MA 137	**Fall 2021**	**Name:** Answer Key
Calculus I with Life Science Applications	12/13/2021	
FINAL EXAM		**Sect. #:** ______________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 12 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ b c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of four open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

GOOD LUCK!

1. ■ b c d e

2. a ■ c d e

3. a ■ c d e

4. a b ■ d e

5. a b c d ■

6. a b c ■ e

7. a b ■ d e

8. a ■ c d e

9. a b c ■ e

10. a b c d ■

11. a b c ■ e

12. ■ b c d e

QUESTION	SCORE	OUT OF
Multiple Choice		60 pts
13.		10 pts
14.		10 pts
15.		10 pts
16.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Time/Lecture Location	Lecturer
001-010	MWF 10:00 am - 10:50 am, CB 106	Alberto Corso
Section #	**Time/ Recitation Location**	**TA**
001	TR 08:00-08:50 AM, CB 339	Nicholas Arsenault
002	TR 09:00-9:50 AM, CB 339	
003	TR 10:00-10:50 AM, CB 339	Katherine (Kat) Henneberger
004	TR 11:00-11:50 AM, CB 339	
005	TR 12:00-12:50 PM, CB 339	Faith Hensley
006	TR 01:00-01:50 PM, CB 339	
007	TR 12:00-12:50 PM, CB 341	Michael Morrow
008	TR 01:00-01:50 PM, CB 341	
009	TR 02:00-02:50 PM, CB 339	Karen Reed
010	TR 03:00-03:50 PM, CB 339	

1. Let $B(t)$ denote the number of bacteria at time t (measured in hours) in a certain culture. The population grows exponential, thus

$$B(t) = B_0 e^{rt}$$

for some positive constants B_0 and r.

Suppose the population has a doubling time of 3 hours. Find the rate of growth r.

B_0 = initial population of bacteria

If T_d is the doubling time, we have

$\underline{2B_0} = B(T_d) = \underline{B_0} e^{rT_d}$ If we simplify B_0

we obtain $2 = e^{rT_d} \iff$

$\ln 2 = \ln(e^{rT_d}) = rT_d$

$\therefore \; r = \dfrac{\ln 2}{T_d}$ = in our case = $\dfrac{\ln 2}{3}$

Possibilities:

(a) $\boxed{r = (\ln 2)/3}$

(b) $r = 3/\ln 2$

(c) $r = 1/2$

(d) $r = 1/3$

(e) $r = \ln(2/3)$

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2. Let

$$f(x) = \begin{cases} -2x + c & x \leq 1 \\ 3x^2 + 2x + 3 & x > 1 \end{cases}$$

For what value(s) of c is this function continuous?

The function is continuous for all $x \neq 1$ as it is given by polynomials. We need to make sure that f is continuous at $x = 1$. Thus we need

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

Hence $\lim_{x \to 1^-} (-2x + c) = \lim_{x \to 1^+} (3x^2 + 2x + 3)$

$\iff -2 + c = 8$ OR $\boxed{c = 10}$

Possibilities:

(a) $c = 8$

(b) $\boxed{c = 10}$

(c) $c = 12$

(d) $c = 14$

(e) $c = 16$

3. Find a function f and a number a so that the following limit represents a derivative $f'(a)$

$$\lim_{h\to 0}\frac{\ln(e+h)-1}{h}$$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Hence $f(x)=\ln x$ and $a=e$

(notice that $\ln e = 1$)

Possibilities:

(a) $f(x)=\ln(x)\quad a=1$

$\boxed{\text{(b) } f(x)=\ln(x)\quad a=e}$

(c) $f(x)=\ln(x+e)\quad a=1$

(d) $f(x)=\ln(x+1)\quad a=e$

(e) None of the above.

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4. Find the linearization of $f(x)=\dfrac{x}{1+x^2}$ at $x=2$.

We recall that $L(x)=f(a)+f'(a)\cdot(x-a)$

Hence: $\boxed{f(2)=\frac{2}{5}}$; $f'(x)=\dfrac{1\cdot(1+x^2)-x(2x)}{(1+x^2)^2}$

$$=\frac{1-x^2}{(1+x^2)^2}$$

$$\boxed{f'(2)=-3/25}$$

Possibilities:

(a) $L(x)=\dfrac{2}{5}$

(b) $L(x)=\dfrac{3}{25}x-\dfrac{1}{25}$

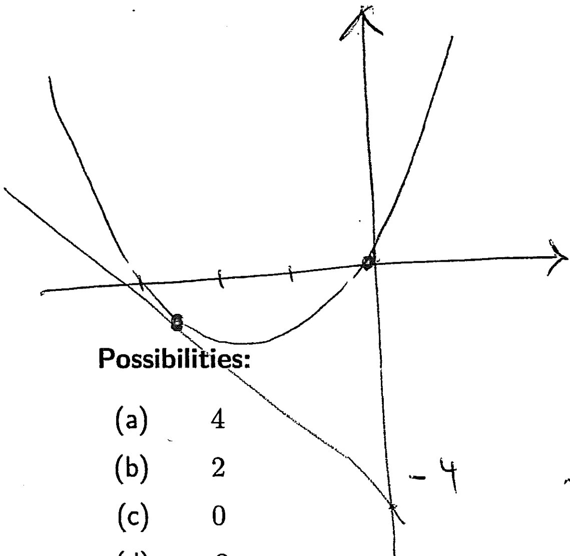
$\boxed{\text{(c) } L(x)=-\dfrac{3}{25}x+\dfrac{16}{25}}$

(d) $L(x)=x+2$

(e) $L(x)=\dfrac{2}{5}x$

$\therefore L(x)=\dfrac{2}{5}-\dfrac{3}{25}(x-2)=-\dfrac{3}{25}x+\dfrac{16}{25}$

5. In the xy-plane, the line $x + y = k$, where k is a constant, is tangent to the graph of $y = x^2 + 3x$. What is the value of k?



the tangent line is $y = -x + k$ and has slope "-1". Thus we need to find the point on the parabola with derivative ($\equiv$ slope) -1!

$f'(x_0) = 2x_0 + 3 = -1 \iff x_0 = -2$ $\quad P(-2, -2)$

as $f(-2) = (-2)^2 + 3(-2) = -2$ Thus k is $(-2) + (-2) = k$

$\boxed{k = -4}$

Possibilities:

(a) 4

(b) 2

(c) 0

(d) -2

$\boxed{\text{(e)} \quad -4}$

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6. Suppose a function $y = y(x)$ is implicitly defined by the equation

$$x^2 + xy + y^2 = 1.$$

Find a formula for dy/dx in terms of x and y.

$$\frac{d}{dx}\left[x^2 + xy + y^2\right] = \frac{d}{dx}[1]$$

$$2x + \underbrace{1 \cdot y + x \cdot \frac{dy}{dx}}_{\text{product rule}} + \underbrace{2y\frac{dy}{dx}}_{\text{chain rule}} = 0$$

$$(x + 2y)\frac{dy}{dx} = -2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{-2x - y}{x + 2y}}$$

Possibilities:

(a) $\dfrac{dy}{dx} = \dfrac{-(2x+y)}{x+y}$

(b) $\dfrac{dy}{dx} = \dfrac{-2x}{x+2y}$

(c) $\dfrac{dy}{dx} = \dfrac{2x+y}{x+2y}$

$\boxed{\text{(d)} \quad \dfrac{dy}{dx} = \dfrac{-(2x+y)}{x+2y}}$

(e) None of the above

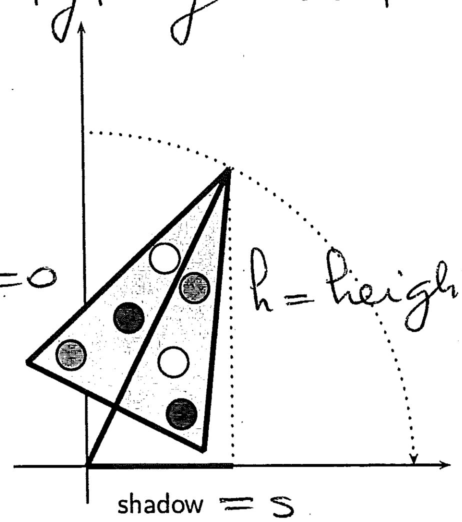
When $h=30$ observe that $s=\sqrt{50^2-30^2}=\underline{\underline{40}}$

7. On Christmas Eve the full moon is directly overhead the town of Whoville, and its 50-foot-tall outdoor Christmas tree. The Grinch wants to ruin the Whos' Christmas so he decides to cut down this Christmas tree. We are watching the dramatic scene of this majestic tree falling over. Since we are mathematically inclined, we observe that at the moment when the top of the tree is 30 feet from the ground it is falling at the rate of 4 feet/sec. At that moment, how rapidly is the shadow of the tree, cast by the moon, lengthening?

s = length of shadow; h = height of top of tree when falling. By Pythagoras's Thm $s^2+h^2=50^2$, thus

$$\frac{d}{dt}\left[s^2+h^2\right]=\frac{d}{dt}\left(50^2\right) \quad \text{OR}$$

$$2s\frac{ds}{dt}+2h\frac{dh}{dt}=0$$



h = height

Possibilities:

(a) 1 feet per second

(b) 2 feet per second

(c) 3 feet per second

(d) 4 feet per second

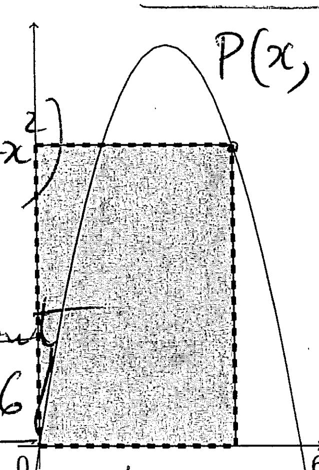
(e) 5 feet per second

$$\boxed{\frac{ds}{dt}=-\frac{h}{s}\frac{dh}{dt}}$$

$h=30$; $s=40$ $\quad \frac{dh}{dt}=-4\,\text{ft/sec}$ $\quad \frac{ds}{dt}=-\frac{30}{40}(-4)=\boxed{3\,\text{ft/sec}}$

8. Consider a rectangle that lies in the first quadrant with one vertex at the origin and two of the sides along the coordinates axis. The fourth vertex lies on the parabola $y=6x-x^2$. Find the area of the largest such rectangle.

The area $A(x)$ of a typical rectangle is: $A(x)=x\cdot(6x-x^2)$
$$=\underline{6x^2-x^3}$$



$P(x, 6x-x^2)$

Since it is in the first quadrant x is such that $\boxed{0\le x\le 6}$

Possibilities:

(a) 40

(b) 32

(c) 27

(d) 3

(e) 0

The global max exists by the Extreme Value Theorem. $A'(x)=12x-3x$

So the critical pts: $x=0,4$ $\quad A(0)=0$

$A(6)=0$ and $A(4)=32$

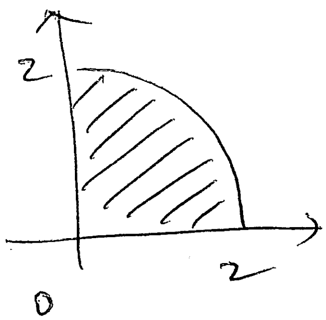
$y = f(x) = \sqrt{4-x^2} \iff x^2+y^2=4$

circle centered at the origin with radius 2

9. Use a geometric argument to evaluate

$$\int_0^2 \sqrt{4-x^2}\,dx$$

We are dealing with a quarter circle of radius 2 in the first quadrant



$$\int_0^2 \sqrt{4-x^2}\,dx = $$

$$= \frac{1}{4}\left[\pi(2)^2\right] = \boxed{\pi}$$

Possibilities:

(a) 4π

(b) 3π

(c) 2π

(d) π

(e) 0

---

10. Let $f(x) = 2x^2 - x^3$. Find the largest open interval on which f is increasing.

We need to find where $f'(x)$ is positive

$f'(x) = 4x - 3x^2 = x(4-3x)$

$f'(x) = 0 \iff x = 0$ OR $x = 4/3$

sign of $f'(x)$: $---\ \underset{0}{|}\ ++++\ \underset{4/3}{|}\ -----$

Possibilities:

(a) $(-2, 0)$

(b) $(0, 2)$

(c) $(-\infty, 2/3)$

(d) $(2/3, \infty)$

(e) $(0, 4/3)$

f is increasing on $(0, 4/3)$

11. Suppose $G(x) = \int_1^{x^2} t^2 e^t \, dt$. Find $G'(2)$.

$$\frac{d}{dx} G(x) = \frac{d}{dx}\left(\int_1^{x^2} t^2 \cdot e^t \, dt\right) = (x^2)^2 \cdot e^{x^2} \cdot (2x)$$

by the chain rule

Possibilities:

(a) $16e^2$

(b) $64e^2$

(c) $16e^4$

$\boxed{\text{(d)} \quad 64e^4}$

(e) $96e^8$

$$= \underline{2x^5 e^{x^2}}$$

Hence $G'(2) = 2 \cdot 2^5 \cdot e^{2^2}$

$$= 64e^4$$

12. Suppose h is a continuous function such that

$h(1) = -2$	$h'(1) = 2$	$h''(1) = 2$
$h(2) = 6$	$h'(2) = 5$	$h''(2) = 13$

and h'' is continuous everywhere.

Evaluate: $\int_1^2 h''(t) \, dt$.

By the fundamental theorem of calculus Part II we have

$$\int_1^2 h''(t) \, dt = h'(t)\Big|_1^2 =$$

Possibilities:

$\boxed{\text{(a) } 3}$

(b) 4

(c) 7

(d) 8

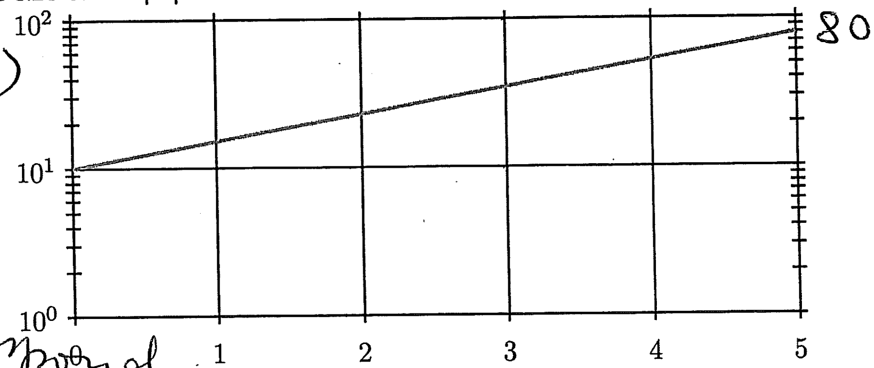
(e) 11

$$= h'(2) - h'(1)$$

$$= 5 - 2 = 3$$

$\boxed{y = 10 \cdot 8^{x/5} = 10 \cdot 1.52^{x}}$

13. (*a*) The graph below is a semilog plot for the size of a population of fruit flies as a function of time. What function describes the size of the population as a function of time?



Points (actual value)
(0,10) & (5,80)

A line in a semilog plot corresponds to an actual exponential relation $y = B \cdot A^x$. The 2 set of values must satisfy the condition; so $10 = B \cdot A^0$ or $B = 10$; $80 = 10 \cdot A^5 \Rightarrow A = \sqrt[5]{8}$ OR $\boxed{y = 10 \cdot 8^{x/5}}$

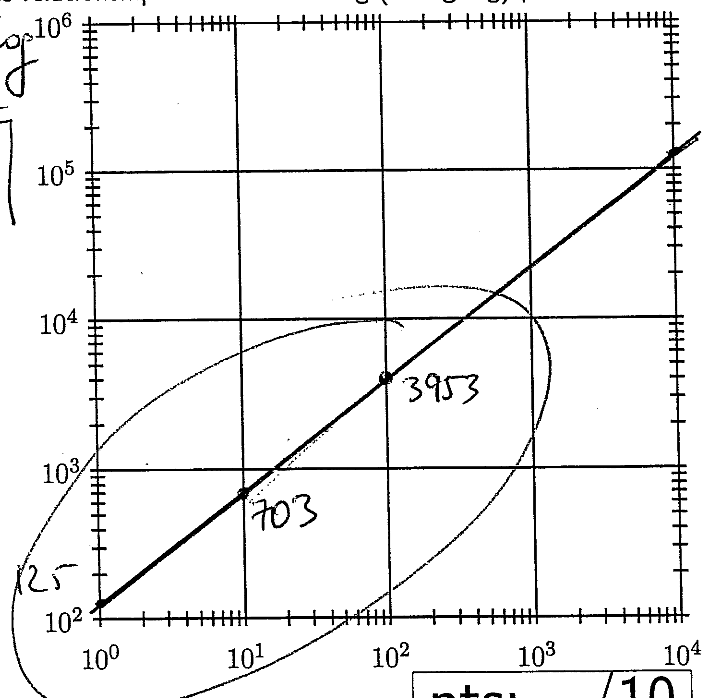
OR $= 10 \cdot 1.52^x$

(*b*) The number of calories an active adult dog requires per day is determined by the mass of the dog. The function that describes this relationship is $C = 125 \cdot M^{3/4}$, where C is the number of kilocalories per day and M is the weight of the dog in kilograms (source: www.dogfoodadvisor.com). Plot this relationship on the double-log (or log-log) plot below[1].

$C = 125 \cdot M^{3/4}$ in the log-log plot become the line

$\boxed{\log C = \frac{3}{4} \log M + \log 125}$

So it has slope $3/4$ and intercept at $\log 125$



pts: /10

[1]The plot is meaningful, say, for the values $1 \leq M \leq 100$. For ease of plotting, we consider however the larger log-log plot above.

14. Use l'Hospital's rule to evaluate the following limits:

(a) $\lim_{x \longrightarrow 2} \dfrac{x^2 - 8x + 12}{4 - x^2} = \dfrac{4 - 16 + 12}{4 - 4} = \dfrac{0}{0}$

Use l'Hôpital's Rule

$= \lim_{x \to 2} \dfrac{2x - 8}{-2x} = \dfrac{4-8}{-4} = \dfrac{-4}{-4} = \boxed{1}$

(b) $\lim_{x \longrightarrow 0} \dfrac{\int_0^x e^h \, dh}{x^2} = \dfrac{\int_0^0 e^h \, dh}{0^2} = \dfrac{0}{0}$

We can use l'Hôpital's rule

$= \lim_{x \to 0} \dfrac{\frac{d}{dx} \int_0^x e^h \, dh}{\frac{d}{dx} x^2} = \lim_{x \to 0} \dfrac{e^x}{2x} = \dfrac{1}{0}$

hence the limit does not exist

$\left[\lim_{x \to 0^+} \dfrac{e^x}{2x} = +\infty \, ; \, \lim_{x \to 0^-} \dfrac{e^x}{2x} = -\infty \right]$

pts: /10

15. (*a*) Compute the indefinite integral $\displaystyle\int \frac{2x^2 - x}{\sqrt{x}}\,dx$

$$\int \left(\frac{2x^2}{\sqrt{x}} - \frac{x}{\sqrt{x}}\right)dx = \int \left(2x^{2-\frac{1}{2}} - x^{1-\frac{1}{2}}\right)dx$$

$$= \int \left(2x^{3/2} - x^{1/2}\right)dx = 2\,\frac{1}{5/2}\,x^{3/2+1} - \frac{1}{3/2}\,x^{1/2+1} + C$$

$$\boxed{= \frac{4}{5}x^{5/2} - \frac{2}{3}x^{3/2} + C}$$

OR $$= \frac{4}{5}x^2\sqrt{x} - \frac{2}{3}x\sqrt{x} + C$$

(*b*) Solve the initial value problem $$\frac{dW}{dt} = e^{-3t}$$

for $t \geq 0$ and $W(0) = 2$.

$$W(t) = \int e^{-3t}\,dt = -\frac{1}{3}e^{-3t} + C$$

We determine C using $W(0) = 2$

$$2 = W(0) = -\frac{1}{3}e^{-3\cdot 0} + C$$

$$\therefore\ 2 = -\frac{1}{3} + C \quad \text{or} \quad C = \frac{7}{3}$$

$$\boxed{W(t) = \frac{1}{3}\left(7 - e^{-3t}\right)}$$

pts: /10

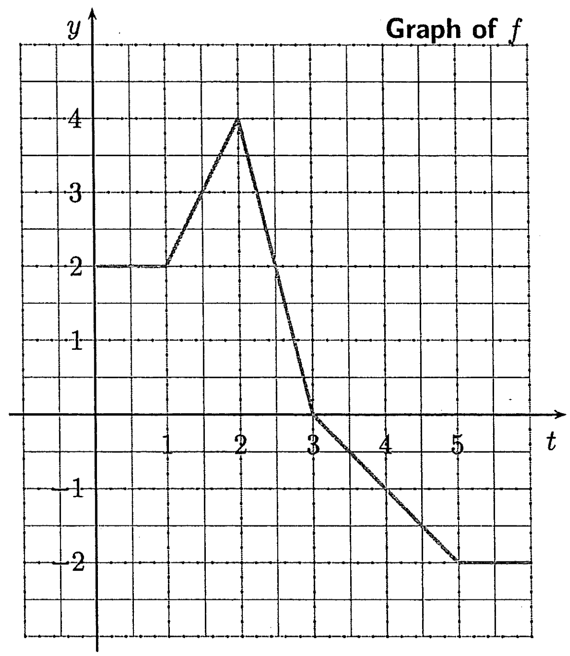
16. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown below.

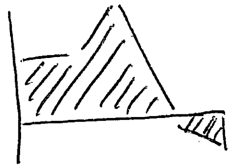
(i) Determine the following values of this new function g:

$g(0) =$ 0 $\qquad g(1) =$ 2

$g(2) =$ 5 $\qquad g(3) =$ 7

$g(4) =$ $13/2$ $\qquad g(5) =$ 5



$\underline{\text{e.g.}}$: $g(4) =$ area 

$= 7 - \frac{1}{2} = \frac{13}{2}$

(ii) Find the derivative of the function g and determine the x value at which g attains its maximum value.

$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_0^x f(t)\,dt = \underline{\underline{f(x)}}$$

by FTC Part I

sign of $g'(x)$ $\qquad$ $+ + + + \quad \overset{}{\underset{3}{|}} \quad - - - -$ $\qquad$ (sign of $f(x)$)

Thus $g(x)$ has a maximum at $x = 3$

pts: $\quad$ /10

Bonus. (*a*) Find both fixed points for the recursive sequence: $a_{n+1} = \dfrac{a_n^2 + 8}{3a_n}$.

$$a = \frac{a^2+8}{3a} \iff 3a^2 = a^2 + 8 \iff 2a^2 = 8$$

$$\iff a^2 = 4 \iff \hat{a} = \pm 2 \text{ fixed pts}$$

(*b*) What does the Stability Criterion say about the fixed points found in (*a*)?

It is given that $f'(x) = \dfrac{x^2 - 8}{3x^2}$ is the derivative of $f(x) = \dfrac{x^2+8}{3x}$.

$$f'(\pm 2) = \frac{(\pm 2)^2 - 8}{3(\pm 2)^2} = -1/3$$

$|f'(\pm 2)| = 1/3 < 1$ hence both are locally stable

(*c*) Given the initial value $a_0 = 1.3$, compute a_1, a_2, $a_3, a_4,$ and a_5:

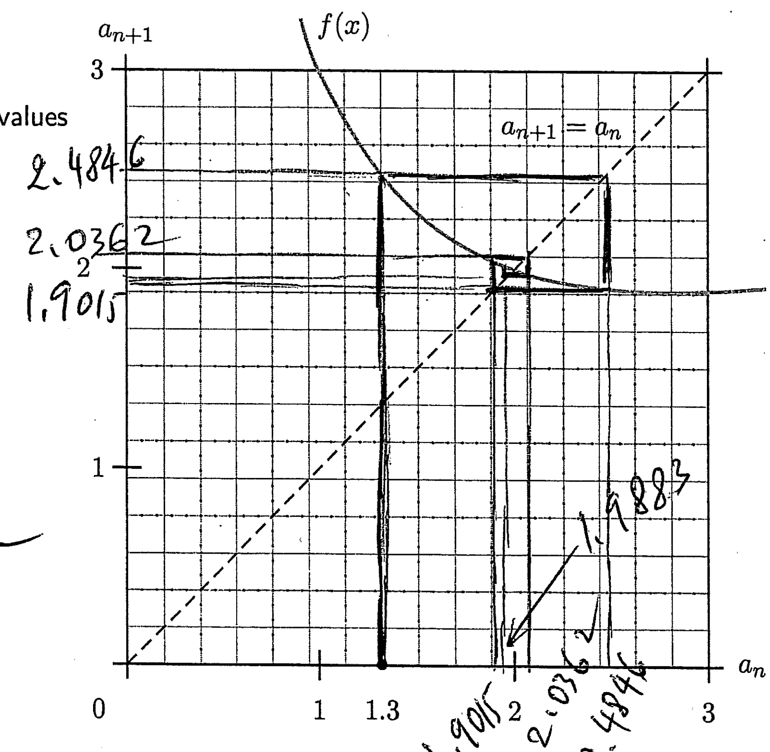
$a_1 =$ 2.4846 $\qquad a_2 =$ 1.9015 $\qquad a_3 =$ 2.0362

$a_4 =$ 1.9883 $\qquad a_5 =$ 2.0039

(*d*) Sketch a cobweb graph starting at $a_0 = 1.3$ on the given plot.

Make sure to mark at least the values $a_1, a_2, a_3, a_4,$ and a_5 found in (*c*).

Use it to determine $\lim_{n\to\infty} a_n$.



$$\lim_{n\to\infty} a_n = 2$$

notice a spiral behaviour around the fixed point as $f'(2) = -1/3$ is negative

pts: /10