + 0.5(0) + 0.5(-1.25) + 0.5(-3)$

$= \boxed{-1.75}$

---

10. Use l'Hôpital's Rule and the Fundamental Theorem of Calculus to compute

$$\lim_{x \to 0} \frac{\int_0^x (1 - \cos t)\,dt}{x^3} = \frac{0}{0}$$

Possibilities:

(a) 0

(b) 1

(c) 1/2

(d) 1/3

(e) 1/6

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x (1-\cos t)\,dt\right)}{\frac{d}{dx}(x^3)} =$$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0} \quad \text{by l'Hôpital's Rule}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}(1) = \boxed{1/6} \quad \text{trig limit}$$

11. Find the function f that satisfies the following equation

$$2 + \int_a^x \frac{f(t)}{t^7}\,dt = 4x^{-3}.$$

[**Hint:** take the derivative of both sides.]

$$\frac{d}{dx}\left[2 + \int_a^x \frac{f(t)}{t^7}\,dt\right] = \frac{d}{dx}\left(4 \cdot x^{-3}\right)$$

Possibilities:

(a) $f(x) = 4x^3$

(b) $\boxed{f(x) = -12x^3}$

(c) $f(x) = 4x^4$

(d) $f(x) = -12x^{-3}$

(e) $f(x) = 4x^{-3}$

By FTC – part I

$$0 + \frac{f(x)}{x^7} = 4(-3) \cdot x^{-4}$$

$$\therefore \quad f(x) = -12\frac{x^7}{x^4} = \boxed{-12x^3}$$

12. Suppose h is a continuous function such that

$$h(1) = -2 \qquad h'(1) = 2 \qquad h''(1) = 2$$

$$h(2) = 6 \qquad h'(2) = 5 \qquad h''(2) = 13$$

and h' and h'' are continuous everywhere.

Evaluate: $\displaystyle\int_1^2 h'(t)\,dt.$

by FTC – part II

$$\int_1^2 h'(t)\,dt = h(t)\Big|_1^2$$

$$= h(2) - h(1) = 6 - (-2)$$

$$= \boxed{8}$$

Possibilities:

(a) 3

(b) 4

(c) 7

(d) $\boxed{8}$

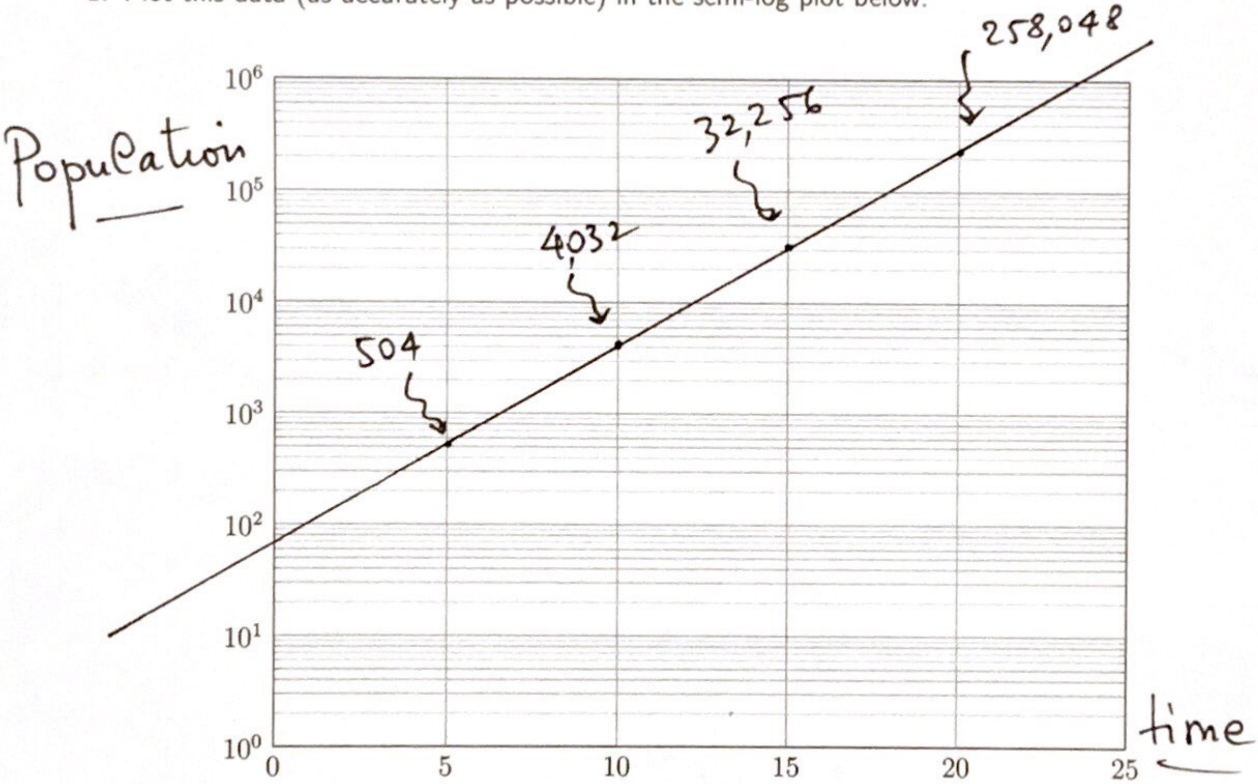
(e) 11

$$\boxed{P(t) = 63 \cdot (1.5157)^t \text{ or } 63 \cdot 8^{t/5}}$$

13. When a new species is introduced into an environment there may be no natural predators. In this case, the population may grow very rapidly. Suppose such an invasive species is introduced into a region and the population is measured at several times.

Time in months	5	10	15	20
Population	504	4,032	32,256	258,048

1. Plot this data (as accurately as possible) in the semi-log plot below.



2. Find a functional relationship between population and time.

We know that a line in a semi-log plot corresponds to an exponential relation of the form $P(t) = B \cdot A^t$

Let us use the 2 exact quantities $(5, 504)$ and $(10, 4{,}032)$ and substitute in the expression

$504 = B \cdot A^5$ and $4{,}032 = B \cdot A^{10}$. We get

$\frac{504}{A^5} = B = \frac{4{,}032}{A^{10}}$ or $A^5 = \frac{4{,}032}{504} = 8$

So $A = \sqrt[5]{8} \approx 1.5157$

pts: /10

and $B = \frac{504}{8} = 63$

14. Use an optimization technique to find two positive numbers whose product is 324 and whose sum is a minimum.

You must justify your answer to receive credit for this problem.

Let x and y be positive numbers such that $x \cdot y = 324$. So $y = \frac{324}{x}$

We want to minimize the sum of $x + y$, or $S = x + \frac{324}{x}$

on $0 < x < +\infty$.

Let's compute S' and study the intervals of increase and decrease for S:

$$S' = 1 - \frac{324}{x^2} = \frac{x^2 - 324}{x^2} = 0$$

$$\Longleftrightarrow x^2 - 324 = 0 \Longleftrightarrow x = \sqrt{324} = 18$$

as x is positive

test 1 test 20

sign of S': 0 - - - - - 18 + + + +

$S'(1) = -323$ $S'(20) = 0.19$

thus S is decreasing on $(0, 18)$ and increasing on $(18, \infty)$. Thus $\boxed{x = y = 18}$ minimize the sum.

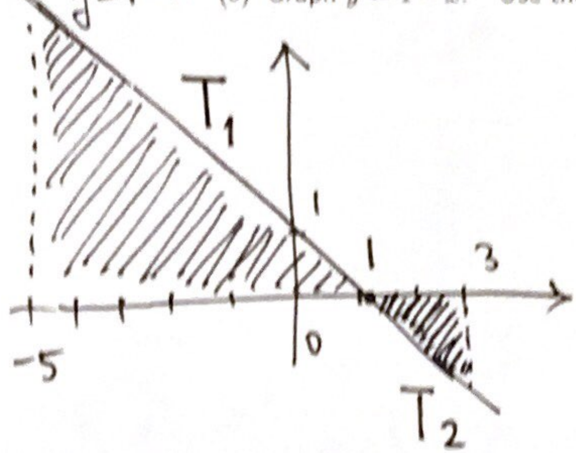
pts: /10

15. (*a*) Evaluate $\int_1^2 (12x^2 + 12x^{-2})\,dx$. by FTC - part II

$$= 4x^3 - \frac{12}{x}\Big]_1^2 = \left(4\cdot 8 - \frac{12}{2}\right) - (4 - 12)$$

$$= 32 - 6 - 4 + 12 = \boxed{34}$$

(*b*) Graph $y = 1 - x$. Use the graph and a geometric argument to evaluate $\int_{-5}^{3} (1 - x)\,dx$.



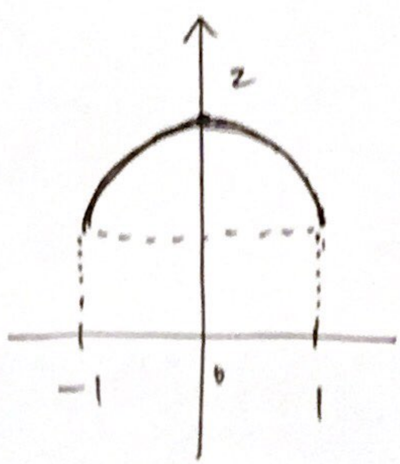
$$= \text{area } T_1 - \text{area } T_2$$

$$= \frac{6\cdot 6}{2} - \frac{2\cdot 2}{2} =$$

$$= 18 - 2 = \boxed{16}$$

(*c*) Graph $y = 1 + \sqrt{1 - x^2}$. Use the graph and a geometric argument to evaluate

$\int_{-1}^{1} (1 + \sqrt{1 - x^2})\,dx$. =



= area of rectangle + area of semicircle with radius 1

$$= 2\cdot 1 + \frac{1}{2}(\pi\cdot 1^2)$$

$$= \boxed{2 + \pi/2 \approx 3.57}$$

pts: /10

16. Use the Fundamental Theorem of Calculus (part 1) to evaluate the following derivatives:

(*a*) If $F(x) = \int_x^{-1} \sqrt{u^3+1}\, du$ then $F'(x)$ equals:

$$F(x) = -\int_{-1}^{x} \sqrt{u^3+1}\, du$$

by FTC - part I we have

$$\frac{d}{dx} F(x) = \frac{d}{dx}\left(-\int_{-1}^{x} \sqrt{u^3+1}\, du\right) = \boxed{-\sqrt{x^3+1}}$$

(*b*) If $G(x) = \int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\, ds$ then $\frac{dG}{dx}$ equals:

We need to use FTC - part I and the Chain Rule (Leibniz formula)

$u = \sqrt{x}$

$$\frac{dG}{dx} = \frac{d}{dx}\left(\int_1^{u} \frac{s^2}{s^2+1}\, ds\right) = \left(\frac{d}{du}\int_1^{u} \frac{s^2}{(s^2+1)}\, ds\right) \cdot \frac{du}{dx}$$

chain rule

$$= \frac{u^2}{u^2+1} \cdot \frac{du}{dx} = \frac{(\sqrt{x})^2}{(\sqrt{x})^2+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \boxed{\frac{x}{x+1} \cdot \frac{1}{2\sqrt{x}}}$$

pts: /10

Bonus. Find the area bounded by the function $y = 2 - x - x^2$ and the x-axis.

You should sketch the graph of the region.

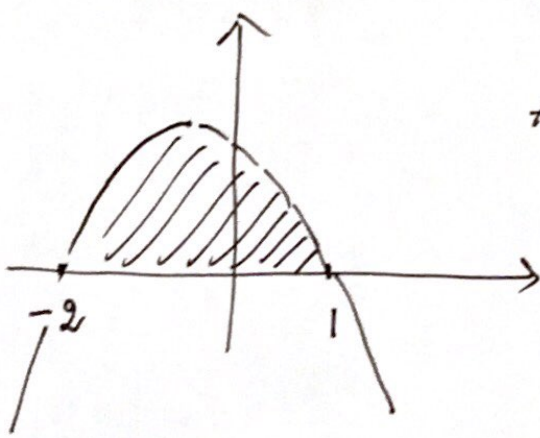
the graph of the function $y = 2 - x - x^2$ is a parabola that opens down.

The y-intercept is $y = 2$ (when $x = 0$)

The x-intercepts are obtained by solving

$$2 - x - x^2 = 0 \quad \text{or} \quad x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0 \quad \text{Hence} \quad x = -2, 1$$



$$\text{Area} = \int_{-2}^{1} (2 - x - x^2)\, dx$$

$$= 2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-2}^{1}$$

by the FTC - part II

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(2(-2) - \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3\right)$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3} = 8 - 3 - \frac{1}{2} = \boxed{4.5}$$

-3

pts: /10