ADDITIVE POWER OPERATIONS IN EQUIVARIANT COHOMOLOGY

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Abstract. Let G be a finite group and E be an H_∞-ring G-spectrum. For any G-space X and positive integer m, we give an explicit description of the smallest Mackey ideal $\underline{J}$ in $\underline{E}^0(X \times B\Sigma_m)$ for which the reduced mth power operation $\underline{E}^0(X) \longrightarrow \underline{E}^0(X \times B\Sigma_m)/\underline{J}$ is a map of Green functors. We obtain this result as a special case of a general theorem that we establish in the context of $G \times \Sigma_m$-Green functors. This theorem also specializes to characterize the appropriate ideal $\underline{J}$ when E is an ultra-commutative global ring spectrum. We give example computations for the sphere spectrum, complex K-theory, and Morava E-theory.

Contents

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1. Introduction

An H_∞-ring structure on a spectrum E gives rise to power operations on the E-cohomology of any space. These operations have played an important role in both our theoretical and our computational understanding of essentially all naturally-occurring cohomology theories. The most useful power operations, Steenrod operations and Adams operations, are both additive power operations. Additive power operations are all built from the universal additive power operation

$$P_m\colon E^0(X) \to E^0(X \times B\Sigma_m)/I_{\mathrm{Tr}}, \tag{1.1}$$

where Σ_m is the symmetric group and I_{Tr} is a specific transfer ideal that can be defined naturally for any spectrum E. Given the effectiveness of these additive power operations, it is desirable to understand their analogues in other contexts. In this paper we focus on the case of equivariant cohomology theories. In particular, in Section 2 we study Borel equivariant cohomology theories, and in Section 3 we tackle genuine and global equivariant cohomology theories.

For any finite group G, the G-Mackey functor of coefficients of a genuine G-spectrum E is given by the formula

$$G/H \mapsto \underline{E}^0(G/H) = [G/H_+, E]^G.$$

More generally, given a G-space X, the E-cohomology of X is a G-Mackey functor by the formula

$$G/H \mapsto \underline{E}^0(X)(G/H) = E^0(G/H \times X) = [(G/H \times X)_+, E]^G$$

If we further assume that E is equipped with the structure of an H_∞-ring in the category of genuine G-spectra, then the associated power operation is a map

$$P_m\colon \underline{E}^0(G/H) \to \underline{E}^0(B\Sigma_m)(G/H),$$

where $B\Sigma_m$ is a G-space with trivial action. In this case, the Mackey functors $\underline{E}^0$ and $\underline{E}^0(B\Sigma_m)$ are both G-Green functors, so both $\underline{E}^0(G/H)$ and $\underline{E}^0(B\Sigma_m)(G/H)$ are commutative rings. The map P_m is multiplicative, but not additive, and it does not respect the induction maps in the G-Mackey functors. The additivity of P_m reduces to a classical problem, which was solved for spectra in complete generality. [BMMS86, Proposition VIII.1.4(iv)] identifies an ideal $I_{\mathrm{Tr}} \subseteq \underline{E}^0(B\Sigma_m)(G/H)$, generated by the image of the transfer maps $\underline{E}^0(B\Sigma_i \times \Sigma_j)(G/H) \to \underline{E}^0(B\Sigma_m)(G/H)$ for $i, j > 0$ with $i + j = m$, with the property that the composite

$$P_m/I_{\mathrm{Tr}}\colon \underline{E}^0(G/H) \to \underline{E}^0(B\Sigma_m)(G/H) \to \underline{E}^0(B\Sigma_m)(G/H)/I_{\mathrm{Tr}}$$

is a map of commutative rings that respects the restriction maps in the G-Mackey functor structure. However, these maps do not necessarily respect the induction maps in the G-Mackey functor structure. The goal of this paper is to identify and study the minimal Mackey ideal $\underline{J} \subseteq \underline{E}^0(B\Sigma_m)$ so that the composite

$$P_m/\underline{J}\colon \underline{E}^0 \to \underline{E}^0(B\Sigma_m) \to \underline{E}^0(B\Sigma_m)/\underline{J}$$

is a map of Green functors.

The ideal $\underline{J}(G/H) \subseteq \underline{E}^0(B\Sigma_p)(G/H)$ is built out of transfer maps. Given a surjective map of finite $G \times \Sigma_m$-sets $X \to Y$, applying homotopy orbits for the Σ_m-action gives a cover of G-spaces $X_{h\Sigma_m} \to Y_{h\Sigma_m}$. This gives rise to a transfer map in E-cohomology

$$\mathrm{Tr}\colon E^0(X_{h\Sigma_m}) \to E^0(Y_{h\Sigma_m}).$$

If $Y = G/H$, with trivial Σ_m-action, then the target of this transfer map is $E^0(G/H \times B\Sigma_m) = \underline{E}^0(B\Sigma_m)(G/H)$.

In the case that $m = p$ is a prime, the ideal $\underline{J}(G/H) \subseteq \underline{E}^0(B\Sigma_p)(G/H)$ is defined to be the ideal generated by the transfer maps induced by the maps of $G \times \Sigma_m$-sets

(i) $(G \times \Sigma_p)/(H \times \Sigma_i \times \Sigma_j) \to G/H$ for $i + j = p$ and $i, j > 0$ and
(ii) $(G \times \Sigma_p)/\Gamma(a) \to G/H$, for all subgroups $S \leq H$ and homomorphisms $a\colon S \to \Sigma_p$ with image containing a p-cycle, where $\Gamma(a) \subseteq G \times \Sigma_p$ is the graph subgroup of a.

By construction, I_{Tr} is contained in $\underline{J}(G/H)$, and $\underline{J}$ is natural in the cohomology theory E. The following result is the special case of the main theorems of this paper when E is an H_∞-ring G-spectra and when $m = p$. It makes use of Proposition 3.22 and is a special case of Theorem 3.24 and Corollary 3.35.

Proposition. *Assume that E is an H_∞-ring in genuine G spectra. The ideals $\underline{J}(G/H)$, defined above, assemble to a Mackey ideal $\underline{J} \subseteq \underline{E}^0(B\Sigma_p)$, minimal with the property that the composite*

$$P_p/\underline{J}\colon \underline{E}^0 \to \underline{E}^0(B\Sigma_p)/\underline{J}$$

is a map of G-Green functors.

In fact, this result holds much more generally. Let $\underline{R}$ be a $G \times \Sigma_m$-Green functor. From $\underline{R}$ we may form the induced G-Green functor $\underline{R}{\uparrow}^G_{G\times\Sigma_m}$ given by the formula

$$\underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/H) = \underline{R}\big((G \times \Sigma_m)/(H \times \Sigma_m)\big).$$

As $\underline{R}$ is a $G \times \Sigma_m$-Green functor, it may be viewed as a functor from the category of finite $G \times \Sigma_m$-sets to commutative rings that admits transfers along surjections. In this situation, we define $\underline{J}(G/H) \subseteq \underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/H)$ to be the ideal generated by the images of certain transfer maps generalizing the maps in (i) and (ii). These maps are described explicitly in Section 3.2 and make use of a group extension of $\Gamma(a)$ by a product of symmetric groups described in Section 2.2. The following result is Theorem 3.24.

Theorem. *Let $\underline{R}$ be a $G\times\Sigma_m$-Green functor. The ideals $\underline{J}(G/H) \subseteq \underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/H)$ assemble into a G-Mackey ideal $\underline{J} \subseteq \underline{R}{\uparrow}^G_{G\times\Sigma_m}$.*

When E is a homotopy commutative ring G-spectrum, we get a $G \times \Sigma_m$-Green functor $\underline{R}$ via the formula

$$\underline{R}\big((G \times \Sigma_m)/\Lambda\big) = E^0\Big(\big((G \times \Sigma_m)/\Lambda\big)_{h\Sigma_m}\Big).$$

In this case, the restriction of $\underline{R}$ to G satisfies $\underline{R}{\downarrow}^{G\times\Sigma_m}_G = \underline{E}^0$, the induced G-Green functor satisfies $\underline{R}{\uparrow}^G_{G\times\Sigma_m} = \underline{E}^0(B\Sigma_m)$ and, when $m = p$, the Mackey ideals called $\underline{J}$ in the proposition and theorem above agree.

Global equivariant homotopy theory furnishes us with further examples of $G \times \Sigma_m$-Green functors. If E is a homotopy commutative ring global spectrum in the sense of [Sch18], there is an associated $G \times \Sigma_m$-functor $\underline{R}$ given by

$$\underline{R}((G \times \Sigma_m)/\Lambda) = \pi_0^\Lambda E.$$

In this case, the induced G-Green functor is given by

$$\underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/H) = \pi_0^{H\times\Sigma_m} E.$$

The above theorem furnishes us with a Mackey ideal $\underline{J} \subseteq \underline{R}{\uparrow}^G_{G\times\Sigma_m}$. The restricted G-Green functor satisfies $\underline{R}{\downarrow}^{G\times\Sigma_m}_G(G/H) = \pi_0^H E$.

In case E is either an H_∞-ring in genuine G-spectra or an ultra-commutative global spectrum, then the mth power operation is a map

$$P_m\colon \underline{R}{\downarrow}^{G\times\Sigma_m}_G(G/H) \to \underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/H).$$

In Section 3.3, we introduce the notion of an mth total power operation on a $G \times \Sigma_m$-Green functor that captures the mth power operation in each of these examples. Although both the source and target of P_m are Green functors, the operation P_m is not a map of Green functors before passing to a quotient. The proposition above is then a special case of the following theorem (see Corollary 3.35 and Corollary 3.41).

Theorem. *Let E be an H_∞-ring in genuine G-spectra or an ultra-commutative global spectrum and let $\underline{R}$ be the associated $G \times \Sigma_m$-Green functor defined above. The composite*

$$P_m/\underline{J}\colon \underline{R}{\downarrow}_G^{G\times\Sigma_m} \longrightarrow \underline{R}{\uparrow}_{G\times\Sigma_m}^G \longrightarrow \underline{R}{\uparrow}_{G\times\Sigma_m}^G/\underline{J}$$

is a map of G-Green functors.

The general case of a $G \times \Sigma_m$-Green functor with mth power operation is treated in Theorem 3.30.

Since Borel equivariant cohomology theories are examples of genuine equivariant cohomology theories, if E is an ordinary H_∞-ring spectrum, then the proposition above may be applied to the Borel equivariant cohomology theory associated to E. However, in this setting, it is also natural to ask for the smallest ideal with the property that the transfer from a specific subgroup commutes with the power operation after taking the quotient by the ideal. If $H \subseteq G$, then $BH \to BG$ is equivalent to a finite cover of spaces. We would like the smallest ideal $J_H^G \subseteq E^0(BG \times B\Sigma_m)$ such that the following diagram commutes

$$\begin{tikzcd}
E^0(BH) \arrow[r] \arrow[d, "\mathrm{Tr}"'] & E^0(BH \times B\Sigma_m)/I_{\mathrm{Tr}} \arrow[d, "\mathrm{Tr}"] \\
E^0(BG) \arrow[r] & E^0(BG \times B\Sigma_m)/J_H^G.
\end{tikzcd}$$

Specializing the case of genuine $G \times \Sigma_m$-spectra to Borel equivariant $G \times \Sigma_m$-spectra, the $G \times \Sigma_m$-Green functor associated to E is given by the formula

$$\underline{R}((G \times \Sigma_m)/\Lambda) = E^0\Big(\big((G \times \Sigma_m)/\Lambda\big)_{hG\times\Sigma_m}\Big) \cong E^0(B\Lambda)$$

and $\underline{R}{\uparrow}_{G\times\Sigma_m}^G(G/H) \cong E^0(BH \times B\Sigma_m)$. For the case where H is a normal subgroup of G, in Section 2.4 we explicitly describe a subset of the transfer maps that go into the construction of $\underline{J}(G/G) \subseteq \underline{R}{\uparrow}_{G\times\Sigma_m}^G(G/G)$ and show that J_H^G is the sub-ideal generated by the image of this subset. As a consequence of this description of J_H^G, we learn that, when H is normal, if m and $|G/H|$ are relatively prime, then $J_H^G = I_{\mathrm{Tr}} \subseteq E^0(BG \times B\Sigma_m)$.

1.1. Conventions.

- By G, we will always mean a finite group.
- By a *graph subgroup* $\Gamma \leq G \times \Sigma_m$, we will mean a subgroup such that $\Gamma \cap \Sigma_m = \{e\}$. Such a subgroup is the graph of a homomorphism $K \longrightarrow \Sigma_m$, where $K = \pi_G(\Gamma)$ and $\pi_G\colon G \times \Sigma_m \longrightarrow G$ is the projection.
- We will use the notation $\underline{n} = \{1, \ldots, n\}$.
- A G-spectrum will always mean in the "genuine" sense. In other words, our G-spectra are indexed over a complete G-universe.
- We will use transfer maps in cohomology throughout, and in order to help orient the reader, we always display such transfer maps in orange.
- In Section 3.2, we abbreviate an induced Mackey functor $\underline{R}{\uparrow}_{G\times\Sigma_m}^G$ to $\underline{R}{\uparrow}$.
- In Section 4, we abbreviate an induced Mackey functor $(\underline{A}_{G\times\Sigma_m}){\uparrow}_{G\times\Sigma_m}^G$ to $\underline{A}{\uparrow}_{G\times\Sigma_m}^G$.

1.2. **Organization.** We begin Section 2 by considering the Borel equivariant case. Key results about the ideal $\underline{J}$ are given in Section 2.3 and Section 2.4; these results are specialized to the case $m = p$ is prime in Section 2.5. Our main results about power operations appear in Section 3. We introduce the notion of an mth total power operation for a $G \times \Sigma_m$-Green functor in Section 3.3. One of the central results of the article, Theorem 3.24, is that $\underline{J}$ is a Mackey ideal. Section 4 gives a number of examples. We consider the sphere spectrum, KU-theory, Eilenberg-Mac Lane spectra, and height 2 Morava E-theory.

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2. The Borel equivariant case

The purpose of this section is to understand the relationship between the additive power operations for an H_∞-ring spectrum E and transfers along finite covers of the form $BH \to BG$ for $H < G$ a subgroup. In particular, the goal is to describe, as explicitly as possible, the smallest ideal

$$J_H^G \subset E^0(BG \times B\Sigma_m)$$

containing I_{Tr} such that the diagram

$$\begin{tikzcd} E^0(BH) \arrow[r, "P_m/I_{\mathrm{Tr}}"] \arrow[d] & E^0(BH \times B\Sigma_m)/I_{\mathrm{Tr}} \arrow[d] \\ E^0(BG) \arrow[r, "P_m/J_H^G"] & E^0(BG \times B\Sigma_m)/J_H^G \end{tikzcd} \tag{2.1}$$

commutes and the horizontal maps are additive, so that in particular this a commuting square of ring maps.

We will also describe the absolute ideal $J^G \subseteq E^0(BG \times B\Sigma_m)$, which is the sum of the ideals J_H^G as H varies. In terms of the notation from Section 1, J^G is what was denoted there as $\underline{J}(G/G)$. This provides the smallest ideal such that the reduced power operation P_m/J^G commutes with transfers $BH \to BG$ for all subgroups $H \leq G$.

In Section 2.1, we outline this story. In the remaining subsections, we describe explicitly the ideals J_H^G and J^G in various cases. First, in Section 2.2, we study the stabilizers for the $G \times \Sigma_m$-action on $(G/H)^m$ for $H \leq G$ and $m \geq 0$. In Section 2.3, we provide an explicit description of J^G. In Section 2.4, we provide an explicit description of J_H^G in the case where $H \trianglelefteq G$ is a normal subgroup. In Section 2.5, we provide a more concrete description in the case that m is prime. Finally, in Section 2.6, we consider the simpler case where m and $|G|$ are relatively prime.

2.1. **Overview.** We have two tasks: first, to ensure that the power operation is additive, and second, to ensure it commutes with transfer maps. As we saw in Section 1, in order for the power operation to be additive and thus a map of commutative rings, one must quotient by the ideal generated by transfers along the proper partition subgroups $G \times \Sigma_i \times \Sigma_j \subset G \times \Sigma_m$; that is, we must have $I_{\mathrm{Tr}} \subseteq J_H^G$. However, this ideal is not necessarily sufficient to make diagram (2.1) commute, as we demonstrate in Section 4. The problem boils down to the relationship between the transfer along the inclusion $H \wr \Sigma_m \subseteq G \wr \Sigma_m$ and the diagonal map $G \times \Sigma_m \to G \wr \Sigma_m$. We are particularly interested in studying the power operation that lands in the product because of the relationship to power operations for genuine equivariant cohomology theories.

In [BMMS86], the power operation $P_m\colon E^0(BH) \to E^0(BH \times B\Sigma_m)$ is defined as a composite

$$P_m\colon E^0(BH) \xrightarrow{\mathbb{P}_m} E^0(BH \wr \Sigma_m) \xrightarrow{\Delta^*} E^0(BH \times B\Sigma_m)$$

where $\mathbb{P}_m$ is the *total power operation*. The operation $\mathbb{P}_m$ is functorial on all stable maps, and thus every subgroup $H \subseteq G$ gives rise to a commutative diagram

$$\begin{tikzcd} E^0(BH) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\mathrm{Tr}"] & E^0(BH \wr \Sigma_m) \arrow[d, "\mathrm{Tr}"] \\ E^0(BG) \arrow[r, "\mathbb{P}_m"] & E^0(BG \wr \Sigma_m). \end{tikzcd} \tag{2.2}$$

After composing the bottom arrow with the map in E-cohomology induced by the diagonal $BG \times B\Sigma_m \to BG \wr \Sigma_m$, we may extend the diagram above by considering a homotopy pullback. We may do this by making use of the fact (see [Ada78, Chapter 4], for instance) that, given a homotopy pullback of spaces

$$\begin{tikzcd} Y \arrow[d] & H \arrow[l] \arrow[d] \\ X & A \arrow[l] \end{tikzcd}$$

in which $Y \to X$ is a finite cover, there is a commutative diagram

$$\begin{tikzcd} E^0(Y) \arrow[r] \arrow[d, "\mathrm{Tr}"] & E^0(H) \arrow[d, "\mathrm{Tr}"] \\ E^0(X) \arrow[r] & E^0(A), \end{tikzcd} \tag{2.3}$$

where the horizontal maps are restriction maps and the vertical maps are transfer maps. For any subgroups $H, K \subseteq G$, the homotopy pullback of the span $BH \to BG \leftarrow BK$ is equivalent to $(G/H)_{hK}$.

Making use of the isomorphism of $\Sigma_m \times G$-sets

$$(G \wr \Sigma_m) / (H \wr \Sigma_m) \cong (G/H)^{\times m},$$

we get the following proposition:

Proposition 2.4. *We have the following homotopy pullback of spaces*

$$\begin{tikzcd} BH \wr \Sigma_m \arrow[d] & (G/H)^{\times m}_{h(G \times \Sigma_m)} \arrow[l] \arrow[d] \\ BG \wr \Sigma_m & BG \times B\Sigma_m. \arrow[l] \end{tikzcd}$$

Applying E-cohomology and composing the resulting diagram (2.3) with the total power operation diagram (2.2) gives the commutative diagram

$$\begin{tikzcd} E^0(BH) \arrow[r] \arrow[d, "\mathrm{Tr}"'] & E^0((G/H)^{\times m}_{h(G \times \Sigma_m)}) \arrow[d, "\mathrm{Tr}"'] \\ E^0(BG) \arrow[r, "P_m"] & E^0(BG \times B\Sigma_m). \end{tikzcd} \tag{2.5}$$

The subset $\Delta(G/H) \coloneqq \{(gH, \ldots, gH) \mid g \in G\} \subset (G/H)^{\times m}$ is closed under the action of $G \times \Sigma_m$, and there is an equivalence

$$\Delta(G/H)_{h(G\times\Sigma_m)} \simeq BH \times B\Sigma_m.$$

Thus we have a decomposition of spaces

$$(G/H)^{\times m}_{h(G\times\Sigma_m)} \simeq \big(BH \times B\Sigma_m\big) \coprod Z^{G,H}_{h(G\times\Sigma_m)},$$

where

$$Z^{G,H} = (G/H)^{\times m} \smallsetminus \Delta(G/H). \tag{2.6}$$

Applying E-cohomology, there is an isomorphism

$$E^0((G/H)^{\times m}_{h(G\times\Sigma_m)}) \cong E^0(BH \times B\Sigma_m) \times E^0(Z^{G,H}_{h(G\times\Sigma_m)}). \tag{2.7}$$

We can obtain $E^0(BH \times B\Sigma_m)/I_{\mathrm{Tr}}$ from this product by taking the quotient by the ideal generated by transfers along $H \times \Sigma_i \times \Sigma_j \subseteq H \times \Sigma_m$ for $i, j > 0$ and $i + j = m$ and also the transfer along the component $Z^{G,H}_{h(G\times\Sigma_m)} \subseteq (G/H)^{\times m}_{h(G\times\Sigma_m)}$ (ie. the entire right factor). We thus make the following definition.

Definition 2.8. Define $J^G_H \subseteq E^0(BG \times B\Sigma_m)$ to be the ideal generated by the image of the transfers along

(i) $G \times \Sigma_i \times \Sigma_j \subseteq G \times \Sigma_m$ for $i, j > 0$ and $i + j = m$, and
(ii) the composite

$$Z^{G,H}_{h(G\times\Sigma_m)} \subseteq (G/H)^{\times m}_{h(G\times\Sigma_m)} \to BG \times B\Sigma_m. \tag{2.9}$$

The following result is then immediate from the above discussion.

Proposition 2.10. *Let $J^G_H \subseteq E^0(BG \times B\Sigma_m)$ be the ideal defined above. After taking the quotient by J^G_H, the transfer and additive power operation are compatible in the sense that the following diagram commutes:*

$$\begin{tikzcd} E^0(BH) \arrow[r, "P_m/I_{\mathrm{Tr}}"] \arrow[d, "\mathrm{Tr}"'] & E^0(BH \times B\Sigma_m)/I_{\mathrm{Tr}} \arrow[d, "\mathrm{Tr}"] \\ E^0(BG) \arrow[r, "P_m/J^G_H"] & E^0(BG \times B\Sigma_m)/J^G_H. \end{tikzcd}$$

Proof. Consider the commutative diagram (2.5). According to (2.7), the top right vertex decomposes as a product, one factor of which is the desired $E^0(BH\times B\Sigma_m)$. Thus in order for the right vertical transfer in (2.5) to factor through the projection onto $E^0(BH \times B\Sigma_m)/I_{\mathrm{Tr}}$, we must collapse the image in $E^0(BG \times B\Sigma_m)$ of the complementary factor $E^0(Z^{G,H}_{h(G\times\Sigma_m)})$ and I_{Tr}; these desiderata motivated the definition of J^G_H. □

We give a complete description of J^G_H in the case where $H \trianglelefteq G$ is a normal subgroup: the following is a direct consequence of Proposition 2.34. See Notation 2.19 for a description of the group $\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/H})$.

Theorem 2.11. *Fix a normal subgroup $H \triangleleft G$. Then $J^G_H \subseteq E^0(BG \times B\Sigma_m)$ is the ideal generated by I_{Tr} and the images of the transfers along*

$$\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/H}) \longrightarrow G \times \Sigma_m \tag{2.12}$$

for all $m = nq$ and $H < S \leq G$ with $[S : H] = n \neq 1$, where $a_{S/H}\colon S \to \mathrm{Aut}_{\mathrm{Set}}(S/H) \cong \Sigma_n$ is the action map by left multiplication.

Note that although the definition of $a_{S/H}$ depends on a choice of ordering of S/H, the choice will not affect the image of the transfer.

We also consider the related absolute ideal, to ensure compatibility with transfers from all subgroups of G.

Definition 2.13. Define $J^G \subseteq E^0(BG \times B\Sigma_m)$ to be the ideal generated by J_H^G for all $H \leq G$. More explicitly, J^G is the ideal generated by the image of the transfers along

(i) $G \times \Sigma_i \times \Sigma_j \subseteq G \times \Sigma_m$ for $i, j > 0$ and $i + j = m$, and
(ii) the composites
$$Z^{G,H}_{h(G\times\Sigma_m)} \subseteq (G/H)^{\times m}_{h(G\times\Sigma_m)} \to BG \times B\Sigma_m \tag{2.14}$$
for all $H < G$.

Proposition 2.10 implies the following.

Corollary 2.15. *Let $J^G \subseteq E^0(BG \times B\Sigma_m)$ be the ideal defined above. Taking the quotient by J^G, the additive power operation is compatible with all transfers in the sense that the following diagram commutes for all $H < G$:*

$$\begin{tikzcd} E^0(BH) \arrow[r, "P_m/I_{\mathrm{Tr}}"] \arrow[d, "\mathrm{Tr}"'] & E^0(BH \times B\Sigma_m)/I_{\mathrm{Tr}} \arrow[d, "\mathrm{Tr}"] \\ E^0(BG) \arrow[r, "P_m/J^G"] & E^0(BG \times B\Sigma_m)/J^G. \end{tikzcd} \tag{2.16}$$

The following description of J^G is a consequence of Proposition 2.30.

Theorem 2.17. *The ideal $J^G \subseteq E^0(BG \times B\Sigma_m)$ is generated by I_{Tr} and the images of the transfers along*
$$\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/K}) \longrightarrow G \times \Sigma_m \tag{2.18}$$
for all $m = nq$ and $K < S \leq G$ with $[S : K] = n \neq 1$, where $a_{S/K}: S \to \mathrm{Aut}_{\mathrm{Set}}(S/K) \cong \Sigma_n$ is the action map by left multiplication.

2.2. **Stabilizers of elements in $(G/H)^{\times m}$.** Our goal is to understand the ideals J_H^G and J^G appearing in Proposition 2.10 and Corollary 2.15 and defined in Definitions 2.8 and 2.13 using two collections of transfers. In general, there can be overlap between these transfers in the following sense: sometimes the map from a component of $Z^{G,H}_{h(G\times\Sigma_m)}$ factors through $BG \times B\Sigma_i \times B\Sigma_j$ for some choice of i and j. It then suffices to describe the components of $Z^{G,H}_{h(G\times\Sigma_m)}$ that do not factor through $BG \times B\Sigma_i \times B\Sigma_j$ for any $i, j > 0$ such that $i + j = m$.

To start, we note that $Z^{G,H}_{h(G\times\Sigma_m)}$ of (2.9) is equivalent to the disjoint union of classifying spaces of the form $B\Lambda$ for $\Lambda \leq G \times \Sigma_m$ the stabilizer of some element in $Z^{G,H}$ (2.6). Moreover, the associated component of $Z^{G,H}_{h(G\times\Sigma_m)}$ does not factor through some $BG \times B\Sigma_i \times B\Sigma_j$ if and only if the image $\pi_{\Sigma_m}(\Lambda) \leq \Sigma_m$ is a *transitive* subgroup, where $\pi_{\Sigma_m}: G \times \Sigma_m \to \Sigma_m$ is the projection. Thus, it suffices to analyze the stabilizers of the $G \times \Sigma_m$-action on $(G/H)^{\times m}$ that have transitive image in Σ_m.

Elements of the diagonal $\Delta(G/H) = \{(gH, \ldots, gH) \mid g \in G\}$ have the simplest stabilizers: $\mathrm{Stab}_{G\times\Sigma_m}(gH, \ldots, gH) = gHg^{-1} \times \Sigma_m$. However, the stabilizers of the elements of $Z^{G,H}$ can be quite complicated.

In this section, we establish some group-theoretic results regarding these stabilizers and set up notation for describing these groups in the sections ahead.

Notation 2.19. Given $\Lambda \leq G \times \Sigma_n$ and a group H, let $H \wr_{\underline{n}} \Lambda \leq G \times (H \wr \Sigma_n)$ denote the preimage

$$H \wr_{\underline{n}} \Lambda = \pi^{-1}\Lambda, \tag{2.20}$$

where $G \times (H \wr \Sigma_n) \xrightarrow{\pi} G \times \Sigma_n$ is the canonical map.

Note that the group $H \wr_{\underline{n}} \Lambda$ is isomorphic to $H^n \rtimes \Lambda$, where Λ acts on H^n via its projection to Σ_n. This follows from the canonical isomorphism $G \times (H \wr \Sigma_n) \cong H^n \rtimes (G \times \Sigma_n)$.

Notation 2.21. Let X be a G-set, and $Y \subseteq X$ a finite subset. We write $S_Y \leq G$ to denote the set-wise stabilizer of Y,

$$S_Y = \{s \in G \mid s \cdot y \in Y \text{ for all } y \in Y\}.$$

A choice of total ordering $Y = \{y_1, y_2, \ldots, y_n\}$ induces an associated action map

$$a_Y \colon S_Y \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(Y) \cong \Sigma_n. \tag{2.22}$$

Different choices of ordering on Y give conjugate action homomorphisms.

In general, S_Y, a_Y, and $\ker(a_Y)$ can be difficult to compute. We give one primary example.

Example 2.23. Let $H \leq G$, $X = G/H$, and $Y = \{g_1 H, \ldots, g_n H\} \subseteq G/H$. Then we have

$$S_Y = \bigcup_{\sigma \in \mathrm{im}(a_Y)} \bigcap_{i=1}^{n} g_{\sigma(i)} H g_i^{-1} \qquad \text{and} \qquad \ker(a_Y) = \bigcap_{i=1}^{n} g_i H g_i^{-1}.$$

If $Y = K/H$ for some $H \leq K \leq G$, then $S_{K/H} = K$. Indeed, if $g \in G$ satisfies $g \cdot eH = kH$, then $g = g \cdot e$ lies in $kH \subseteq K$.

Lemma 2.24. *Let $Y \subseteq X$ be a finite subset of a G-set, equipped with a total ordering $Y = \{y_1, \ldots, y_n\}$.*

(i) Let $\vec{y} = (y_1, y_2, \ldots, y_n) \in X^{\times n}$. Then

$$\mathrm{Stab}_{G \times \Sigma_n}(\vec{y}) = \Gamma(a_Y),$$

where $\Gamma(a_Y)$ is the graph subgroup associated to $a_Y \colon S_Y \longrightarrow \Sigma_n$.

(ii) Let

$$\vec{y}^{*q} = (\underbrace{y_1, \ldots, y_1}_{q}, \underbrace{y_2, \ldots, y_2}_{q}, \ldots, \underbrace{y_n, \ldots, y_n}_{q})$$

be the q-fold shuffle of $\vec{y}$. Then the stabilizer of $\vec{y}^{*q}$ is

$$\mathrm{Stab}_{G \times \Sigma_{qn}}(\vec{y}^{*q}) = \Sigma_q \wr_{\underline{n}} \Gamma(a_Y) \leq G \times (\Sigma_q \wr \Sigma_n) \leq G \times \Sigma_{qn}.$$

Proof. For (i), (g, σ) is in $\mathrm{Stab}(\vec{y})$ if and only if $g y_i = y_{\sigma i}$ for all i. This defines an action of g on Y, and thus g is in S_Y and $\sigma = a_Y(g)$.

For (ii), it is clear that $(\Sigma_q)^{\times n}$ stabilizes the q-fold shuffle $\vec{y}^{*q}$. The group Σ_n acts by permuting the blocks of size q. Given $(g, (\vec{\tau}, \sigma)) \in G \times (\Sigma_q \wr \Sigma_n)$, we have $((g, (\vec{\tau}, \sigma)) \cdot \vec{y}^{*q})_{i+kq} = g y_{\sigma^{-1} k}$ for $1 \leq i \leq q$. Thus if $(g, (\vec{\tau}, \sigma))$ is in the stabilizer, we must have $(g, \sigma) \in \Gamma(a_Y)$, while the $\tau_i \in \Sigma_q$ have no restrictions. $\square$

2.3. **The absolute transfer ideal J^G.** In Proposition 2.30, we give a complete description of the subgroups of $G \times \Sigma_m$ that we must transfer along to form J^G. This description will be used in the proof of Theorem 3.24.

First, we establish a restricted case. Let $\Delta^{\text{fat}}(G/H) \subseteq (G/H)^{\times n}$ be the fat diagonal, which consists of tuples of cosets in which two or more of the cosets are identical.

Proposition 2.25. *Fix $n \geq 1$. Then the assignment of the stabilizer $\text{Stab}_{G\times\Sigma_n}(-)$ admits a section ζ*

$$\begin{array}{c} \left\{ \vec{g}H \in (G/H)^{\times n} \smallsetminus \Delta^{\text{fat}}(G/H) \;\middle|\; H \leq G,\ \pi_{\Sigma_n}(\text{Stab}(\vec{g}H)) \leq \Sigma_n \textit{ transitive} \right\} \\ \zeta \Big\uparrow \Big\downarrow \text{Stab} \\ \left\{ \Gamma(\phi) \leq G \times \Sigma_n \;\middle|\; \phi \colon S \to \Sigma_n,\ S \leq G,\ \text{im}(\phi) \leq \Sigma_n \textit{ transitive} \right\} \end{array} \tag{2.26}$$

and is therefore surjective.

Proof. According to Lemma 2.24(i), the stabilizer is a graph subgroup. Since the cosets $g_i H$ are all distinct and G/H is a transitive G-set, the image of ϕ in Σ_n must be a transitive subgroup.

Now, any ϕ as above encodes a transitive action of S on $\{1, \ldots, n\}$. In particular, letting $K = \text{Stab}_S(1)$, the action provides a bijection $S/K \cong \{1, \ldots, n\}$, which specifies an ordering of S/K. Thus we may consider S/K as an element of $(G/K)^{\times n}$, and the assignment $\Gamma(\phi) \mapsto S/K \in (G/K)^{\times n}$ is a section of (2.26) by Example 2.23. ☐

Remark 2.27. In light of the description of the section ζ to (2.26) given in the proof above, after passing to Σ_n-conjugacy classes, we may replace the target of Stab in (2.26) with

$$\left\{ [\Gamma(a_{S/K}) \leq G \times \Sigma_n] \;\middle|\; K \leq S \leq G,\ [S{:}K] = n \right\}, \tag{2.28}$$

where $a_{S/K} \colon S \longrightarrow \Sigma_n$ is the action homomorphism specified in (2.22). Note that different choices of orderings $S/K \xrightarrow{\cong} \underline{n}$ induce the same Σ_n-conjugacy class $[\Gamma(a_{S/K})]$.

We note that the source in (2.26) runs over *all* subgroups of G. One might hope for a similar result with a fixed $H \leq G$. However,

$$K = \text{Stab}_S(1) = a_{\vec{g}H}^{-1}(\Sigma_1 \times \Sigma_{n-1}) = g_1 H g_1^{-1} \cap S_{\vec{g}H}$$

need not equal H, as we show in Example 2.29. Therefore, we cannot expect a section if we first fix $H \leq G$. We will show in Proposition 2.32 that such a section does exist if H is normal in G.

Example 2.29. Let $G = D_8$, generated by a rotation r and a reflection s. Let $H = \langle rs \rangle$, and consider $(eH, sH) \in (G/H)^{\times 2}$. We check by hand that $S = \langle s \rangle$ and $K = e$, so the section ζ sends $\Gamma(a_{(eH,sH)})$ to the S-set $S/e = (e, s)$.

Note in particular that $K \neq H$, $m = 2$ is prime, and $r^2 \in H$. Thus in general we cannot a priori fix $H \leq G$ in (2.26).

We will now give a complete description of the stabilizers which appear in J^G. Recall that $Z^{G,H}$ is $(G/H)^{\times m} \smallsetminus \Delta(G/H)$.

Proposition 2.30. *Fix $m \geq 1$. Then the assignment of the stabilizer $\mathrm{Stab}_{G\times\Sigma_m}(-)$ gives a surjection*

$$\begin{array}{c} \left\{ \vec{g}H \in Z^{G,H} \;\middle|\; H < G,\ \pi_{\Sigma_m}(\mathrm{Stab}(\vec{g}H)) \leq \Sigma_m \textit{ transitive} \right\} \\ \Big\downarrow{\scriptstyle \mathrm{Stab}} \\ \left\{ [\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/K}) \leq G \times \Sigma_m] \;\middle|\; m = nq,\ K < S \leq G,\ [S:K] = n \neq 1 \right\}, \end{array} \tag{2.31}$$

where $\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/K})$ is defined as in (2.20) and $[\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/K})]$ denotes the Σ_m-conjugacy class of the subgroup in $G \times \Sigma_m$.

Proof. Suppose given $\vec{g}H$ in $Z^{G,H} \subset (G/H)^{\times m}$. For $(g,\sigma) \in \mathrm{Stab}_{G\times\Sigma_m}(\vec{g}H)$, we have

$$g_iH = g_jH \Leftrightarrow g\cdot g_iH = g\cdot g_jH \Leftrightarrow g_{\sigma(i)}H = g_{\sigma(j)}H.$$

Since $\pi_{\Sigma_m}(\mathrm{Stab}(\vec{g}H))$ is a transitive subgroup of Σ_m, we conclude that after reordering if necessary, $\vec{g}H$ is the q-fold shuffle of $(g_1H,\dots,g_nH)$, where the cosets g_1H, …, g_nH are distinct. Using both parts of Lemma 2.24, we see that

$$\mathrm{Stab}_{G\times\Sigma_m}(\vec{g}H) = \Sigma_q \wr_{\underline{n}} \mathrm{Stab}_{G\times\Sigma_n}((g_1H,\dots,g_nH)),$$

and so the arrow (2.31) is well-defined.

We wish to show that (2.31) is surjective. A choice of $a_{S/K}\colon S \to \Sigma_n$ specifies an ordering of S/K. Given $\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/K})$, the assignment

$$\zeta\colon \Sigma_q \wr_{\underline{n}} \Gamma(a_{S/K}) \longmapsto (S/K)^{*q},$$

is a section of (2.31) by an argument similar to that used in the proof of Proposition 2.25. ☐

2.4. **The relative ideal J^G_H for normal subgroups $H \trianglelefteq G$.** The relative transfer ideal J^G_H was defined in Definition 2.8 and appears in Proposition 2.10. Propositions 2.25 and 2.30 can be used to obtain a description of the subgroups that we must transfer along to form J^G_H in the case that H is normal in G.

Proposition 2.32. *Fix $n \geq 1$ and $H \triangleleft G$ normal. The map (2.26) restricts to give a surjection*

$$\begin{array}{c} \left\{ \vec{g}H \in (G/H)^{\times n} \setminus \Delta^{\mathrm{fat}}(G/H) \;\middle|\; \pi_{\Sigma_n}(\mathrm{Stab}(\vec{g}H)) \leq \Sigma_n \textit{ transitive} \right\} \\ \Big\downarrow{\scriptstyle \mathrm{Stab}} \\ \left\{ [\Gamma(a_{S/H}) \leq G \times \Sigma_n] \;\middle|\; H \leq S \leq G,\ [S:H] = n \right\}, \end{array} \tag{2.33}$$

where $[-]$ denotes the Σ_n-conjugacy class of the subgroup in $G \times \Sigma_n$.

Proof. Since H is normal, the G-stabilizer of each $g_iH \in G/H$ is H. Thus, if S denotes the set-wise stabilizer of $\vec{g}H \subset G/H$ and $K \leq S$ denotes the stabilizer in S of $g_1H \in G/H$, then $K = H$. Furthermore, since $K = H$, the section ζ of (2.26) restricts to a section for fixed, normal H. It follows that (2.33) is surjective. ☐

The next result is an analogue of Proposition 2.30.

Proposition 2.34. *Fix a normal subgroup $H \lhd G$. Then the assignment of the stabilizer (2.31) restricts to a surjection*

$$\begin{array}{c} \left\{ \vec{g}H \in Z^{G,H} \;\middle|\; \pi_{\Sigma_m}(\mathrm{Stab}(\vec{g}H)) \le \Sigma_m \text{ transitive} \right\} \\ \Big\downarrow \mathrm{Stab} \\ \left\{ [\Sigma_q \wr_{\underline{n}} \Gamma(a_{S/H}) \le G \times \Sigma_m] \;\middle|\; m = nq,\; H < S \le G,\; [S:H] = n \neq 1 \right\} \end{array} \tag{2.35}$$

where $\Sigma_q \wr_{\underline{n}} \Gamma(\phi)$ is defined as in (2.20) and $[-]$ denotes the Σ_m-conjugacy class of the subgroup in $G \times \Sigma_m$.

2.5. **Specializing to a prime.** In this section, we give more explicit identifications of the subgroups of $G \times \Sigma_p$ which appear in J^G and J^G_H, where p is a prime. We identify the relevant tuples in $Z^{G,H}$ (Proposition 2.36) and give closed-form descriptions of their stabilizers.

Working at a prime has the advantage that transitive subgroups of Σ_p are exactly those which contain a p-cycle σ_p. This is an immediate consequence of Cauchy's theorem.

We first classify, for general m, those tuples $\vec{g}H \in (G/H)^{\times m}$ for which $\pi_{\Sigma_m}(\mathrm{Stab}(\vec{g}H))$ contains a long cycle.

Proposition 2.36. *Assume that $\vec{g}H = (eH, g_1H, \dots, g_{m-1}H)$, and let $\sigma_m = (1\ 2 \dots\ m)$ be the long cycle. Then (g, σ_m) lies in $\mathrm{Stab}(\vec{g}H)$ if and only if $g^m \in H$ and*

$$\vec{g}H = (eH, gH, g^2H, \dots, g^{m-1}H).$$

Proof. First we will prove the forward direction. By direct observation, we see

$$g \in (Hg_{m-1}^{-1}) \cap (g_{m-1}Hg_{m-2}^{-1}) \cap \dots \cap (g_1H).$$

Thus, there exists $h_i \in H$ such that $g = g_ih_ig_{i-1}^{-1}$ when $0 < i \le m$ (where we set $g_0 = g_m = e$). With this convention we see that

$$g^iH = g_ih_ig_{i-1}^{-1} \cdot g_{i-1}h_{i-1}g_{i-2}^{-1} \cdots g_1H = g_iH.$$

Thus we have that (g, σ_m) stabilizes $\vec{g}H = (eH, gH, g^2H, \dots, g^{m-1}H)$, so $g^m \in H$.

For the reverse direction, it suffices to note that under the condition that g^m is in H,

$$\begin{aligned} (g, \sigma_m) \cdot (eH, gH, g^2H, \dots, g^{m-1}H) &= (gg^{m-1}H, geH, ggH, \dots, gg^{m-2}H) \\ &= (eH, gH, g^2H, \dots, g^{m-1}H). \end{aligned}$$ □

Corollary 2.37. *Using the notation of Proposition 2.25, the subgroup $\pi_{\Sigma_p}(\mathrm{Stab}(\vec{g}H)) \le \Sigma_p$ is transitive if and only if $\vec{g}H \in (G/H)^{\times p}$ lies in the same $G \times \Sigma_p$-orbit as the p-tuple $(eH, gH, g^2H, \dots g^{p-1}H)$ for some $g \in G$ such that $g^p \in H$.*

Proposition 2.25 then specializes to the following.

Corollary 2.38. *Fix a prime p. Then the assignment of the stabilizer $\mathrm{Stab}_{G \times \Sigma_p}(-)$ gives a surjection*

$$\begin{array}{c} \left\{ (g^iH) \in Z^{G,H} \subset (G/H)^{\times p} \;\middle|\; H < G,\; g^p \in H,\; g \notin H \right\} \\ \Big\downarrow \mathrm{Stab} \\ \left\{ [\Gamma(a_{S/K}) \le G \times \Sigma_p] \;\middle|\; K < S \le G,\; [S:K] = p \right\}, \end{array} \tag{2.39}$$

where $[-]$ denotes the Σ_p-conjugacy class of the subgroup in $G \times \Sigma_p$.

Remark 2.40. The codomain of (2.39) can be described more simply as

$$\left\{ \ [\Gamma(S \xrightarrow{a} \Sigma_p) \leq G \times \Sigma_p.] \ \middle| \ S \leq G, \mathrm{im}(a) \text{ contains a } p\text{-cycle} \ \right\}$$

Example 2.29 shows that we still cannot restrict to a fixed $H < G$ on either side for general subgroups H. However, a cleaner description of $\Gamma(a_{S/K})$ does occur for H normal in G.

Notation 2.41. For $\vec{g}H = (g^i H) \in (G/H)^{\times p}$ with $g^p \in H$ and $H \lhd G$ normal, we write $a_g\colon S_g \to \Sigma_m$ for the action map (so $a_g = a_{(g^i H)}$, $S_g = S_{(g^i H)}$).

The following result is a specialization of Proposition 2.32.

Corollary 2.42. *Fix a prime p and a normal subgroup $H \trianglelefteq G$. The assignment $(g^i H) \mapsto \mathrm{Stab}_{G\times\Sigma_p}\big((g^i H)\big) = \Gamma(a_g)$ induces a surjection*

$$\begin{array}{c} \left\{ \ (g^i H) \in Z^{G,H} \subset (G/H)^{\times p} \ \middle| \ g^p \in H, \ g \notin H \ \right\} \\ \Big\downarrow \mathrm{Stab} \\ \left\{ \ [\Gamma(a_{S/H}) \leq G \times \Sigma_p] \ \middle| \ H < S \leq G, \ [S:H] = p \ \right\}. \end{array} \tag{2.43}$$

Moreover, the action map a_g is precisely

$$a_g\colon \langle H, g\rangle \longrightarrow C_p \leq \Sigma_p, \qquad g^i H \longmapsto \sigma_p^i$$

with $\sigma_p = (1\ 2\ \ldots p)$ the long cycle.

Proof. It remains to describe the action map a_g. First, note that, given $(g^i H)$, the set-wise stabilizer of $(g^i H) \subset G/H$ is the subgroup $\langle H, g\rangle \leq G$. Thus $S = S_g = \langle H, g\rangle$. The formula then follows from the fact that $S/H = \langle H, g\rangle/H$ is isomorphic to C_p. □

Generally speaking, the action maps $a_{S/K}$ are difficult to understand. However, in Corollary 2.42, we explicitly describe the action map when H is normal in G and $m = p$ is prime. This will be useful later when we compute power operations.

2.6. **The relatively prime case.** In this section, we record conditions on the integers m, $|G|$, and $|G/H|$ that force every component of $Z^{G,H}_{h(G\times\Sigma_m)}$ to factor through $BG \times B\Sigma_i \times B\Sigma_j$ for some $i, j > 0$ with $i + j = m$.

Corollary 2.44. *Suppose that m and $|G|$ are relatively prime. Then $J^G = I_{\mathrm{Tr}}$.*

Proof. By Theorem 2.17, it suffices to show that the codomain of (2.31) is empty. Suppose not; then we would have subgroups $K < S \leq G$ with $[S\colon K] \neq 1$ dividing m. But $[S : K]$ divides $|S|$ and hence $|G|$, a contradiction. □

When $H \lhd G$ is normal, we have the following specification of Corollary 2.44.

Corollary 2.45. *Let $H \lhd G$ be normal, and suppose m and $|G/H|$ are relatively prime. Then $J^G_H = I_{\mathrm{Tr}}$.*

Proof. Similarly, by Theorem 2.11 it suffices to show that the codomain of (2.35) is empty. Suppose not; then there exists $H < S \leq G$ such that $[S : H]$ is larger than 1 and divides m. But $[S : H]$ also divides $[G : H] = |G/H|$, a contradiction. □

Remark 2.46. We record that this result fails if H is not a normal subgroup. Consider $G = \Sigma_3$ with $H = \{e, (12)\}$ so that $|G/H| = 3$, and let $m = 2$. We note that $((13),(12)) \in G \times \Sigma_2$ is in the stabilizer of $(eH, (13)H) \in (G/H)^{\times 2}$. Therefore, letting $\Gamma \leq G \times \Sigma_2$ be the order two subgroup generated by the element $((13),(12))$, the ideal J_H^G contains the image of the transfer along $\Gamma \longrightarrow G \times \Sigma_2$, which is not contained in I_{Tr}.

3. Additive power operations and Green functors

Making use of the group-theoretic results in Section 2, we provide in this section a general framework for additive power operations in the equivariant setting. In Section 3.1 we recall the notion of a G-Green functor and describe two sources of examples from equivariant homotopy theory. Motivated by the discussion in Section 2.1, in Section 3.2 we prove that the induced G-Green functor associated to a $G \times \Sigma_m$-Green functor contains a canonical Mackey ideal $\underline{J}$. In Section 3.3, we introduce the notion of a $G \times \Sigma_m$-Green functor with mth total power operation and show in Theorem 3.30 that taking the quotient by the Mackey ideal $\underline{J}$ leads to a reduced power operation that is a map of Green functors. In the final two subsections, we show that H_∞-rings in G-spectra and ultra-commutative ring spectra provide two classes of examples of $G \times \Sigma_m$-Green functors with mth total power operation.

3.1. Reminder on Mackey functors and Green functors.

Let G be a finite group. Recall that a G-Mackey functor $\underline{M}$ consists of abelian groups $\underline{M}(G/H)$ for each subgroup $H \leq G$, together with restriction and induction maps

$$\mathrm{Res}\colon \underline{M}(G/K) \longrightarrow \underline{M}(G/H) \qquad \text{and} \qquad \mathrm{Tr}\colon \underline{M}(G/H) \longrightarrow \underline{M}(G/K)$$

for each map $G/H \to G/K$, satisfying a number of axioms. The most notable axiom is the double-coset formula, which we describe in Remark 3.1 below.

A G-Mackey functor can be extended (uniquely up to canonical isomorphism) to all finite G-sets via

$$\underline{M}(A_1 \amalg \ldots A_n) = \bigoplus_i \underline{M}(A_i) \cong \bigoplus_i \underline{M}(G/H_i)$$

for G-orbits $A_1, \ldots, A_n$.

Remark 3.1. The double-coset formula says that for any pullback of G-sets

$$\begin{tikzcd} A \arrow[d, two heads] & P \arrow[l] \arrow[d, two heads] \\ C & B, \arrow[l] \end{tikzcd}$$

in which the vertical maps are surjective, the diagram of abelian groups

$$\begin{tikzcd} \underline{M}(A) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{Tr}"] & \underline{M}(P) \arrow[d, "\mathrm{Tr}"] \\ \underline{M}(C) \arrow[r, "\mathrm{Res}"] & \underline{M}(B) \end{tikzcd} \tag{3.2}$$

commutes.

Definition 3.3. A *G-Green functor* is a G-Mackey functor $\underline{R}$ such that each $\underline{R}(G/H)$ is a commutative ring, each restriction map is a ring homomorphism, and each induction map $\mathrm{Tr}\colon \underline{R}(G/H) \longrightarrow \underline{R}(G/K)$ is an $\underline{R}(G/K)$-module map. The condition that induction is a module map is also referred to as "Frobenius reciprocity". A *Mackey ideal* in a Green functor is a sub-Mackey functor which is levelwise an ideal.

Equivalently, a G-Green functor is a commutative monoid in the category of G-Mackey functors under the box product; see e.g. [Lew, Prop. 1.4], [Shu10, Lemma 2.17].

Example 3.4. If E is a homotopy-commutative ring (genuine) G-spectrum, then the G-Green functor of coefficients is given by

$$\underline{E}^0(G/H) = E^0(G/H).$$

The restriction and transfer maps are defined via naturality on the maps of G-sets $G/K \to G/H$ and the G-transfers $\Sigma^\infty_G G/H_+ \xrightarrow{\mathrm{Tr}} \Sigma^\infty_G G/K_+$. Moreover, the double coset formula (3.2) is a special case of Nishida's push-pull property for equivariant cohomology [Nis78, Prop. 4.4] (see also [LMSM86, IV.1]).

We will pay particular attention to $G \times \Sigma_m$-Green functors and to the following associated G-Green functors.

Definition 3.5. If $\underline{R}$ is any $G \times \Sigma_m$-Green functor, we define $\underline{R}\uparrow^G_{G\times\Sigma_m}$ to be the G-Green functor given by the induction of $\underline{R}$ along the projection $G \times \Sigma_m \to G$ (see [TW95, Lemma 5.4(ii)]). The value of $\underline{R}\uparrow^G_{G\times\Sigma_m}$ on G/H is given by

$$\underline{R}\uparrow^G_{G\times\Sigma_m}(G/H) = \underline{R}\big((G \times \Sigma_m)/(H \times \Sigma_m)\big).$$

In other words, given a G-set A, we define $\underline{R}\uparrow^G_{G\times\Sigma_m}(A)$ to be $\underline{R}(A)$, where in $\underline{R}(A)$ we equip A with a trivial Σ_m-action. For example, there is an isomorphism of $G \times \Sigma_m$-sets $(G \times \Sigma_m)/(H \times \Sigma_m) \cong G/H$, where $G \times \Sigma_m$ acts on G/H through the projection $G \times \Sigma_m \to G$.

Definition 3.6. If $\underline{R}$ is any $G \times \Sigma_m$-Green functor, we define $\underline{R}\downarrow^{G\times\Sigma_m}_G$ to be the restriction of $\underline{R}$ along the inclusion $G \cong G \times \{e\} \le G \times \Sigma_m$. Explicitly,

$$\underline{R}\downarrow^{G\times\Sigma_m}_G(G/H) = \underline{R}\big((G \times \Sigma_m)/(H \times e)\big).$$

Example 3.7. If E is a homotopy-commutative ring G-spectrum, then the assignment

$$(G \times \Sigma_m)/\Lambda \mapsto E^0\Big(\big((G \times \Sigma_m)/\Lambda\big)_{h\Sigma_m}\Big), \tag{3.8}$$

for Λ a subgroup of $G \times \Sigma_m$, is a $G \times \Sigma_m$-Green functor. In this case, the induced G-Green functor as in Definition 3.5 is given by $\underline{E}^0(B\Sigma_m)$, or explicitly the assignment

$$G/H \mapsto E^0(G/H \times B\Sigma_m), \tag{3.9}$$

where $B\Sigma_m$ has a trivial G-action. The restricted G-Green functor as in Definition 3.6 is naturally isomorphic to $\underline{E}^0$, as

$$G/H \mapsto E^0\Big(\big(G/H \times \Sigma_m\big)_{h\Sigma_m}\Big) = E^0(G/H \times E\Sigma_m) \cong E^0(G/H). \tag{3.10}$$

Example 3.11. If E is a homotopy commutative global ring spectrum, there is an associated global Green functor $\underline{E}^0$ ([Sch18, Definition 5.1.3, Theorem 5.1.11], [Gan13, Definition 3.1]). By [Gan13, Lemma 2.10], this gives a G-Green functor $\underline{E}^0_G$ and a $G \times \Sigma_m$-Green functor $\underline{E}^0_{G\times\Sigma_m}$ (see Section 3.5). In this case, the induced G-Green functor as in Definition 3.5 is given by the assignment

$$G/H \mapsto \underline{E}^0_{G\times\Sigma_m}(G/H),$$

where G/H is given a trivial Σ_m-action.

We highlight a particular case of the double-coset formula.

Corollary 3.12. *For any $G \times \Sigma_m$-Green functor $\underline{R}$, the following square of abelian groups commutes*

$$\begin{tikzcd} \underline{R}((G/H)^{\times m}) \arrow[r, "i^*"] \arrow[d, "\mathrm{Tr}"'] & \underline{R}((G/H)^{\times_{G/L} m}) \arrow[d, "\mathrm{Tr}"] \\ \underline{R}((G/L)^{\times m}) \arrow[r, "\Delta^*"] & \underline{R}(G/L) \end{tikzcd}$$

for any map of G-sets $G/H \longrightarrow G/L$.

Proof. This follows from Remark 3.1, as we have a pullback square of $G \times \Sigma_m$-sets

$$\begin{tikzcd} (G/H)^{\times m} \arrow[d] & (G/H)^{\times_{G/L} m} \arrow[l, "i"'] \arrow[d] \\ (G/L)^{\times m} & G/L, \arrow[l, "\Delta"'] \end{tikzcd}$$

in which the vertical maps are surjective. □

3.2. **Certain ideals in $\underline{R}\uparrow^G_{G\times\Sigma_m}$.** Given a $G \times \Sigma_m$-Green functor $\underline{R}$, Definition 3.5 produces a G-Green functor $\underline{R}\uparrow^G_{G\times\Sigma_m}$.

Notation 3.13. Since the induced Mackey functor $\underline{R}\uparrow^G_{G\times\Sigma_m}$ will appear many times in this subsection, we will abbreviate it to $\underline{R}\uparrow$.

In this subsection, we describe two Mackey ideals in the G-Green functor $\underline{R}\uparrow$ that depend on the fact that $\underline{R}\uparrow$ is induced from $\underline{R}$. The definitions of these Mackey ideals are motivated by considerations coming from power operations as in Section 2; however, they make sense in any G-Green functor of the form $\underline{R}\uparrow$.

We begin with the transfer Mackey ideal.

Definition 3.14. Fix a $G \times \Sigma_m$-Green functor $\underline{R}$. Define $\underline{I}_{\mathrm{Tr}}(G/H) \subseteq \underline{R}\uparrow(G/H)$ to be the image of the transfers

$$\bigoplus_{\substack{i+j=m \\ i,j>0}} \underline{R}\big((G \times \Sigma_m)/(H \times \Sigma_i \times \Sigma_j)\big) \xrightarrow{\mathrm{Tr}} \underline{R}\big((G \times \Sigma_m)/(H \times \Sigma_m)\big).$$

We note that the target is $\underline{R}\uparrow(G/H)$.

Lemma 3.15. *The ideals $\underline{I}_{\mathrm{Tr}}(G/H)$ of Definition 3.14 fit together to define a Mackey ideal of $\underline{R}\uparrow$.*

Proof. Frobenius reciprocity implies that $\underline{I}_{\mathrm{Tr}}(G/H)$ is an ideal of $\underline{R}\uparrow(G/H)$.

It remains to show that $\underline{I}_{\mathrm{Tr}}$ is a sub-Mackey functor. To see that $\underline{I}_{\mathrm{Tr}}$ is closed under restriction maps, note that

$$\begin{tikzcd} G/H \times \Sigma_m/(\Sigma_i \times \Sigma_j) \arrow[r] \arrow[d] & G/K \times \Sigma_m/(\Sigma_i \times \Sigma_j) \arrow[d] \\ G/H \times \Sigma_m/\Sigma_m \arrow[r] & G/K \times \Sigma_m/\Sigma_m \end{tikzcd}$$

is a pullback square of $G \times \Sigma_m$-sets and apply Remark 3.1. Finally, $\underline{I}_{\mathrm{Tr}}$ is closed under inductions since the composition of inductions is again an induction. □

Now we will define a Mackey ideal $\underline{J} \subseteq \underline{R}\!\uparrow$, inspired by the ideal J^G of Section 2, with the property that $\underline{I}_{\mathrm{Tr}} \subseteq \underline{J}$.

We have a diagonal inclusion of $G \times \Sigma_m$-sets

$$G/H \xrightarrow{\Delta} (G/H)^{\times_{G/L} m}$$

with complementary $G \times \Sigma_m$-set (cf. (2.6))

$$Z_G^{L,H} = Z^{L,H} = (G/H)^{\times_{G/L} m} \smallsetminus \Delta(G/H).$$

In other words, there is a decomposition of $G \times \Sigma_m$-sets

$$(G/H)^{\times_{G/L} m} \;\cong\; G/H \;\amalg\; Z_G^{L,H}, \tag{3.16}$$

where G/H has a trivial Σ_m-action. Note that $Z_G^{G,H}$ is what was previously called $Z^{G,H}$ in (2.6). We will often suppress the subscript G in the notation when there is no likelihood for confusion. The following description of $Z_G^{L,H}$ will be useful below.

Proposition 3.17. *The $G \times \Sigma_m$-set $Z_G^{L,H}$ is induced from the subgroup $L \times \Sigma_m$:*

$$Z_G^{L,H} \cong G \times_L [(L/H)^{\times m} \smallsetminus \Delta(L/H)] = G \times_L Z_L^{L,H}.$$

Proof. Since the G-set induction functor $G \times_L (-)\colon L\,\mathrm{Set} \longrightarrow G\,\mathrm{Set}$ preserves pullbacks, it follows that $(G/H)^{\times_{G/L} m}$ is isomorphic to $G \times_L \big((L/H)^{\times m}\big)$. Since $G \times_L L/H \cong G/H$, applying induction to the $L = G$ case of the decomposition (3.16) produces a decomposition $G \times_L Z_L^{L,H} \cong\; G/H \amalg G \times_L Z_H^{L,H}$. □

The decomposition (3.16) induces an isomorphism of commutative rings

$$\underline{R}\,((G/H)^{\times_{G/L} m}) \cong \underline{R}\!\uparrow(G/H) \times \underline{R}(Z^{L,H}). \tag{3.18}$$

We may obtain $\underline{R}\!\uparrow(G/H)/\underline{I}_{\mathrm{Tr}}(G/H)$ from $\underline{R}((G/H)^{\times_{G/L} m})$ by taking the quotient by $\underline{I}_{\mathrm{Tr}}(G/H)$ in the first factor of (3.18) and taking the quotient with respect to the entire second factor. This inspires the definition of $\underline{J}$:

Definition 3.19. Fix a $G \times \Sigma_m$-Green functor $\underline{R}$. Define $\underline{J}(G/L) \subseteq \underline{R}\!\uparrow(G/L)$ to be the ideal generated by the images of the transfers along:

- the quotients $G/L \times \Sigma_m/(\Sigma_i \times \Sigma_j) \to G/L \times \Sigma_m/\Sigma_m$ for $i + j = m$ and
- the composition $Z^{L,H} \hookrightarrow (G/H)^{\times_{G/L} m} \to G/L \times \Sigma_m/\Sigma_m$ for $H \le L$.

By construction, we have the following compatibility.

Proposition 3.20. *For H subconjugate to L, we have the following commutative diagram*

$$\begin{tikzcd}
\underline{R}\,((G/H)^{\times_{G/L} m}) \arrow[r, two heads] \arrow[d, "\mathrm{Tr}"] & \underline{R}\!\uparrow(G/H)\,/\underline{I}_{\mathrm{Tr}}(G/H) \arrow[d, "\mathrm{Tr}"] \\
\underline{R}\!\uparrow(G/L) \arrow[r, two heads] & \underline{R}\!\uparrow(G/L)/\underline{J}(G/L),
\end{tikzcd}$$

in which the top map is given by first projecting to the left factor in (3.18) and then taking the quotient by the ideal $\underline{I}_{\mathrm{Tr}}(G/H)$.

The analogue of Theorem 2.17 in this context reads as follows. Like Theorem 2.17, it is an immediate consequence of Proposition 2.30.

Proposition 3.21. *Let $\underline{R}$ be a $G\times\Sigma_m$-Green functor and fix $m\geq 1$. Then $\underline{J}(G/L)\subseteq \underline{R}\!\uparrow\!(G/L)$ is generated by $\underline{I}_{\mathrm{Tr}}(G/L)$ and the images of the transfers*

$$\underline{R}\Big(G\times\Sigma_m\Big/\Sigma_q\wr_{\underline{n}}\Gamma(a_{S/K})\Big)\xrightarrow{\mathrm{Tr}}\underline{R}(G/L)=\underline{R}\!\uparrow\!(G/L)$$

for all $m=nq$, $K<S\leq L$ with $[S:K]=n\neq 1$, and $a_{S/K}\colon S\to \mathrm{Aut}_{\mathrm{Set}}(S/K)\cong\Sigma_n$ the action map by left multiplication.

In the case that m is prime, Corollary 2.38 gives the following simplified form.

Proposition 3.22. *Let $\underline{R}$ be a $G\times\Sigma_m$-Green functor and let $m=p$ be prime. Then $\underline{J}(G/L)\subseteq\underline{R}\!\uparrow\!(G/L)$ is generated by $\underline{I}_{\mathrm{Tr}}(G/L)$ and the images of the transfers*

$$\underline{R}\Big(G\times\Sigma_p\Big/\Gamma(a)\Big)\xrightarrow{\mathrm{Tr}}\underline{R}(G/L)=\underline{R}\!\uparrow\!(G/L)$$

for all subgroups $S\leq L$ and homomorphisms $a\colon S\longrightarrow\Sigma_p$ whose images contain a p-cycle.

The proof of the following corollary is the same as for Corollary 2.44.

Corollary 3.23. *If m is relatively prime to the order of G, then $\underline{J}=\underline{I}_{\mathrm{Tr}}$.*

Now that we have described the ideals $\underline{J}(G/L)$ for fixed L, we turn to the question of how they interact as L varies.

Theorem 3.24. *The ideals $\underline{J}(G/L)$ of Definition 3.19 fit together to define a Mackey ideal of $\underline{R}\!\uparrow$.*

Proof. It suffices to show that $\underline{J}$ is a sub-Mackey functor, i.e. that the image of the transfers in Definition 3.19 is closed under restriction and induction. By Lemma 3.15, it suffices to show that if $H\leq K\leq L$ (up to conjugacy), then

(1) the image of

$$\underline{R}(Z^{L,H})\xrightarrow{\mathrm{Tr}}\underline{R}(G/L)=\underline{R}\!\uparrow\!(G/L)\xrightarrow{\mathrm{Res}}\underline{R}\!\uparrow\!(G/K)$$

lands in $\underline{J}(G/K)\subset\underline{R}\!\uparrow\!(G/K)$ and

(2) the image of

$$\underline{R}(Z^{K,H})\xrightarrow{\mathrm{Tr}}\underline{R}(G/K)=\underline{R}\!\uparrow\!(G/K)\xrightarrow{\mathrm{Tr}}\underline{R}\!\uparrow\!(G/L)$$

lands in $\underline{J}(G/L)\subset\underline{R}\!\uparrow\!(G/L)$.

We begin with (2), as it is much simpler to verify. We have a commutative diagram of $G\times\Sigma_m$-sets

$$\begin{tikzcd}
Z^{K,H} \arrow[r, hook] \arrow[d, hook, dashed] & G/H^{\times_{G/K}m} \arrow[r, two heads] \arrow[d, hook] & G/K \arrow[d, two heads] \\
Z^{L,H} \arrow[r, hook] & G/H^{\times_{G/L}m} \arrow[r, two heads] & G/L,
\end{tikzcd}$$

which yields the commutative diagram

$$\begin{tikzcd}
\underline{R}(Z^{K,H}) \arrow[r, "\mathrm{Tr}"] \arrow[d, hook] & \underline{R}(G/K) \arrow[r, equal] \arrow[d, "\mathrm{Tr}"] & \underline{R}\!\uparrow\!(G/K) \arrow[d, "\mathrm{Tr}"] \\
\underline{R}(Z^{L,H}) \arrow[r, "\mathrm{Tr}"] & \underline{R}(G/L) \arrow[r, equal] & \underline{R}\!\uparrow\!(G/L).
\end{tikzcd}$$

It follows that $\underline{J}$ is closed under Mackey induction.

We now turn to (1), which is more difficult to handle. It suffices to show that we have a factorization

$$\begin{tikzcd} \bigoplus_{H<L} \underline{R}\left(Z^{L,H}\right) \arrow[rr, "\mathrm{Tr}"] \arrow[d, dashed] & & \underline{R}\!\uparrow\!(G/L) \arrow[d, "\mathrm{Res}"] \\ \bigoplus_{i+j=m} \underline{R}\Big(G/K \times \Sigma_m/(\Sigma_i \times \Sigma_j)\Big) \oplus \bigoplus_{H'<K} \underline{R}\left(Z^{K,H'}\right) \arrow[rr, "\mathrm{Tr}"] & & \underline{R}\!\uparrow\!(G/K). \end{tikzcd} \tag{3.25}$$

We have a pullback diagram of $G \times \Sigma_m$-sets

$$\begin{tikzcd} Z^{L,H} \arrow[r] & G/L \\ (Z^{L,H} \times_{G/L} G/K) \arrow[r] \arrow[u, "q"] & G/K \arrow[u, "p"'] \end{tikzcd} \tag{3.26}$$

in which all maps are surjective. Then Remark 3.1 gives a commutative diagram

$$\begin{tikzcd} \underline{R}\left(Z^{L,H}\right) \arrow[r, "\mathrm{Tr}"] \arrow[d, "q^*"'] & \underline{R}\!\uparrow\!(G/L) \arrow[d, "\mathrm{Res}=p^*"] \\ \underline{R}\left(Z^{L,H} \times_{G/L} G/K\right) \arrow[r, "\mathrm{Tr}"] & \underline{R}\!\uparrow\!(G/K). \end{tikzcd}$$

It remains to show that the bottom transfer map factors through the sum

$$\bigoplus_{i+j=m} \underline{R}(G/K \times \Sigma_m/(\Sigma_i \times \Sigma_j)) \oplus \bigoplus_{H'<K} \underline{R}\left(Z^{K,H'}\right).$$

We may decompose $Z^{L,H} \times_{G/L} G/K$ into $G \times \Sigma_m$-orbits, and it suffices to produce the factorization at the level of $G \times \Sigma_m$-sets on each orbit.

Thus let $U \subset Z^{L,H} \times_{G/L} G/K$ be such an orbit, and choose $(\vec{x}, y) \in U$. Then if $\Lambda \leq G \times \Sigma_m$ is the stabilizer of $(\vec{x}, y)$, the presence of the factor G/K forces $\pi_G(\Lambda)$ to be subconjugate to K. We now consider two cases.

In the first case, suppose that $\pi_{\Sigma_m}(\Lambda)$ is not a transitive subgroup. Then $\pi_{\Sigma_m}(\Lambda)$ is subconjugate to $\Sigma_i \times \Sigma_j$ for some positive i and j summing to m. It follows that there exists a map of $G \times \Sigma_m$-sets of the form

$$\begin{array}{ccccc} U & \xrightarrow{\cong} & (G \times \Sigma_m)/\Lambda & \longrightarrow & (G \times \Sigma_m)/(K \times \Sigma_i \times \Sigma_j) \\ (\vec{x}, y) & \longmapsto & e\Lambda & \longmapsto & (g, \sigma)(K \times \Sigma_i \times \Sigma_j) \end{array}$$

where $y = gK$. Then composing this map with the projection onto G/K produces a map of $G \times \Sigma_m$-sets sending $(\vec{x}, y)$ to $gK = y$. It follows that the image of the $\underline{R}$-transfer along $U \longrightarrow G/K$ is contained in the image of the $\underline{R}$-transfer along the G-cover $G/K \times \Sigma_m/(\Sigma_i \times \Sigma_j) \longrightarrow G/K$.

In the second case, we suppose that $\pi_{\Sigma_m}(\Lambda)$ is a transitive subgroup of Σ_m. We further assume for simplicity that H and K are subgroups of L, rather than merely subconjugate. The general case is similar but notationally more complex.

We now reduce to the case $L = G$: recall from Proposition 3.17 that the $G \times \Sigma_m$-set $Z_G^{L,H}$ is induced up from the subgroup $L \times \Sigma_m$. Since the G-set induction $G \times_L (-)\colon L\,\mathrm{Set} \longrightarrow G\,\mathrm{Set}$

preserves pullbacks, the pullback square of L-sets

$$\begin{tikzcd} Z_L^{L,H} \times L/K \arrow[r] \arrow[d] & L/K \arrow[d] \\ Z_L^{L,H} \arrow[r] & L/L \end{tikzcd}$$

gives rise to an isomorphism of $G \times \Sigma_m$-sets

$$Z_G^{L,H} \times_{G/L} G/K \cong G \times_L \left(Z_L^{L,H} \times L/K\right).$$

Moreover, the projection $Z_G^{L,H} \times_{G/L} G/K \longrightarrow G/K$ is the induction from L to G of the projection $Z_L^{L,H} \times L/K \longrightarrow L/K$. We may therefore restrict to the case $L = G$.

We now return to the $G \times \Sigma_m$-orbit U with chosen point $(\vec{x}, y)$ and $\Lambda = \mathrm{Stab}_{G\times\Sigma_m}(\vec{x}, y)$. Up to Σ_m-conjugacy, the tuple $\vec{x}$ is a q-fold shuffle. We have assumed that $\pi_{\Sigma_m}(\Lambda) \le \Sigma_m$ is transitive. Let $y = gK$. Then

$$\Lambda = \mathrm{Stab}(\vec{x}, y) = \mathrm{Stab}(\vec{x}) \cap \mathrm{Stab}(y) = \mathrm{Stab}(\vec{x}) \cap \left(gKg^{-1} \times \Sigma_m\right).$$

Lemma 2.24 explicitly describes $\mathrm{Stab}(\vec{x})$ as $\Sigma_q \wr_{\underline{n}} \Gamma(a)$ for some homomorphism $a: S \longrightarrow \Sigma_n$. The intersection $\Lambda = \mathrm{Stab}(\vec{x}) \cap (gKg^{-1} \times \Sigma_m)$ is now $\Sigma_q \wr_{\underline{n}} \Gamma(a\,|_{S\cap gKg^{-1}})$, where $a\,|_{S\cap gKg^{-1}}$ is the restriction of a to $S \cap gKg^{-1}$. This is a subgroup of $gKg^{-1} \times \Sigma_m$ that projects onto a transitive subgroup of Σ_m by assumption. We may use the surjectivity statement of Proposition 2.30 to describe this as a stabilizer of some element of

$$(gKg^{-1}/H')^{\times m} \smallsetminus \Delta(gKg^{-1}/H')$$

for some $H' \le gKg^{-1}$. It follows that $U \cong (G \times \Sigma_m)/\Lambda$ appears as an orbit of $Z^{gKg^{-1},H'} \cong Z^{K,g^{-1}H'g}$. Therefore the image of the $\underline{R}$-transfer from U is contained in the image of the $\underline{R}$-transfer from $Z^{K,g^{-1}H'g}$. □

Remark 3.27. By construction, a map of $G \times \Sigma_m$-Green functors $\underline{R} \longrightarrow \underline{S}$ gives rise to a map of G-Green functors $\underline{R}\!\uparrow\!/\underline{J} \longrightarrow \underline{S}\!\uparrow\!/\underline{J}$.

3.3. **Power operations on Green functors.** For any $m \ge 0$, let

$$F^m: G\text{–}\,\mathrm{Set} \longrightarrow G \times \Sigma_m\text{–}\,\mathrm{Set}$$

be the mth power functor $F^m(A) = A^{\times m}$, where Σ_m permutes the factors and G acts diagonally. Note that F^m preserves pullbacks. It follows that, for any $G \times \Sigma_m$-Green functor $\underline{R}$, the composition $\underline{R} \circ F^m$ is a G-Green functor.

Following Definition 3.14, let $\mathbb{I}_{\mathrm{Tr}} \subseteq \underline{R} \circ F^m$ be the Mackey ideal defined by letting $\mathbb{I}_{\mathrm{Tr}}(G/H)$ be the image of the transfers

$$\bigoplus_{\substack{i+j=m \\ i,j>0}} \underline{R}\big((G/H)^{\times m} \times \Sigma_m/(\Sigma_i \times \Sigma_j)\big) \xrightarrow{\mathrm{Tr}} \underline{R}\big((G/H)^{\times m} \times \Sigma_m/\Sigma_m)\big).$$

Note that the target is $\underline{R} \circ F^m(G/H)$.

Definition 3.28. An mth *total power operation* on a $G \times \Sigma_m$-Green functor $\underline{R}$ is a natural transformation

$$\mathbb{P}_m: \underline{R}\!\downarrow_G^{G\times\Sigma_m} \longrightarrow \underline{R} \circ F^m$$

of G-Mackey functors of sets which preserves the multiplicative structure and such that $\mathbb{P}_m/\mathbb{I}_{\mathrm{Tr}}$ is a map of G-Green functors.

We will consider two sources of $G \times \Sigma_m$-Green functors with mth total power operation in the following two subsections. These are H_∞-rings in genuine G-spectra (Section 3.4) and ultra-commutative ring spectra in the sense of [Sch18, Chapter 5] (Section 3.5).

For any G-set A, the diagonal inclusion $\Delta\colon A \hookrightarrow A^{\times m} = F^m(A)$ is $G \times \Sigma_m$-equivariant. Pulling back along Δ defines a map of coefficient systems of commutative rings

$$\underline{R} \circ F^m \xrightarrow{\Delta^*} \underline{R}{\uparrow}^G_{G\times\Sigma_m}.$$

The image of $\mathbb{I}_{\mathrm{Tr}}$ under Δ^* is $\underline{I}_{\mathrm{Tr}}$ (Definition 3.14). We then define the power operation P_m as the composition

$$P_m\colon \underline{R}{\downarrow}^{G\times\Sigma_m}_G \xrightarrow{\mathbb{P}_m} \underline{R} \circ F^m \xrightarrow{\Delta^*} \underline{R}{\uparrow}^G_{G\times\Sigma_m}.$$

In general, the power operation P_m is not additive, and it does not commute with the Mackey induction maps and is therefore not a map of Mackey functors. Making use of [BMMS86, VIII.1.4] and Lemma 3.15, additivity can be arranged by taking the quotient with respect to $\underline{I}_{\mathrm{Tr}}$.

Proposition 3.29. *Let $\underline{R}$ be a $G \times \Sigma_m$-Green functor with an mth total power operation. Then the composition*

$$\underline{R}{\downarrow}^{G\times\Sigma_m}_G \xrightarrow{P_m} \underline{R}{\uparrow}^G_{G\times\Sigma_m} \twoheadrightarrow \underline{R}{\uparrow}^G_{G\times\Sigma_m}/\underline{I}_{\mathrm{Tr}}$$

is a map of coefficient systems of commutative rings.

Further passing to the quotient with respect to $\underline{J}$ produces a map of G-Green functors.

Theorem 3.30. *Let $\underline{R}$ be a $G \times \Sigma_m$-Green functor with an mth total power operation, and let $\underline{J}$ be as in Definition 3.19. Then the reduced mth power operation*

$$P_m/\underline{J}\colon \underline{R}{\downarrow}^{G\times\Sigma_m}_G \xrightarrow{\mathbb{P}_m} \underline{R} \circ F^m \xrightarrow{\Delta^*} \underline{R}{\uparrow}^G_{G\times\Sigma_m} \twoheadrightarrow \underline{R}{\uparrow}^G_{G\times\Sigma_m}/\underline{J}$$

is a map of G-Green functors.

The argument below follows the strategy outlined in Section 2.1.

Proof. By Theorem 3.24, $\underline{J}$ is a Mackey functor ideal in $\underline{R}{\uparrow}^G_{G\times\Sigma_m}$. Thus $\underline{R}{\uparrow}^G_{G\times\Sigma_m}/\underline{J}$ is a G-Green functor. Since $\underline{I}_{\mathrm{Tr}}$ is contained in $\underline{J}$, Proposition 3.29 implies that it remains to show that $P_m/\underline{J}$ commutes with induction maps. Thus suppose that $H \le G$ is subconjugate to $L \le G$.

Since the total power operation $\mathbb{P}_m$ is a map of Mackey functors (of sets), Corollary 3.12 implies that we have the following commuting diagram:

$$\begin{tikzcd}
\underline{R}{\downarrow}^{G\times\Sigma_m}_G(G/H) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\mathrm{Tr}"'] & \underline{R}((G/H)^{\times m}) \arrow[r, "i^*"] \arrow[d, "\mathrm{Tr}"'] & \underline{R}((G/H)^{\times_{G/L} m}) \arrow[d, "\mathrm{Tr}"] \\
\underline{R}{\downarrow}^{G\times\Sigma_m}_G(G/L) \arrow[r, "\mathbb{P}_m"] & \underline{R}((G/L)^{\times m}) \arrow[r, "\Delta^*"] & \underline{R}(G/L).
\end{tikzcd} \tag{3.31}$$

Note that in the bottom right corner, G/L has a trivial Σ_m-action, so that $\underline{R}(G/L)$ is $\underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/L)$. Proposition 3.20 states that we have a commuting diagram

$$\begin{tikzcd}
\underline{R}((G/H)^{\times_{G/L} m}) \arrow[r, two heads] \arrow[d, "\mathrm{Tr}"] & \underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/H)/\underline{I}_{\mathrm{Tr}}(G/H) \arrow[d, "\mathrm{Tr}"] \\
\underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/L) \arrow[r, two heads] & \underline{R}{\uparrow}^G_{G\times\Sigma_m}(G/L)/\underline{J}(G/L).
\end{tikzcd}$$

The result then follows by the factorization

$$\begin{tikzcd} \underline{R}\uparrow^G_{G\times\Sigma_m}(G/H)/\underline{I}_{\mathrm{Tr}}(G/H) \arrow[r, two heads] \arrow[d, "\mathrm{Tr}"] & \underline{R}\uparrow^G_{G\times\Sigma_m}(G/H)/\underline{J}(G/H) \arrow[d, "\mathrm{Tr}"] \\ \underline{R}\uparrow^G_{G\times\Sigma_m}(G/L)/\underline{J}(G/L) \arrow[r, equal] & \underline{R}\uparrow^G_{G\times\Sigma_m}(G/L)/\underline{J}(G/L), \end{tikzcd}$$

which occurs since $\underline{J}$ is a Mackey ideal containing $\underline{I}_{\mathrm{Tr}}$. □

3.4. **Power operations on G-spectra.** We first consider H_∞-ring spectra, in the sense of [BMMS86], in the equivariant category.

Definition 3.32. An *H_∞-ring G-spectrum* is a G-spectrum E equipped with G-equivariant maps $E^{\wedge m}_{h\Sigma_m} \longrightarrow E$ which make the diagrams of [BMMS86, I.3] commute in the equivariant stable homotopy category.

As noted in Example 3.4, every homotopy-commutative ring G-spectrum E induces a Green functor-valued equivariant cohomology theory on G-spaces, defined by

$$\underline{E}^0(X)(G/H) = E^0(G/H \times X) = [(G/H \times X)_+, E]^G,$$

where $[-,-]^G$ denotes the abelian group of maps in the equivariant stable homotopy category.

If E is moreover an H_∞-ring G-spectrum, more is true:

Proposition 3.33. *If E is an H_∞-ring G-spectrum, then the $G\times\Sigma_m$-Green functor given by $\underline{R}(A) = E^0(X \times A_{h\Sigma_m})$, as in Example 3.7, has an mth total power operation.*

Proof. In this case,

$$\underline{R}\downarrow^{G\times\Sigma_m}_G(G/H) = E^0(X \times G/H)$$

and

$$\underline{R}\circ F^m(G/H) = E^0(X \times (G/H)^m_{h\Sigma_m}).$$

We define $\mathbb{P}_m$ levelwise to be the composite

$$[(X\times G/H)_+, E]^G \to [(X\times G/H)^{\times m}_{h\Sigma_m,+}, E^{\wedge m}_{h\Sigma_m}]^G \xrightarrow{\mu} [(X\times G/H)^{\times m}_{h\Sigma_m,+}, E]^G \xrightarrow{\Delta_X^*} [(X\times (G/H)^{\times m}_{h\Sigma_m})_+, E]^G.$$

The map $\mathbb{P}_m$ satisfies the requirements of Definition 3.28: it is natural in all stable maps by construction, and so in particular is a map of G-Mackey functors of sets; it is multiplicative; and it is additive after passing to the quotient by $\mathbb{I}_{\mathrm{Tr}}$ by [BMMS86, VIII.1.1]. □

Remark 3.34. We note that the function spectrum E^X is an H_∞-ring G-spectrum whenever E is an H_∞-ring G-spectrum and X is a G-space.[1] Thus the power operation $\mathbb{P}_m$ for E defined at X in Proposition 3.33 agrees with the power operation $\mathbb{P}_m$ for E^X at G/G. Thus, without loss of generality we may assume $X = G/G$ throughout.

The target of the power operation P_m is $\underline{E}^0(B\Sigma_m)$ (3.9). Thus, Definition 3.19 yields an ideal $\underline{J}(G/L) \subseteq E^0(G/L \times B\Sigma_m)$ generated by the images of the transfers along:

- the covers $G/L \times B\Sigma_i \times B\Sigma_j \to G/L \times B\Sigma_m$ for $i + j = m$ and
- the composition $Z^{L,H}_{h\Sigma_m} \hookrightarrow (G/H)^{\times_{G/L} m}_{h\Sigma_m} \to G/L \times B\Sigma_m$ for $H \le L$.

The following is then an immediate corollary of Theorem 3.30.

[1] This is false if we replace X with a G-spectrum Y, as the diagonal map $X \to X^{\times n}$ plays a key role in this structure.

Corollary 3.35. *For any H_∞-ring G-spectrum E, the reduced mth power operation*

$$P_m/\underline{J}\colon \underline{E}^0 \xrightarrow{P_m} \underline{E}^0(B\Sigma_m) \twoheadrightarrow \underline{E}^0(B\Sigma_m)/\underline{J}$$

is a map of G-Green functors.

Remark 3.36. By Remark 3.34, for any G-space X, we get a map of G-Green functors

$$P_m/\underline{J}\colon \underline{E}^0(X) \longrightarrow \underline{E}^0(X \times B\Sigma_m)/\underline{J},$$

where $\underline{J} \subset \underline{E}^0(X \times B\Sigma_m)$ is defined as above by crossing with X.

3.5. **Power operations on global spectra.** Let E be an ultra-commutative ring spectrum in the sense of [Sch18, Chapter 5]. By [Sch18, Theorem 5.1.11], $\underline{E}^0$ is a global power functor. Following [Gan13], we will view $\underline{E}^0$ as a contravariant functor from the category of finite groupoids to commutative rings satisfying the properties of [Gan13, Sections 2.2, 3.1, and 4.1]. Most notably, applying $\underline{E}^0$ to fibrations of finite groupoids gives rise to transfer maps satisfying the push-pull identity of Remark 3.1.

Notation 3.37. For a group G and G-set A, we write $A/\!\!/G$ for the *action groupoid* $G/\!\!/G{\times_G}A$. The objects of $A/\!\!/G$ are the elements of A, and a morphism $a \longrightarrow b$ in $A/\!\!/G$ consists of an element $g \in G$ such that $g \cdot a = b$.

We note that geometric realization translates action groupoids to homotopy orbits:

$$\left|{}^{\mathcal{C}}/\!\!/_{G}\right| \simeq |\mathcal{C}|_{hG}. \tag{3.38}$$

In particular, we have equivalences $|*/\!\!/G| \simeq BG$ and $|G/H/\!\!/G| \simeq BH$.

Proposition 3.39. *If E is an ultra-commutative ring spectrum, then the $G \times \Sigma_m$-Green functor given by $\underline{R}(A) = \underline{E}^0_{G\times\Sigma_m}(A) = \underline{E}^0(A/\!\!/(G\times\Sigma_m))$, as in Example 3.11, has an mth total power operation.*

Proof. Given a finite groupoid $\mathcal{G}$, there is an operation

$$\underline{E}^0(\mathcal{G}) \to \underline{E}^0(\mathcal{G} \wr \Sigma_m), \tag{3.40}$$

where $\mathcal{G} \wr \Sigma_m$ is the finite groupoid of [Gan13, Definition 3.3]. In the case $\mathcal{G} = G/H/\!\!/G$, the groupoid $\mathcal{G}\wr\Sigma_m$ is the action groupoid $(G/H)^{\times m}/\!\!/(G\wr\Sigma_m)$. From (3.40) we can construct the power operation

$$\mathbb{P}_m\colon \underline{E}^0(G/H/\!\!/G) \longrightarrow \underline{E}^0((G/H)^{\times m}/\!\!/(G\wr\Sigma_m)) \to \underline{E}^0((G/H)^{\times m}/\!\!/(G\times\Sigma_m))$$

by restricting the action along the diagonal map $G\times\Sigma_m \to G\wr\Sigma_m$. Note that the source is $\underline{E}^0_G(G/H) = (\underline{E}^0_{G\times\Sigma_m})\downarrow_G^{G\times\Sigma_m}(G/H)$ and the target is $(\underline{E}^0_{G\times\Sigma_m}\circ F^m)(G/H)$.
The requirements of Definition 3.28 now follow from [Sch18, Section 5.1]. $\square$

In this case, Definition 3.19 specializes to an ideal $\underline{J}(G/L) \subseteq \underline{E}^0_{G\times\Sigma_m}(G/L)$ generated by the images of the transfers along:

- the functors $(G/L)/\!\!/(G\times\Sigma_i\times\Sigma_j) \to (G/L)/\!\!/(G\times\Sigma_m)$ for $i+j=m$ and
- the composition

  $$Z^{L,H}/\!\!/(G\times\Sigma_m) \hookrightarrow (G/H)^{\times_{G/L}m}/\!\!/(G\times\Sigma_m) \to (G/L)/\!\!/(G\times\Sigma_m)$$

  for $H \le L$.

The following is then an immediate corollary of Theorem 3.30.

Corollary 3.41. *Let E be an ultra-commutative ring spectrum. Then the reduced mth power operation*

$$\underline{E}^0_G \xrightarrow{P_m} (\underline{E}^0_{G\times\Sigma_m})\uparrow^G_{G\times\Sigma_m} \twoheadrightarrow (\underline{E}^0_{G\times\Sigma_m})\uparrow^G_{G\times\Sigma_m}/\underline{J}$$

is a map of G-Green functors.

4. Examples

In this section, we calculate the ideal $\underline{J}$ in a number of examples. In certain cases, we identify $\underline{J}$ for all finite groups G. Throughout this section, Mackey induction homomorphisms will be displayed in orange, whereas power operations will be displayed in blue. We will deal with many global Green functors in this section, and we will abbreviate an induction such as $(\underline{A}_{G\times\Sigma_m})\uparrow^G_{G\times\Sigma_m}$ to $\underline{A}\uparrow^G_{G\times\Sigma_m}$.

4.1. **Ordinary cohomology.** We begin with ordinary cohomology. This case turns out to be degenerate, in the sense that $\underline{J}$ is equal to $\underline{I}_{\mathrm{Tr}}$.

Let $\underline{R}$ be a G-Green functor and $H\underline{R}$ the G-equivariant Eilenberg-Mac Lane spectrum. We will show that the composition

$$\underline{R} = \mathrm{H}\underline{R}^0 \xrightarrow{P_m} \mathrm{H}\underline{R}^0(B\Sigma_m) \longrightarrow \mathrm{H}\underline{R}^0(B\Sigma_m)/\underline{I}_{\mathrm{Tr}}$$

is already a map of Mackey functors in the case of ordinary cohomology, and thus $\underline{J} = \underline{I}_{\mathrm{Tr}}$. At a G-orbit G/K, the mth power operation P_m is $\underline{R}(G/K) \xrightarrow{(-)^m} \underline{R}(G/K)$.

Lemma 4.1. *If $m = p^r$, where p is prime, then for any $K \subset G$, the ideal $\underline{I}_{\mathrm{Tr}}(G/K) \subset R(G/K)$ is the principal ideal (p). On the other hand, if m has multiple prime factors, then $\underline{I}_{\mathrm{Tr}}(G/K) = (1)$.*

Proof. Let $m = p^r$. We will abbreviate $\underline{I}_{\mathrm{Tr}}(G/K)$ simply to I_{Tr}. For $i + j = m$, the cover $B\Sigma_i \times B\Sigma_j \longrightarrow B\Sigma_m$ induces a transfer map

$$\mathrm{H}\underline{R}^0(B\Sigma_i \times B\Sigma_j)(G/K) \cong \underline{R}(G/K) \xrightarrow{\mathrm{Tr}} \mathrm{H}\underline{R}^0(B\Sigma_m)(G/K) \cong \underline{R}(G/K),$$

which is multiplication by the index of the cover, the binomial coefficient $\binom{m}{i} = \frac{m!}{i!\,j!}$. Since $0 < i < p^r$, all of these coefficients are multiples of p, and we conclude that $I_{\mathrm{Tr}} \subset (p)$. Taking $i = 1$ shows that $p^r \in I_{\mathrm{Tr}}$. In the case $i = p^{r-1}$, the coefficient $\binom{p^r}{p^{r-1}}$ is congruent to p modulo p^r. It follows that $p \in I_{\mathrm{Tr}}$.

Suppose, on the other hand, that m is divisible by distinct primes p_i. By Lucas' Theorem, if r_i is the largest integer such that $p_i^{r_i}$ divides m, then $\binom{m}{p_i^{r_i}}$ is prime to p_i. It follows that the collection of coefficients $\{\binom{m}{p_i^{r_i}}\}$ are relatively prime to $\binom{m}{1} = n$, and therefore generate the ideal (1). $\square$

Thus in the interesting case $m = p^r$, the composition $\underline{R} \xrightarrow{(-)^m} \underline{R} \longrightarrow \underline{R}/\underline{I}_{\mathrm{Tr}}$ is equivalent to the quotient map $\underline{R} \longrightarrow \underline{R}/p$, which is a map of Mackey functors.

4.2. **The sphere spectrum.** We start with the global sphere spectrum $\mathbb{S}$, which is an ultra-commutative ring spectrum [Sch18, Example 4.2.7]. Recall that $\underline{\pi}^0_G = \underline{\mathbb{S}}^0_G$ is isomorphic to the Burnside ring Mackey functor, $\underline{A}_G$ [Seg71, Corollary to Proposition 1]. Thus

$$\underline{\pi}^0_G(G/H) \cong \underline{A}_G(G/H) \cong A(H)$$

is the commutative ring of isomorphism classes of finite H-sets. The restriction and induction maps, in the case that $H \le K$, are given by considering a finite K-set as a finite H-set, on the one hand, and by inducing an H-set up to a K-set, on the other.

The mth power operation associated to the ultra-commutative ring spectrum $\mathbb{S}$ as in Section 3.5 takes the form

$$P_m \colon \underline{A}_G \longrightarrow \underline{A}{\uparrow}^G_{G\times\Sigma_m},$$

where we recall that $\underline{A}{\uparrow}^G_{G\times\Sigma_m}$ is the G-Green functor given by $G/H \mapsto A(H \times \Sigma_m)$. On the other hand, the mth power operation associated to the H_∞-ring G-spectrum $\mathbb{S}^0_G$ takes the form

$$P_m \colon \underline{A}_G \longrightarrow \underline{\pi}^0_G(B\Sigma_m)$$

as in Section 3.4. The map of $G \times \Sigma_m$-spaces $E\Sigma_m \longrightarrow *$ induces a map of G-green functors $\underline{A}{\uparrow}^G_{G\times\Sigma_m} \longrightarrow \underline{\pi}^0_G(B\Sigma_m)$ which is a completion map in the sense that, at an orbit G/H,

$$\underline{\pi}^0_G(B\Sigma_m)(G/H) \cong \pi^0_H(B\Sigma_m) \cong A(\Sigma_m \times H)^\wedge_{I_{\Sigma_m}} \cong \underline{A}{\uparrow}^G_{G\times\Sigma_m}(G/H)^\wedge_{I_{\Sigma_m}}.$$

This observation appears without proof in [May96, Chapter XX]; we provide a proof for completeness. Recall that for groups L and W, the Burnside module

$$A(L, W) \subset A(L \times W)$$

is free abelian on $(L \times W)$-sets for which the action of W is free.

Proposition 4.2. *For any groups H and L, there is an isomorphism*

$$A(L \times H) \cong \bigoplus_{[K\subset H]} A(L, W_H(K))$$

of $A(L)$-modules, where the sum runs over conjugacy classes of subgroups. This induces an isomorphism of commutative rings

$$\pi^0_H(BL) \cong A(L \times H)^\wedge_{I_L},$$

where I_L is the augmentation ideal of $A(L)$.

Proof. We define an $A(L)$-module map

$$\Phi \colon A(L \times H) \longrightarrow \bigoplus_{[K\subset H]} A(L, W_H(K))$$

by

$$\Phi\big((L \times H)/\Gamma\big) = \big(L \times W_H(K_\Gamma)\big)/(\Gamma/K_\Gamma),$$

where $K_\Gamma = \Gamma \cap H$. Writing $p_H \colon L \times H \longrightarrow H$ for the projection, we have $K_\Gamma \trianglelefteq p_H(\Gamma)$, so that Γ/K_Γ is a subgroup of $L \times W_H(K_\Gamma)$. For the reverse direction, we send a $(L \times W_H(K))$-set Y to $Y \times_{W_H(K)} H/K$. Here, L acts on Y, and the quotient (coequalizer) is formed in the category of H-sets, where H is acting trivially on Y. This assignment is inverse to Φ, and it follows that Φ is an isomorphism of $A(L)$-modules. We therefore deduce an isomorphism

$$A(L \times H)^\wedge_{I_L} \cong \bigoplus_{[K\subset H]} A(L, W_H(K))^\wedge_{I_L}$$

upon completion.

Now consider the ring map $A(L \times H) \longrightarrow \pi^0_H(BL)$ defined by sending the $(L \times H)$-set X to the composition

$$\Sigma^\infty_H(BL)_+ \xrightarrow{\mathrm{Tr}} \Sigma^\infty_H(X_{hL})_+ \longrightarrow S^0_H.$$

This ring map factors as in the commutative diagram

$$\begin{tikzcd} A(L\times H) \arrow[rr] \arrow[d, "\cong"', "\Phi"] & & \pi^0_H(BL) \arrow[d, "\cong"', "\text{tom Dieck splitting}"] \\ \bigoplus_{[K\subset H]} A(L, W_H(K)) \arrow[r] & \bigoplus_{[K\subset H]} [\Sigma^\infty BL_+, \Sigma^\infty BW_H(K)_+]. \end{tikzcd}$$

The lower horizontal arrow is completion at I_L according to the version of the Segal conjecture given in [LMM82]. ☐

The mth power operation $P_m\colon \underline{A}_G \longrightarrow \underline{A}\uparrow^G_{G\times\Sigma_m}$, when evaluated at an orbit G/H, takes the form

$$P_m\colon A(H) \longrightarrow A(H\times\Sigma_m).$$

On an H-set X, it is given by $X \mapsto X^{\times m}$, where the output is considered as an $(H\times\Sigma_m)$-set.

In the case $m=2$, we have a complete description of $\underline{J}$ as follows.

Proposition 4.3. *In the case $m=2$, the Mackey ideal $\underline{J}\subset \underline{A}\uparrow^G_{G\times\Sigma_2}$ is the kernel of the Σ_2-fixed point homomorphism $\underline{A}\uparrow^G_{G\times\Sigma_2}\longrightarrow \underline{A}_G$, and the composition*

$$\underline{A}_G \xrightarrow{P_2} \underline{A}\uparrow^G_{G\times\Sigma_2} \longrightarrow \underline{A}\uparrow^G_{G\times\Sigma_2}/\underline{J} \cong \underline{A}_G$$

is the identity map.

Proof. It suffices to consider the case $H=G$. The kernel of the Σ_2-fixed point homomorphism has generators $(G\times\Sigma_2)/\Gamma$ where Γ is a graph subgroup. If Γ is a non-transitive graph subgroup, then it is of the form $K\times\Sigma_1\times\Sigma_1$ and therefore in $\underline{J}(G/G)$. On the other hand, if it is transitive, then it is in $\underline{J}(G/G)$ according to Proposition 3.22.

Now we consider the composition of the power operation P_2 and fixed points. The Σ_2-fixed points of $P_2(G/H)=(G/H)^{\times 2}$ are simply the diagonal $G/H\cong\Delta(G/H)\subset(G/H)^{\times 2}$. This shows that the composition is the identity as claimed. ☐

When $m=p>2$ is prime, we have the following result.

Proposition 4.4. *If $p>2$ is prime and $L\le G$, then the orbit $(L\times\Sigma_p)/\Gamma$ lies in the ideal $\underline{J}(G/L)\subset A(L\times\Sigma_p)$ if and only if either Γ is a graph subgroup of $L\times\Sigma_p$ such that $\pi_{\Sigma_p}(\Gamma)$ contains some C_p, or Γ is subconjugate to $L\times\Sigma_i\times\Sigma_j$, for i and j positive and summing to p.*

Proof. This follows from Proposition 3.22. ☐

Example 4.5. Consider the case $G=C_2$ and $m=2$. We will describe the power operation $P_2\colon\underline{A}_{C_2}\longrightarrow\underline{A}\uparrow^{C_2}_{C_2\times\Sigma_2}$, which is only a map of coefficient systems (of sets) over C_2. We write $\Gamma=C_2\times\Sigma_2$ and $D<\Gamma$ for the diagonal subgroup. Writing 1 for the one-point orbit (of any group), we have

$$A(C_2)=\mathbb{Z}\{C_2,1\},$$
$$A(\Sigma_2)=\mathbb{Z}\{\Sigma_2,1\},$$

and

$$A(C_2\times\Sigma_2)=\mathbb{Z}\{\Gamma,\Gamma/C_2,\Gamma/\Sigma_2,\Gamma/D,1\}.$$

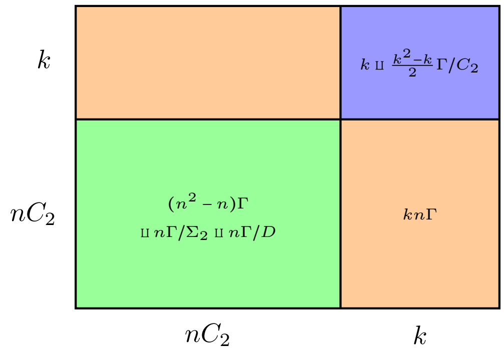
Proposition 4.6. *The power operations*

$$P_2^e\colon A(e)\longrightarrow A(\Sigma_2)$$

and

$$P_2^{C_2}\colon A(C_2)\longrightarrow A(C_2\times\Sigma_2)$$

FIGURE 1. The $(C_2 \times \Sigma_2)$-set $(nC_2 + k) \times (nC_2 + k)$



are given by

$$P_2^e(k) = \frac{k^2-k}{2}\Sigma_2 + k$$

and

$$P_2^{C_2}(nC_2 + k) = (n^2 - n + kn)\Gamma \,+\, \frac{k^2-k}{2}\Gamma/C_2 \,+\, n\Gamma/\Sigma_2 \,+\, n\Gamma/D + k,$$

respectively.

Proof. For P_2^e, this is simply a matter of observing that the diagonal of $k \times k$ is fixed by the Σ_2-action, and the rest is free. The case of $P_2^{C_2}$ is displayed in Figure 1. The key here is that $C_2 \times C_2 \cong \Gamma/\Sigma_2 \amalg \Gamma/D$. $\square$

The images of the transfer maps $A(e) \longrightarrow A(\Sigma_2)$ and $A(C_2) \longrightarrow A(C_2 \times \Sigma_2)$ are $\mathbb{Z}\{\Sigma_2\} \subset A(\Sigma_2)$ and $\mathbb{Z}\{\Gamma, \Gamma/C_2\} \subset A(C_2 \times \Sigma_2)$. Thus, after modding out by the images of these transfer maps, we have

$$\begin{tikzcd}
\mathbb{Z}\{C_2, 1\} \arrow[rr, "\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, bend right, "\begin{pmatrix} 2 & 1 \end{pmatrix}"'] & & \mathbb{Z}\{\Gamma/\Sigma_2, \Gamma/D, 1\} \arrow[d, bend right, "\begin{pmatrix} 2 & 0 & 1 \end{pmatrix}"'] \\
\mathbb{Z} \arrow[rr, "1"] \arrow[u, bend right, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & & \mathbb{Z} \arrow[u, bend right, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"']
\end{tikzcd}$$

The power operation does not commute with the Mackey induction homomorphisms. But after collapsing out the image of the additional transfer $A(D) \longrightarrow A(C_2 \times \Sigma_2)$, the power operation becomes the identity map of the Burnside Green functor, as it must be according to Proposition 4.3.

Example 4.7. Consider $G = \Sigma_3$ and $m = 2$. We will write ρ for a 3-cycle and τ for a transposition in Σ_3. We will let $\sigma \in \Sigma_2$ be the transposition. We begin by restricting attention to $C_3 < \Sigma_3$. The decomposition of $(C_3 \times \Sigma_2)$-sets

$$C_3^{\times 2} \cong (C_3 \times \Sigma_2)/\Sigma_2 \amalg (C_3 \times \Sigma_2)$$

gives the following result.

Proposition 4.8. *Writing $\Gamma = C_3 \times \Sigma_2$, the second power operation $P_2^{C_3}\colon A(C_3) \longrightarrow A(C_3 \times \Sigma_2)$ is given by*

$$P_2^{C_3}(nC_3 + k) = \left(nk + \frac{3n^2 - n}{2}\right)\Gamma + \frac{k^2 - k}{2}\Gamma/C_3 + n\Gamma/\Sigma_2 + k.$$

On the other hand, now writing $\Gamma = \Sigma_3 \times \Sigma_2$, we have

$$\Sigma_3^{\times 2} \cong \Gamma/\Sigma_2 \amalg \Gamma \amalg 3(\Gamma/D),$$

where $D < \Gamma$ is the order two subgroup generated by the element (τ, σ). We have

$$A(\Sigma_3) = \mathbb{Z}\{\Sigma_3, \Sigma_3/C_3, \Sigma_3/C_2, 1\}$$

and

$$A(\Sigma_3 \times \Sigma_2) = \mathbb{Z}\{\Gamma, \Gamma/C_3, \Gamma/C_2, \Gamma/\Sigma_3, \Gamma/\Sigma_2, \Gamma/(C_3 \times \Sigma_2), \Gamma/(C_2 \times \Sigma_2), \Gamma/D, \Gamma/DC_3, 1\}.$$

Proposition 4.9. *Writing $\Gamma = \Sigma_3 \times \Sigma_2$, the second power operation $P_2^{\Sigma_3}\colon A(\Sigma_3) \longrightarrow A(\Sigma_3 \times \Sigma_2)$ is given by*

$$\begin{aligned} P_2^{\Sigma_3}(n\Sigma_3 \amalg i\Sigma_3/C_3 \amalg j\Sigma_3/C_2 \amalg k) &= \left(3n^2 - 2n + 2ni + 3nj + nk + ij + \frac{j^2 - j}{2}\right)\Gamma \\ &\amalg (i^2 - i + ik)\Gamma/C_3 \amalg \left(\frac{j^2 - j}{2} + jk\right)\Gamma/C_2 \amalg \frac{k^2 - k}{2}\Gamma/\Sigma_3 \amalg n\Gamma/\Sigma_2 \\ &\amalg i\Gamma/(C_3 \times \Sigma_2) \amalg j\Gamma/(C_2 \times \Sigma_2) \amalg (3n + j)\Gamma/D \amalg i\Gamma/DC_3 \amalg k. \end{aligned}$$

In addition to the transfers already discussed in Example 4.5, the images of the transfers $A(C_3) \longrightarrow A(C_3 \times \Sigma_2)$ and $A(\Sigma_3) \longrightarrow A(\Sigma_3 \times \Sigma_2)$ are

$$\mathbb{Z}\{C_3 \times \Sigma_2, (C_3 \times \Sigma_2)/C_3\} \subset A(C_3 \times \Sigma_2)$$

and

$$\mathbb{Z}\{\Gamma, \Gamma/C_3, \Gamma/C_2, \Gamma/\Sigma_3\} \subset A(\Sigma_3 \times \Sigma_2).$$

Thus, after modding out by the images of these transfer maps, we have

$$\begin{tikzcd}
 & \mathbb{Z}\{\Sigma_3, \Sigma_3/C_3, \Sigma_3/C_2, 1\} \arrow[rrr, "\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\3&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}"] \arrow[dl, bend right, "\begin{pmatrix}2&0&1&0\\0&2&0&1\end{pmatrix}"'] \arrow[ddr, bend right, "\begin{pmatrix}3&1&1&0\\0&0&1&1\end{pmatrix}"] & & & \mathbb{Z}\{\Gamma/\Sigma_2, \Gamma/(C_3 \times \Sigma_2), \Gamma/(C_2 \times \Sigma_2), \Gamma/D, \Gamma/DC_3, 1\} \arrow[dl, bend right, "\begin{pmatrix}2&0&1&0&0&0\\0&2&0&0&0&1\end{pmatrix}"'] \arrow[ddr, bend right, "\begin{pmatrix}3&1&0&0&0&0\\0&0&0&3&1&0\\0&0&3&0&0&1\end{pmatrix}"'] & \\
\mathbb{Z}\{C_3, 1\} \arrow[ur, bend right, "\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}"'] \arrow[rrr, bend left, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"] \arrow[ddr, bend right, "\begin{pmatrix}3&1\end{pmatrix}"'] & & & \mathbb{Z}\{(C_3 \times \Sigma_2)/\Sigma_2, 1\} \arrow[ur, bend right, "\begin{pmatrix}1&0\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix}"'] \arrow[dd, bend right, "\begin{pmatrix}3&1\end{pmatrix}"'] & & \\
 & & \mathbb{Z}\{C_2, 1\} \arrow[uul, bend right, "\begin{pmatrix}1&0\\0&0\\0&1\\0&0\end{pmatrix}"'] \arrow[rrr, bend left, "\begin{pmatrix}1&0\\1&0\\0&1\end{pmatrix}"] \arrow[dl, bend right, "\begin{pmatrix}2&1\end{pmatrix}"'] & & & \mathbb{Z}\{(C_2 \times \Sigma_2)/\Sigma_2, (C_2 \times \Sigma_2)/D, 1\} \arrow[uul, bend right, "\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\\0&1&0\\0&0&0\\0&0&0\end{pmatrix}"'] \arrow[dll, bend right, "\begin{pmatrix}2&2&1\end{pmatrix}"'] \\
 & \mathbb{Z} \arrow[uul, bend right, "\begin{pmatrix}1\\0\end{pmatrix}"'] \arrow[ur, bend right, "\begin{pmatrix}1\\0\end{pmatrix}"'] \arrow[rr, "1"] & & \mathbb{Z} \arrow[uu, bend right, "\begin{pmatrix}1\\0\end{pmatrix}"'] \arrow[urr, bend right, "\begin{pmatrix}1\\0\\0\end{pmatrix}"'] & &
\end{tikzcd}$$

In order to make the power operations commute with Mackey induction, we must further collapse $\mathbb{Z}\{(C_2 \times \Sigma_2)/D\} \subset A(C_2 \times \Sigma_2)$ and $\mathbb{Z}\{\Gamma/D, \Gamma/DC_3\} \subset A(\Sigma_3 \times \Sigma_2)$. The resulting power operation of Green functors is the identity on $\underline{A}_{\Sigma_3}$, as it must be according to Proposition 4.3.

Example 4.10. Consider $G = C_3$ and $m = 3$. Then

$$A(\Sigma_3) = \mathbb{Z}\{\Sigma_3, \Sigma_3/C_3, \Sigma_3/C_2, 1\}.$$

For $\Gamma = C_3 \times \Sigma_3$, we write C_3^R for the order 3 subgroup of Σ_3, and we write C_2 for a choice of order two subgroup of Σ_3 and Δ for the order 3 subgroup generated by (ρ, σ_3), where ρ generates C_3 and σ_3 is a 3-cycle. Then

$$A(C_3 \times \Sigma_3) \cong \mathbb{Z}\{\Gamma, \Gamma/C_3, \Gamma/C_2, \Gamma/\Sigma_3, \Gamma/C_3^R, \Gamma/(C_3 \times C_2), \Gamma/C_3^{\times 2}, \Gamma/\Delta, \Gamma/\Delta C_2, 1\},$$

where ΔC_2 is the internal product in $C_3 \times \Sigma_3$.

Proposition 4.11. *Writing $\Gamma = C_3 \times \Sigma_3$, the power operations*

$$P_3^e \colon A(e) \longrightarrow A(\Sigma_3)$$

and

$$P_3^{C_3} \colon A(C_3) \longrightarrow A(C_3 \times \Sigma_3)$$

are given by

$$P_3^e(k) = \binom{k}{3}\Sigma_3 + k(k-1)\Sigma_3/C_2 + k\Sigma_3/\Sigma_3$$

and

$$\begin{aligned} P_3^{C_3}(nC_3 + k) = \Big[n\binom{k}{2} + k\frac{3n^2 - n}{2} + 6\binom{n}{2} + 9\binom{n}{3}\Big]\Gamma \;+\; \binom{k}{3}\Gamma/C_3 \;+\; n\Gamma/\Sigma_3 \\ +\; k(k-1)\Gamma/(C_3 \times C_2) \;+\; \Big[2nk + 2n + 6\binom{n}{2}\Big]\Gamma/C_2 \;+\; n\Gamma/\Delta \;+\; k\Gamma/\Gamma \end{aligned}$$

respectively.

By Proposition 4.4, $\underline{J}(C_3/C_3) \subset A(C_3 \times \Sigma_3)$ is generated by the orbits Γ, Γ/C_3, $\Gamma/(C_3 \times C_2)$, Γ/C_2, and Γ/Δ, which are precisely the terms appearing in Proposition 4.11 with a nonlinear coefficient. The resulting power operation of Green functors is then an inclusion

$$\underline{A}_{C_3} \xrightarrow{P_3/\underline{J}} \underline{A}\uparrow^{C_3}_{C_3 \times \Sigma_3}/\underline{J} \cong \underline{A}_{C_3} \oplus \underline{A}_{C_3} \oplus \begin{array}{c} \mathbb{Z}\{\Gamma/\Delta\} \\ \downarrow \uparrow \\ 0 \end{array}$$

Here the first copy of $\underline{A}_{C_3}$ contains the orbits Γ/Σ_3 and $1 = \Gamma/\Gamma$, whereas the second copy contains the orbits Γ/C_3^R and $\Gamma/(C_3)^2$.

4.3. **Global** KU**.** Consider the ultra-commutative ring spectrum KU ([Sch18, 6.4.9]). The associated G-Green functor $\underline{KU}^0_G$ is the representation ring Green functor, with $\underline{KU}^0_G(G/H) \cong RU(H)$. The restriction and induction maps correspond to restriction and induction of representations, respectively. The mth power operation associated to the ultra-commutative ring spectrum KU as in Section 3.5 takes the form

$$P_m \colon \underline{RU}_G \longrightarrow \underline{RU}\uparrow^G_{G \times \Sigma_m}$$

and is given at G/H by $V \mapsto V^{\otimes m}$, where the latter is consider as a $(H \times \Sigma_m)$-representation. On the other hand, the mth power operation associated to the H_∞-ring G-spectrum KU_G takes the form

$$P_m \colon \underline{RU}_G = \underline{KU}^0_G \longrightarrow \underline{KU}^0_G(B\Sigma_m).$$

as in Section 3.4. The map of $G \times \Sigma_m$-spaces $E\Sigma_m \longrightarrow *$ induces a map of G-green functors $\underline{RU}\uparrow^G_{G \times \Sigma_m} \longrightarrow \underline{KU}^0_G(B\Sigma_m)$. At an orbit G/H, this is the map

$$\underline{RU}\uparrow^G_{G \times \Sigma_m}(G/H) \cong RU(H \times \Sigma_m) \xrightarrow{(-)_{h\Sigma_m}} KU^0_H(B\Sigma_m) \cong \underline{KU}^0_G(B\Sigma_m)(G/H)$$

which takes an $H \times \Sigma_m$-representation and passes to homotopy orbits with respect to the Σ_m-action. As we show in the following proposition, this map is a completion, in the sense that

$$\underline{KU}^0_G(B\Sigma_m)(G/H) \cong KU^0_H(B\Sigma_m) \cong RU(H \times \Sigma_m)^\wedge_{I_{\Sigma_m}} \cong \underline{RU}\uparrow^G_{G\times\Sigma_m}(G/H)^\wedge_{I_{\Sigma_m}}.$$

Proposition 4.12. *For any groups H and L, the map*

$$(-)_{hL} \colon RU(H \times L) \longrightarrow KU^0_H(BL)$$

is completion at the augmentation ideal $I_L \subset RU(L)$.

Proof. We have a commuting square

$$\begin{tikzcd} RU(H\times L) \arrow[r] \arrow[d, "\cong"'] & KU^0_H(BL) \arrow[d, "\cong"] \\ RU(H)\otimes RU(L) \arrow[r] & RU(H)\otimes KU(BL), \end{tikzcd}$$

where the right vertical map is an isomorphism because H acts trivially on BL. Now the Atiyah-Segal Completion Theorem [AS69] states that $RU(L) \longrightarrow KU(BL)$ is completion at I_L. The isomorphism

$$\begin{aligned} RU(H)\otimes RU(L)^\wedge_{I_L} &\cong RU(H)\otimes RU(L)\otimes_{RU(L)} RU(L)^\wedge_{I_L} \\ &\cong \big(RU(H)\otimes RU(L)\big)\otimes_{RU(L)} RU(L)^\wedge_{I_L} \end{aligned}$$

finishes the proof. □

As in Section 4.2, we focus on the power operation with target $\underline{RU}\uparrow^G_{G\times\Sigma_m}$. Denote by $\mathrm{ev}_\sigma \colon RU(\Sigma_m) \longrightarrow \mathbb{Z}$ the homomorphism that evaluates the character of a representation at an m-cycle. This homomorphism is precisely the quotient map $KU^0_{\Sigma_m}(*) \longrightarrow KU^0_{\Sigma_m}(*)/I_{\mathrm{Tr}}$ by the transfer ideal.

Proposition 4.13 ([Ati66])**.** *The composition*

$$KU^0_G(X) \xrightarrow{P_m} KU^0_{G\times\Sigma_m}(X) \cong KU^0_G(X)\otimes RU(\Sigma_m) \xrightarrow{\mathrm{id}\otimes \mathrm{ev}_\sigma} KU^0_G(X)\otimes\mathbb{Z}$$

is the Adams operation ψ^m.

In other words,

$$\underline{RU}_G \xrightarrow{P_m/\underline{I}_{\mathrm{Tr}}} \underline{RU}\uparrow^G_{G\times\Sigma_m}/\underline{I}_{\mathrm{Tr}} \cong \underline{RU}_G$$

is levelwise the Adams operation ψ^m. As in Proposition 3.29, this is a map of coefficient systems of commutative rings but not a map of Green functors.

Example 4.14. Consider the case $G = C_2$ and $m = 2$. Then $RU(C_2) = \mathbb{Z}\{1, s\}$, where s is the sign representation, satisfying $s^2 = 1$. Then the Mackey induction sends $1 \in RU(e) = \mathbb{Z}$ to the regular representation $1 + s$. The ring homomorphism ψ^2 squares both of the 1-dimensional representations 1 and s, so the diagram

$$\begin{tikzcd} RU(C_2) \arrow[r, "\psi^2"] & RU(C_2) \\ RU(e)=\mathbb{Z} \arrow[u] \arrow[r, "\psi^2=\mathrm{id}"] & RU(e)=\mathbb{Z} \arrow[u] \end{tikzcd}$$

does *not* commute, since

$$\begin{tikzcd} 1+s \arrow[r, mapsto] & 1^2+s^2=2\neq 1+s \\ 1 \arrow[u, mapsto] \arrow[r, mapsto] & 1. \arrow[u, mapsto] \end{tikzcd} \tag{4.15}$$

Following Proposition 3.22, we observe that $\underline{J}$ is generated by $\underline{I}_{\mathrm{Tr}}$ as well as an additional transfer: we must further collapse the image of

$$\begin{array}{ccccc} RU(D) & \longrightarrow & RU(C_2\times\Sigma_2)\cong RU(C_2)\otimes RU(\Sigma_2) & \xrightarrow{1\otimes\mathrm{ev}_\sigma} & RU(C_2)\otimes\mathbb{Z}, \\ 1 & \longmapsto & 1+s\bar{s} & \longmapsto & 1-s \end{array}$$

where $D\leq C_2\times\Sigma_2$ is once again the diagonal subgroup and $\bar{s}\in RU(\Sigma_2)$ is the sign representation. Thus, collapsing the ideal $\underline{J}$ imposes the relation $s\sim 1$, in particular making (4.15) commute. The resulting quotient Green functor is the constant Mackey functor $\underline{\mathbb{Z}}$, and the power operation of Green functors

$$P_2/\underline{J}\colon \underline{RU}_{C_2}\longrightarrow \underline{RU}\uparrow_{C_2\times\Sigma_2}^{C_2}/\underline{J}\cong\underline{\mathbb{Z}}$$

is the augmentation, given by restricting to the trivial subgroup.

Example 4.16. Consider now $G=\Sigma_3$ and $m=2$. We have

$$RU(\Sigma_3)=\mathbb{Z}[s,W]/(s^2-1,sW-W,W^2-W-s-1),$$

where W is the reduced standard representation, and

$$RU(C_3)=\mathbb{Z}[\lambda]/(\lambda^3-1).$$

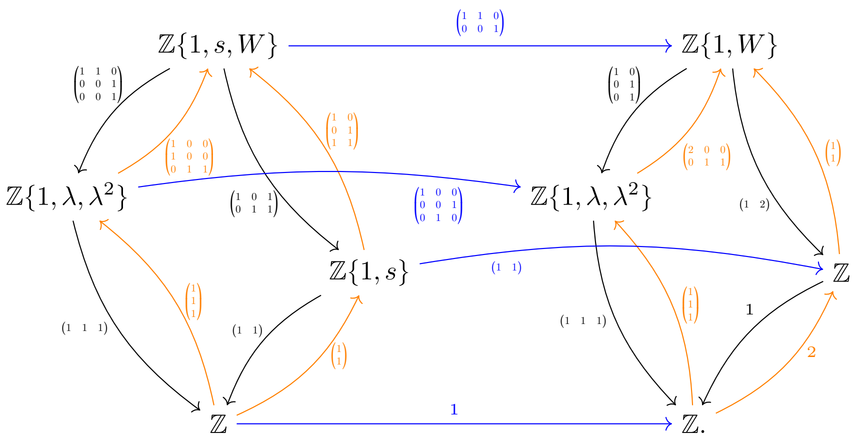
By Proposition 3.22, the ideal $\underline{J}$ is generated by $\underline{I}_{\mathrm{Tr}}$ and three additional transfers:

$$\begin{array}{ccccc} RU(D) & \longrightarrow & RU(\Sigma_3\times\Sigma_2)\cong RU(\Sigma_3)\otimes RU(\Sigma_2) & \xrightarrow{1\otimes\mathrm{ev}_\sigma} & RU(\Sigma_3)\otimes\mathbb{Z}, \\ 1 & \longmapsto & 1+s\bar{s}+W+W\bar{s} & \longmapsto & 1-s \\ RU(\Gamma(\mathrm{sgn})) & \longrightarrow & RU(\Sigma_3\times\Sigma_2)\cong RU(\Sigma_3)\otimes RU(\Sigma_2) & \xrightarrow{1\otimes\mathrm{ev}_\sigma} & RU(\Sigma_3)\otimes\mathbb{Z}, \quad \text{and} \\ 1 & \longmapsto & 1+s\bar{s} & \longmapsto & 1-s \\ RU(D) & \longrightarrow & RU(C_2\times\Sigma_2)\cong RU(C_2)\otimes RU(\Sigma_2) & \xrightarrow{1\otimes\mathrm{ev}_\sigma} & RU(C_2)\otimes\mathbb{Z}, \\ 1 & \longmapsto & 1+s\bar{s} & \longmapsto & 1-s \end{array}$$

where $C_2\leq\Sigma_3$ is a choice of order two subgroup, $D\leq C_2\times\Sigma_2\leq\Sigma_3\times\Sigma_2$ is the diagonal order 2 subgroup, $\mathrm{sgn}\colon\Sigma_3\to\Sigma_2$ is the sign homomorphism, and $\Gamma(\mathrm{sgn})\leq\Sigma_3\times\Sigma_2$ the associated graph subgroup. As in Example 4.14, we conclude that we must impose the relation $s\sim 1$ in the cyclic 2-subgroups and in $RU(\Sigma_3)$. The resulting power operation of Green functors

$$P_2/\underline{J}\colon \underline{RU}_{\Sigma_3}\longrightarrow \underline{RU}\uparrow_{\Sigma_3\times\Sigma_2}^{\Sigma_3}/\underline{J}$$

is given by



Here, the value of $\psi_2(W)$ may be deduced by using the (character) embedding of the representation ring into the ring of class functions and using the formula for the Adams operation ψ_2 given in Proposition 4.19. Since all other representations that appear are 1-dimensional, the operation ψ_2 simply squares them.

Example 4.17. Consider now $G = \Sigma_3$ and $m = 3$. We begin with the same source Green functor as in Example 4.16. As in Example 4.16, $\underline{J}$ is generated by $\underline{I}_{\mathrm{Tr}}$ and three additional transfers:

$$\begin{aligned}
RU(D_{C_3}) &\longrightarrow RU(\Sigma_3\times\Sigma_3) \cong RU(\Sigma_3)\otimes RU(\Sigma_3) \xrightarrow{1\otimes \mathrm{ev}_\sigma} RU(\Sigma_3)\otimes\mathbb{Z},\\
1 &\longmapsto 1+s+\bar{s}+s\bar{s}+2W\bar{W} \longmapsto 2+2s-2W\\
RU(D_{\Sigma_3}) &\longrightarrow RU(\Sigma_3\times\Sigma_3) \cong RU(\Sigma_3)\otimes RU(\Sigma_3) \xrightarrow{1\otimes \mathrm{ev}_\sigma} RU(\Sigma_3)\otimes\mathbb{Z}, \quad \text{and}\\
1 &\longmapsto 1+s\bar{s}+W\bar{W} \longmapsto 1+s-W\\
RU(D_{C_3}) &\longrightarrow RU(C_3\times\Sigma_3) \cong RU(C_3)\otimes RU(\Sigma_3) \xrightarrow{1\otimes \mathrm{ev}_\sigma} RU(C_3)\otimes\mathbb{Z},\\
1 &\longmapsto 1+\bar{s}\to\lambda\bar{W}+\lambda^2\bar{W} \longmapsto 2-\lambda-\lambda^2
\end{aligned}$$

where $D_{C_3} \le C_3\times\Sigma_3$ is the order 3 subgroup generated by (ρ,σ_3) for ρ a generator of C_3 and σ_3 a 3-cycle, and $D_{\Sigma_3} \le \Sigma_3\times\Sigma_3$ is the diagonal subgroup. Thus we collapse the ideals $(W-s-1)\subset RU(\Sigma_3)$ and $(\lambda^2+\lambda-2)\subset RU(C_3)$. The quotients are

$$RU(\Sigma_3)/(W-s-1)\cong\mathbb{Z}\{1,s\} \qquad \text{and} \qquad RU(C_3)/(\lambda^2+\lambda-2)\cong\mathbb{Z}\{1\}\oplus\mathbb{Z}/3\{\bar{\lambda}\},$$

where $\bar{\lambda}=\lambda-1$.

The resulting power operation of Green functors

$$P_3/\underline{J}\colon \underline{RU}_{\Sigma_3} \longrightarrow \underline{RU}\uparrow^{\Sigma_3}_{\Sigma_3\times\Sigma_3}/\underline{J}$$

is given by

$$\begin{tikzcd}
& \mathbb{Z}\{1,s,W\} \arrow[rrrr, "\left(\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}\right)"] \arrow[dl, bend right, "\left(\begin{smallmatrix}1&1&0\\0&0&1\\0&0&1\end{smallmatrix}\right)"'] \arrow[ddr, "\left(\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}\right)"'] & & & & \mathbb{Z}\{1,s\} \arrow[dl, bend right, "\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)"'] \arrow[ddr, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"'] & \\
\mathbb{Z}\{1,\lambda,\lambda^2\} \arrow[ur, bend right, "\left(\begin{smallmatrix}1&0&0\\1&0&0\\0&1&1\end{smallmatrix}\right)"'] \arrow[rrrr, "\left(\begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}\right)"] \arrow[ddr, "\left(\begin{smallmatrix}1&1&1\end{smallmatrix}\right)"'] & & & & \mathbb{Z}\oplus\mathbb{Z}/3\{\bar{\lambda}\} \arrow[ur, bend right, "\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)"'] \arrow[ddr, "\left(\begin{smallmatrix}1&0\end{smallmatrix}\right)"'] & & \\
& & \mathbb{Z}\{1,s\} \arrow[uul, bend right, "\left(\begin{smallmatrix}1&0\\0&1\\1&1\end{smallmatrix}\right)"'] \arrow[rrrr, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] \arrow[dl, bend right, "\left(\begin{smallmatrix}1&1\end{smallmatrix}\right)"'] & & & & \mathbb{Z}\{1,s\} \arrow[uul, bend right, "\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)"'] \arrow[dl, bend right, "\left(\begin{smallmatrix}1&1\end{smallmatrix}\right)"'] \\
& \mathbb{Z} \arrow[uul, bend right, "\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)"'] \arrow[ur, bend right, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] \arrow[rrrr, "1"] & & & & \mathbb{Z}. \arrow[uul, bend right, "\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right)"'] \arrow[ur, bend right, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] &
\end{tikzcd}$$

Again, the value of ψ_3 on representations may be deduced via the character embedding.

4.4. **Class functions.** Rings of class functions appear in homotopy theory as approximations to cohomology theories. In particular, equivariant KU-theory is approximated by the ring of class functions $Cl(G,\mathbb{C})$, which is the ring of $\mathbb{C}$-valued functions on the set of conjugacy classes of G. Further, Hopkins, Kuhn, and Ravenel [HKR00] have shown that the Morava E-theories, which are generalizations of p-adic KU-theory, all admit similar approximations by a ring of "generalized class functions". We introduce this ring for the "height 2" Morava E-theories in Section 4.4.2.

The rings of class functions fit together to give Green functors with restriction and induction maps compatible with the restriction and induction maps for the equivariant cohomology theory that they approximate.

4.4.1. $\mathbb{C}$*-valued class functions.* We begin by considering $Cl(G) = Cl(G,\mathbb{C})$, the ring of class functions that arises in the representation theory of finite groups. For any group G, we have a Mackey functor $\underline{Cl}_G$ defined by $\underline{Cl}_G(G/H) = Cl(H)$. For $H \leq K$, the restriction map is given by simply restricting class functions along the map $H/\mathrm{conj} \to K/\mathrm{conj}$. For $H \leq K$, the induction homomorphism

$$\mathrm{Tr}\colon Cl(H) \longrightarrow Cl(K)$$

is (cf. [Ser77, Theorem 12])

$$\mathrm{Tr}(f)(k) = \sum_{gH \in (K/H)^k} f(g^{-1}kg), \tag{4.18}$$

where $(K/H)^k$ denotes the k-fixed points under the left action of K on K/H.

Conjugacy classes in Σ_m correspond to partitions of m. Let $\{m_1,\dots,m_j\}$ be a partition of m, so that $m_1 + \dots + m_j = m$. The power operation $P_m\colon Cl(G) \longrightarrow Cl(G\times\Sigma_m)$ is given by

$$P_m(f)(g,\{m_1,\dots,m_j\}) = \prod_{i=1}^{j} f(g^{m_i}).$$

The following result is well-known.

Proposition 4.19. *The quotient $Cl(G\times\Sigma_m)/I_{\mathrm{Tr}}$ is isomorphic to $Cl(G)$, and the composition*

$$Cl(G) \xrightarrow{P_m} Cl(G\times\Sigma_m) \longrightarrow Cl(G\times\Sigma_m)/I_{\mathrm{Tr}} \cong Cl(G)$$

is the Adams operation ψ_m, given by $\psi_m(f)(g) = f(g^m)$.

Proof. For any proper partition $\{m_1, \ldots, m_j\}$ of m and conjugacy class $\overline{g}$ in G, class functions on $G \times \Sigma$ supported on $(\overline{g}, \{m_1, \ldots, m_j\})$ are in the image of the transfer along $G \times \prod_i \Sigma_{m_i} \hookrightarrow G \times \Sigma_m$. It follows that the quotient $Cl(G \times \Sigma_m)/I_{\mathrm{Tr}}$ can be identified with functions supported on $(\overline{g}, \sigma)$, where $\sigma = (1 \cdots m)$. This identifies the quotient with $Cl(G)$.

By the previous discussion, after passing to the quotient by the transfer ideals for subgroups $G \times \Sigma_i \times \Sigma_j$, only the value of the class function on the long cycle is retained, and

$$P_m(f)(g, (1 \cdots m)) = f(g^m) = \psi_m(f)(g). \qquad \square$$

Let $G_{p\text{-div}} \subset G$ denote the set of elements whose orders are divisible by p, and let $G_{p\text{-prime}} \subset G$ denote the set of elements whose orders are *not* divisible by p. Similarly, we denote by $Cl_{p\text{-div}}$ and $Cl_{p\text{-prime}}$ the rings of $\mathbb{C}$-valued functions on $G_{p\text{-div}}/\mathrm{conj}$ and $G_{p\text{-prime}}/\mathrm{conj}$, respectively. Then the decomposition $G = G_{p\text{-div}} \amalg G_{p\text{-prime}}$ induces an isomorphism of commutative rings $Cl(G) \cong Cl_{p\text{-div}}(G) \times Cl_{p\text{-prime}}(G)$.

Proposition 4.20. *The Green functor structure on $\underline{Cl}_G$ descends to Green functor structures on $\underline{Cl_{p\text{-div}}}_G$ and $\underline{Cl_{p\text{-prime}}}_G$.*

Proof. This follows from the fact that any subgroup inclusion $H \hookrightarrow G$ induces inclusions $H_{p\text{-div}} \hookrightarrow G_{p\text{-div}}$ and $H_{p\text{-prime}} \hookrightarrow G_{p\text{-prime}}$. $\square$

Proposition 4.21. *If $m = p$ is prime, the image of $\underline{J}$ under the isomorphism of Proposition 4.19 is the Mackey ideal $\underline{Cl_{p\text{-div}}}_G$. The power operation of Green functors*

$$P_p/\underline{J} \colon \underline{Cl}_G \longrightarrow \underline{Cl}\uparrow^G_{G \times \Sigma_p}/\underline{J} \cong \underline{Cl_{p\text{-prime}}}_G$$

is the composition

$$\underline{Cl}_G \xrightarrow{\text{restrict}} \underline{Cl_{p\text{-prime}}}_G \xrightarrow{\psi_p} \underline{Cl_{p\text{-prime}}}_G.$$

Proof. By Proposition 4.19, it suffices to show that for any $g \in G_{p\text{-div}}$, any class function f on $G \times \Sigma_p$ supported at $[g, \sigma]$ is in the image of the transfer from a graph subgroup as in Corollary 2.38. Let S be the cyclic subgroup of G generated by g, and let $a \colon S \longrightarrow \Sigma_p$ send g to σ. Then (4.18) shows that $\mathrm{Tr}(f)$ is also supported at $[g, \sigma]$. Furthermore, the value of $\mathrm{Tr}(f)$ at $[g, \sigma]$ is a positive integer multiple of $f([g, \sigma])$. $\square$

If G is a p-group, we get the following result.

Corollary 4.22. *Suppose that G is a p-group. Then the G-Green functor $\underline{Cl}\uparrow^G_{G \times \Sigma_p}/\underline{J}$ is isomorphic to the constant Mackey functor at $\mathbb{C}$. The power operation of Green functors*

$$\underline{Cl}_G \xrightarrow{P_p/\underline{J}} \underline{Cl}\uparrow^G_{G \times \Sigma_p}/\underline{J} \cong \underline{\mathbb{C}}$$

is given by evaluating a character at the identity element.

The following examples all follow from Proposition 4.21. We include them for comparison with the examples of Section 4.3.

Example 4.23. For the group $G = \mathbb{Z}/2$ and $m = 2$, we have

$$\begin{tikzcd}
Cl(\mathbb{Z}/2) \arrow[r, "\psi_2"] \arrow[d, bend right] & Cl(\mathbb{Z}/2 \times \Sigma_2)/I_{\mathrm{Tr}} \cong Cl(\mathbb{Z}/2) \arrow[d, bend right] \\
Cl(0) = \mathbb{C} \arrow[r, "\psi_2"] \arrow[u, bend right] & Cl(\Sigma_2)/I_{\mathrm{Tr}} \cong \mathbb{C}. \arrow[u, bend right]
\end{tikzcd}$$

Tracing the diagram using induction maps gives

$$\begin{tikzcd}
\boxed{\begin{matrix} 0 \mapsto 2a \\ 1 \mapsto 0 \end{matrix}} \arrow[rr] & & \boxed{\begin{matrix} 0,\sigma \mapsto 2a \\ 1,\sigma \mapsto 2a \end{matrix}} \arrow[r, phantom, "\neq"] & \boxed{\begin{matrix} 0,\sigma \mapsto 2a \\ 1,\sigma \mapsto 0 \end{matrix}} \\
a \arrow[u, mapsto] \arrow[rr, mapsto] & & a. \arrow[ur, mapsto] &
\end{tikzcd}$$

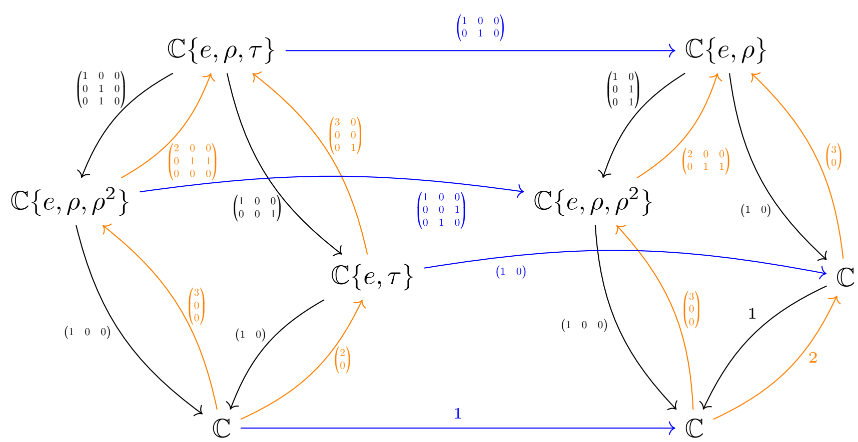
If we further quotient by the image of the transfer from the diagonal subgroup $C_2 \xrightarrow{\Delta} \mathbb{Z}/2 \times \Sigma_2$, then the induction diagram commutes, so that we have a map of Green functors. The quotient Green functor is constant at $\mathbb{C}$, and the resulting map of Green functors $P_2 \colon \underline{Cl}_{\mathbb{Z}/2} \longrightarrow \underline{\mathbb{C}}$ is given by restriction of class functions to the identity element.

Example 4.24. Consider $G = \Sigma_3$ and $m = 2$. We use ρ to denote a 3-cycle and τ to denote (any choice of) transposition. We write C_3 and C_2 for the subgroups generated by ρ and τ

Then by Proposition 4.21, in order for the power operation to be a map of Mackey functors, we must quotient $Cl(\Sigma_3)$ by the values on the conjugacy class of τ, and we must also quotient $Cl(C_2)$ by the same conjugacy class. The resulting power operation of Green functors

$$P_3/\underline{J} \colon \underline{Cl}_{\Sigma_3} \longrightarrow \underline{Cl}\uparrow^{\Sigma_3}_{\Sigma_3 \times \Sigma_2}/\underline{J}$$

is given by



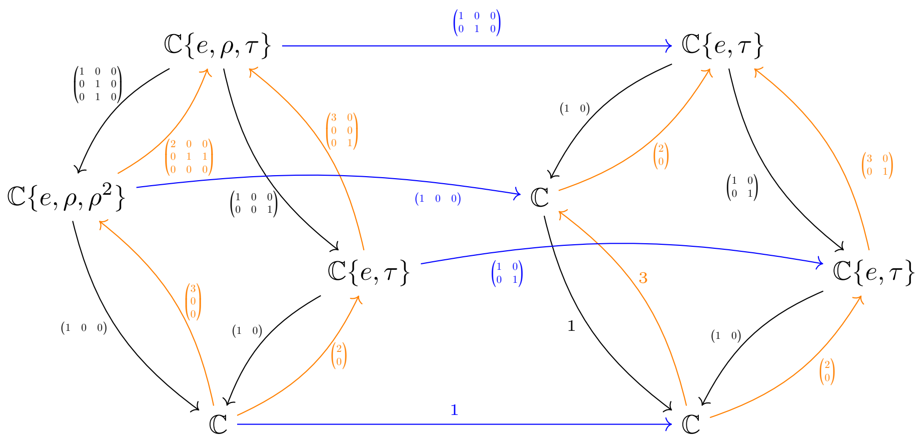
The target Green functor has been made constant on the 2-torsion subgroup.

Example 4.25. Consider again $G = \Sigma_3$, but with $m = 3$. We again use ρ to denote a 3-cycle and τ to denote (any choice of) transposition. We continue to abuse notation by writing C_3 and C_2 for the subgroups generated by ρ and τ

Then by Proposition 4.21, in order for the power operation to be a map of Mackey functors, we must quotient $Cl(\Sigma_3)$ by the values on the conjugacy classes of ρ and ρ^2, and we must also quotient $Cl(C_3)$ by the (collapsed) conjugacy class of ρ. The resulting power operation of Green functors

$$P_3/\underline{J} \colon \underline{Cl}_{\Sigma_3} \longrightarrow \underline{Cl}\uparrow^{\Sigma_3}_{\Sigma_3 \times \Sigma_3}/\underline{J}$$

is given by



The target Mackey functor has been made constant on the 3-torsion subgroup.

4.4.2. *Height* 2. We now turn our attention to height 2. Let E be height 2 Morava E-theory at the prime p, so that $E^0 \cong \mathbb{W}(k)[[u_1]]$, where k is a perfect field of characteristic p. Hopkins, Kuhn, and Ravenel [HKR00] introduced a rational E^0-algebra C_0 and produced an isomorphism of C_0-algebras

$$C_0 \otimes_{E^0} E^0(BG) \cong Cl_{2,p}(G, C_0),$$

where $Cl_{2,p}(G) = Cl_{2,p}(G, C_0)$ denotes the character ring of C_0-valued functions on conjugacy classes of commuting pairs of p-power order elements of G. This extends to a Green functor $\underline{Cl_{2,p}}_G$, where the restriction and induction maps are similar to those described in Section 4.4.1. See also [HKR00, Theorem D].

A description of the power operation

$$P_m\colon Cl_{2,p}(G) \longrightarrow Cl_{2,p}(G \times \Sigma_m)$$

can be found in the introduction to [Sta], which is a specialization of the main result of [BS17]. One important point is that the power operation on class functions depends on a *choice* of ordered set of generators for sublattices of $\mathbb{Z}_p^{\times 2}$.

Example 4.26. Let $G = \mathbb{Z}/2$, $p = 2$, and $m = 2$. We are to consider

$$\begin{tikzcd}
Cl_{2,p}(\mathbb{Z}/2) \arrow[r, "P_2"] \arrow[d, bend right] & Cl_{2,p}(\mathbb{Z}/2 \times \Sigma_2) \arrow[d, bend right] \\
Cl_{2,p}(e) \arrow[r, "P_2"] \arrow[u, bend right] & Cl_{2,p}(\Sigma_2). \arrow[u, bend right]
\end{tikzcd}$$

A class function $f \in Cl_{2,p}(\Sigma_2)$ can be displayed as a table of values

(e,e)	(e,σ)	(σ,e)	(σ,σ)
a	b	c	d,

where $a, b, c, d \in C_0$. For simplicity, we will assume that $a, b, c, d \in \mathbb{Z} \subset C_0$ so that certain ring-automorphisms of C_0 that appear in the general formula in [Sta] do not appear here. Following [Sta], the power operation $Cl_{2,p}(e) \xrightarrow{P_2} Cl_{2,p}(\Sigma_2)$ is given by

$a \mapsto$

(e,e)	(e,σ)	(σ,e)	(σ,σ)
a^2	a	a	a.

Collapsing the transfer from the subgroup $\Sigma_1 \times \Sigma_1$ of Σ_2 will eliminate the (nonlinear) value at (e,e).

We will similarly describe a class function $f \in Cl_{2,p}(\mathbb{Z}/2 \times \Sigma_2)$ via a table of values

	(e,e)	(e,σ)	(σ,e)	(σ,σ)
$(0,0)$	a	b	c	d
$(0,1)$	e	f	g	h
$(1,0)$	i	j	k	l
$(1,1)$	m	n	o	$p.$

Then a *choice* of power operation $P_2\colon Cl_{2,p}(\mathbb{Z}/2) \longrightarrow Cl_{2,p}(\mathbb{Z}/2 \times \Sigma_2)$, compatible with the second power operation on height 2 Morava E-theory at the prime 2, is

$$\begin{array}{|cccc} (0,0) & (0,1) & (1,0) & (1,1) \\ \hline a & b & c & d \end{array} \mapsto \begin{array}{c|cccc} & (e,e) & (e,\sigma) & (\sigma,e) & (\sigma,\sigma) \\ \hline (0,0) & a^2 & a & a & a \\ (0,1) & b^2 & a & {\color{blue}b} & {\color{blue}b} \\ (1,0) & c^2 & {\color{blue}c} & a & {\color{blue}b} \\ (1,1) & d^2 & {\color{blue}c} & {\color{blue}b} & a \end{array},$$

where all of the values that depend on a choice are displayed in color. Collapsing the transfer from the subgroup $\mathbb{Z}/2 \times \Sigma_1 \times \Sigma_1$ of $\mathbb{Z}/2 \times \Sigma_2$ will eliminate the first column of values, while collapsing the transfer from the diagonal subgroup of $\mathbb{Z}/2 \times \Sigma_2$ will eliminate the a's on the (slope negative one) diagonal.

Then the diagrams of restriction maps and power operations

$$\begin{tikzcd} \begin{array}{|cccc} (0,0) & (0,1) & (1,0) & (1,1) \\ \hline a & b & c & d \end{array} \arrow[r, mapsto] \arrow[d, mapsto] & \begin{array}{c|ccc} & (e,\sigma) & (\sigma,e) & (\sigma,\sigma) \\ \hline (0,0) & a & a & a \\ (0,1) & & {\color{blue}b} & {\color{blue}b} \\ (1,0) & {\color{blue}c} & & {\color{blue}b} \\ (1,1) & {\color{blue}c} & {\color{blue}b} & \end{array} \arrow[d, mapsto] \\ a \arrow[r, mapsto] & \begin{array}{|ccc} (e,\sigma) & (\sigma,e) & (\sigma,\sigma) \\ \hline a & a & a \end{array}, \end{tikzcd}$$

and the diagram of induction maps and power operations

$$\begin{tikzcd} \begin{array}{|cccc} (0,0) & (0,1) & (1,0) & (1,1) \\ \hline 2a & 0 & 0 & 0 \end{array} \arrow[r, mapsto] & \begin{array}{c|ccc} & (e,\sigma) & (\sigma,e) & (\sigma,\sigma) \\ \hline (0,0) & 2a & 2a & 2a \\ (0,1) & & 0 & 0 \\ (1,0) & 0 & & 0 \\ (1,1) & 0 & 0 & \end{array} \\ a \arrow[r, mapsto] \arrow[u, mapsto] & \begin{array}{|ccc} (e,\sigma) & (\sigma,e) & (\sigma,\sigma) \\ \hline a & a & a \end{array}, \arrow[u, mapsto] \end{tikzcd}$$

both commute.

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