MINIMAL RESOLUTION OF RELATIVELY COMPRESSED LEVEL ALGEBRAS

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Abstract. A relatively compressed algebra with given socle degrees is an Artinian quotient A of a given graded algebra $R/\mathfrak{c}$, whose Hilbert function is maximal among such quotients with the given socle degrees. For us $\mathfrak{c}$ is usually a "general" complete intersection and we usually require that A be level. The precise value of the Hilbert function of a relatively compressed algebra is open, and we show that finding this value is equivalent to the Fröberg Conjecture.

We then turn to the minimal free resolution of a level algebra relatively compressed with respect to a general complete intersection. When the algebra is Gorenstein of even socle degree we give the precise resolution. When it is of odd socle degree we give good bounds on the graded Betti numbers. We also relate this case to the Minimal Resolution Conjecture of Mustaţă for points on a projective variety.

Finding the graded Betti numbers is essentially equivalent to determining to what extent there can be redundant summands (i.e. "ghost terms") in the minimal free resolution, i.e. when copies of the same $R(-t)$ can occur in two consecutive free modules. This is easy to arrange using Koszul syzygies; we show that it can also occur in more surprising situations that are not Koszul. Using the equivalence to the Fröberg Conjecture, we show that in a polynomial ring where that conjecture holds (e.g. in three variables), the possible non-Koszul ghost terms are extremely limited.

Finally, we use the connection to the Fröberg Conjecture, as well as the calculation of the minimal free resolution for relatively compressed Gorenstein algebras, to find the minimal free resolution of general Artinian almost complete intersections in many new cases. This greatly extends previous work of the first two authors.

Contents

1. Introduction

Let k be a field and let $R = k[x_1, \dots, x_n]$ be the homogeneous polynomial ring. We will say that an Artinian k-algebra $A = R/I$ has *socle degrees* $(s_1, \dots, s_t)$ if the minimal generators of its socle (as R-module) have degrees $s_1 \leq \dots \leq s_t$. Thus, the number of

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s_j's that equal i is the dimension of the component of the socle of A in degree i. For fixed socle degrees, a graded Artinian algebra of maximal Hilbert function among all graded Artinian algebras with that socle degrees is said to be *compressed.* We extend this notion as follows:

Definition 1.1. Let $\mathfrak{c} \subset R$ be a homogeneous ideal and let $0 \leq s_1 \leq \ldots \leq s_t$ be integers. Then a graded Artinian k-algebra A is said to be *relatively compressed with respect to* $\mathfrak{c}$ and with socle degrees $(s_1, \ldots, s_t)$ if A has maximal length among all graded Artinian k-algebras R/I satisfying

(i) $\operatorname{Soc} R/I \cong \oplus_{i=1}^{t} k(-s_i)$;
(ii) $\mathfrak{c} \subset I$.

Equivalently, $A = R/I$ is relatively compressed with respect to $\mathfrak{c}$ if it is a quotient of $R/\mathfrak{c}$ having maximal length and the prescribed socle degrees. This is a slight extension (allowing $\mathfrak{c}$ to be Artinian or non-saturated) of the notion of "relatively compressed algebras" introduced in [6], Definition 2.2. The paper [14] introduces a notion of an algebra being "compressed relative to an Artin algebra," but this is unrelated to our notion.

In almost all of our work, $\mathfrak{c}$ will be a complete intersection (only in Section 2 will we extend this to allow $\mathfrak{c}$ to be Gorenstein). *Note that the complete intersection itself is not necessarily Artinian.* In many situations it is important to look for ideals that contain a regular sequence (i.e. complete intersection) in certain degrees, and to ask for such ideals that are maximal in some way. The lex-plus-powers conjecture is an example of such a problem (cf. [22]). We are interested in a similar situation, seeking an ideal that contains a regular sequence of fixed degrees, has fixed socle degrees, and has maximal Hilbert function among all such ideals.

In Section 2 of this paper we give a good motivation for studying such ideals by showing the connection to the famous conjecture of Fröberg on the Hilbert function of an ideal of general forms of fixed degrees. There is a natural guess for the Hilbert function of a relatively compressed Artinian algebra, based on an upper bound coming from the theory of inverse systems (see for instance [9], [15]). However, we also give examples to show that the "natural guess" for the Hilbert function of a relatively compressed algebra need not hold, and we also show that even the choice of the field (in positive characteristic) can affect this Hilbert function. We also show how the choice of the complete intersection can affect this, again in positive characteristic. We do not know if this can happen in characteristic zero, but anyway for many of our results we will assume that $\mathfrak{c}$ is a "general" complete intersection of fixed generator degrees.

The question of when a given Hilbert function can fail to exist for a given socle degrees, despite it satisfying the natural guess mentioned above, has been considered elsewhere. See for instance [6], [10], [26]. Our focus on algebras relatively compressed with respect to a complete intersection, and our consequences of this fact, are new. This part of our work can be viewed as a partial answer to the question asked near the end of Remark 2.9 of [6], to determine an upper bound for the Hilbert function of a relative compressed algebra, and to try to see when it may be sharp.

For most of this paper, however, we are interested in *level* graded Artinian algebras, i.e. in the case where all socle degrees are equal, so the socle is concentrated in one degree. The study of level algebras was initiated by Stanley [24]. Level graded algebras play an important role in many parts of commutative algebra, algebraic geometry and algebraic combinatorics. For instance, a sufficiently general set of points in projective space is often level (it depends on the number of points) – cf. [16]. Even the Gorenstein case, which is just a special case of level algebras, has an extensive literature. See [10] for an extensive bibliography and overview of level algebras.

The level graded Artinian algebras of maximal Hilbert function among all level graded algebras of given codimension and socle degree are called *compressed level algebras* and they fill a non-empty Zariski open set in the natural parameter space. If the socle has dimension one and occurs in degree s then the algebra R/I is Gorenstein and I can be identified with the ideal consisting of partial derivatives of all orders annihilating a general polynomial $f \in R$ of degree s. The notion of a relatively compressed Gorenstein algebra naturally arises by requiring that a priori some partial derivatives of f vanish.

Beyond finding the Hilbert function, a much more subtle question is to understand all of the syzygies of a relatively compressed level Artinian algebra, i.e. to find the minimal free R-resolution of A or, equivalently, the graded Betti numbers. A central part of this is to determine if there can be redundant summands (i.e. "ghost terms") in consecutive free modules in the minimal free resolution. Sometimes this is easy to force with Koszul syzygies. The interesting situation is when there are syzygies that cannot be explained by Koszul relations, the so-called *non-Koszul ghost terms*.

Note that even in the case of compressed level Artinian algebras, very little is known about the minimal free resolution. In [4] Corollary 3.10, Boij showed that there is a well-defined notion of "generic" Betti numbers for compressed level algebras of fixed socle degrees, and in Conjecture 3.13 he guessed what they may be. The main point is that there should be no ghost terms. The first case is that of Gorenstein algebras. When the socle degree is even, the result was well-known, following from the almost purity of the minimal free resolution. In the case of odd socle degree, the result was shown by the first and second authors in [18] Proposition 3.13, as long as the initial degree is sufficiently large. Very little is known beyond this.

In Section 3, our first main result is to give the precise minimal free resolution of a relatively compressed Artinian Gorenstein algebra, A, of even socle degree. Here we assume that k has characteristic zero, or else that the characteristic satisfies a certain numerical condition (see Remark 3.6). We find that any ghost terms in the minimal free resolution of A occur either directly because of Koszul relations, or indirectly because of Koszul relations and duality. We also give a similar result for relatively compressed Artinian Gorenstein algebras of odd socle degree, but here we are not able to give the precise resolution (but we show where there is uncertainty). However, we are able to give the precise resolution in odd socle degree when the embedding dimension is 4, the socle degree is odd, and A is relatively compressed with respect to a general quadric. We also give a nice connection to the Minimal Resolution Conjecture for points on complete intersection varieties (as special case of a conjecture of Mustaţă), showing that if the conjecture holds then we can find the minimal free resolution of a general Gorenstein Artinian algebra of odd socle degree, relatively compressed with respect to a general complete intersection of codimension $\leq n-2$.

Section 4 deals with level algebras of socle dimension ≥ 2. Our main goal is to see how it can happen that the minimal free resolution has non-Koszul ghost terms. We give some conditions that force such ghost terms. We also give several examples and conjectures. Finally, we show that if R satisfies Fröberg's Conjecture then the minimal free resolution of a relatively compressed level algebra can have non-Koszul ghost terms only in a very limited way. This holds, for example, if $n = 3$.

In Section 5 we go in the opposite direction. It is known that in characteristic zero (and slightly more generally) an ideal of $n+1$ general forms satisfies Fröberg's conjecture. It has been conjectured that an ideal of $n+1$ general forms has the "expected" minimal free resolution in the sense that the Betti numbers are the minimal ones consistent with the Hilbert function (i.e. no ghost terms). The first and second authors showed this in several cases in [18] (and also gave some counterexamples). Here, using our result for Gorenstein algebras in Section 3, we show that an ideal of $n+1$ general forms (with generator degrees satisfying certain conditions) must have the predicted graded Betti numbers, extending the known results.

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2. Relatively compressed algebras and Fröberg's conjecture

Throughout this paper we will use the following notation:

Notation 2.1. Let k be an infinite field (often making further assumptions, such as characteristic zero). Let $R = k[x_1, \cdots, x_n]$ and let A be a graded k-algebra. The Hilbert function of A is denoted by $h_A(t) := \dim_k A_t$.

In this section we give some basic results about relatively compressed algebras. The main purpose is to establish the connection between them and Fröberg's conjecture on the Hilbert function of an ideal of general forms. However, we also discuss the failure of the "expected" Hilbert function to occur even with generic choices.

If the Artinian algebra A has socle degrees $(s_1, \ldots, s_t)$ then $s_t = \max\{s_1, \ldots, s_t\}$ is called *the socle degree* of A. It equals the Castelnuovo-Mumford regularity of A. Moreover, by a *general* homogeneous polynomial of degree d we mean a polynomial in a suitable Zariski open and dense subset of R_d. Similarly, a general complete intersection is generated by general polynomials.

We now begin with a simple remark.

Remark 2.2. Let $A = R/I$ be an Artinian algebra with socle degrees $(s_1, \ldots, s_t)$. Let $\mathfrak{c} \subsetneq I$ be an Artinian Gorenstein ideal. Denote by e the socle degree of $R/\mathfrak{c}$. Let $J := \mathfrak{c} : I$ be the residual ideal. Then there are homogeneous forms $G_1, \ldots, G_t \in R$ of degree $e - s_1, \ldots, e - s_t$ such that

$$J = \mathfrak{c} + (G_1, \ldots, G_t).$$

In fact, this follows from the standard mapping cone procedure that relates the resolutions of $I, J, \mathfrak{c}$.

We will say that $e - s_1, \ldots, e - s_t$ are the *expected degrees* of the extra generators of J, i.e. of the minimal generators of J that are not in $\mathfrak{c}$.

We would like to generalize this remark. Let $A = R/I$ be an Artinian algebra with socle degrees $(s_1, \ldots, s_t)$. Let $\mathfrak{c} \subsetneq I$ be a Gorenstein ideal of codimension c. Denote by e the Castelnuovo-Mumford regularity of $R/\mathfrak{c}$ and put

$$s := \max\{s_1, \ldots, s_t\}.$$

Let $F_1, \ldots, F_{n-c} \in I$ be general forms of degree $s + 1$. Then

$$\mathfrak{c}' := \mathfrak{c} + (F_1, \ldots, F_{n-c})$$

is an Artinian Gorenstein ideal with socle degree

$$e' = e + (n - c)s$$

because $I_{s+1} = R_{s+1}$.

Remark 2.3. If $\mathfrak{c}$ is already Artinian then $c = n$, $\mathfrak{c} = \mathfrak{c}'$, $e = e'$ and there are no forms F_i needed.

Now we link. Let $J := \mathfrak{c}' : I$ be the residual ideal. Then there are homogeneous forms $G_1, \ldots, G_t \in R$ of degree $e' - s_1, \ldots, e' - s_t$ such that

$$J = \mathfrak{c}' + (G_1, \ldots, G_t) = \mathfrak{c} + (F_1, \ldots, F_{n-c}) + (G_1, \ldots, G_t).$$

In fact, this follows from the standard mapping cone procedure that relates the resolutions of I, J and $\mathfrak{c}$.

Hence, keeping the notation above, we get an upper bound for the Hilbert function of a relatively compressed algebra with respect to $\mathfrak{c}$.

Lemma 2.4. *Let $A = R/I$ be a relatively compressed algebra with respect to $\mathfrak{c}$ and with socle degrees $(s_1, \ldots, s_t)$. Let $G'_1, \ldots, G'_t \in R$ be general homogeneous forms of degree $e' - s_1, \ldots, e' - s_t$ and put $J' := \mathfrak{c}' + (G'_1, \ldots, G'_t)$. Then we have for all integers j*

$$h_A(j) \leq h_{R/\mathfrak{c}'}(j) - h_{R/J'}(e' - j).$$

Proof. By Liaison theory we have the formula

$$h_A(j) = h_{R/\mathfrak{c}'}(j) - h_{R/J}(e' - j).$$

Using Remark 2.2 we get for all integers j

$$h_{R/J}(j) \geq h_{R/J'}(j)$$

by the choice of the generators of J'. The claim follows. □

Note that the Hilbert function of $R/\mathfrak{c}'$ is determined by the Hilbert function of $R/\mathfrak{c}$ and s. For example, if $c = n - 1$ then

$$h_{R/\mathfrak{c}'}(j) = h_{R/\mathfrak{c}}(j) - h_{R/\mathfrak{c}}(j - s - 1)$$

for all integers j.

If we start with J' we get:

Corollary 2.5. *Let $F_1, \ldots, F_{n-c} \in R$ be general forms of degree $s + 1$. If the union of a minimal basis of $\mathfrak{c}$, $\{F_1, \ldots, F_{n-c}\}$, and $\{G'_1, \ldots, G'_t\}$ is a minimal basis of J' then we have equality in Lemma 2.4, i.e.*

$$h_A(j) = h_{R/\mathfrak{c}'}(j) - h_{R/J'}(e - j).$$

Proof. The assumption on the minimal generators of J' guarantees that $A = R/I$, where $I := \mathfrak{c}' : J'$, has socle degrees $(s_1, \ldots, s_t)$. Hence Lemma 2.4 shows that A is compressed with respect to $\mathfrak{c}$. □

Remark 2.6. Note that by semicontinuity, the Hilbert function of J' does not depend on the choice of the polynomials G'_i. It is determined by $R/\mathfrak{c}$ and the degrees $e' - s_1, \ldots, e' - s_t$ of the extra generators (since these were chosen generally). In general there are no explicit formulas. However, if $\mathfrak{c}$ is a general complete intersection then Fröberg's conjecture (see below) predicts the precise value of the Hilbert function. But independently of Fröberg, if the assumption of Corollary 2.5 is satisfied then in principle it allows us to compute (at least on the computer) the Hilbert function of a relatively compressed algebra with respect to $\mathfrak{c}$ and with socle degrees $(s_1, \ldots, s_t)$ from the given data.

Thanks to a conjecture of Fröberg ([7], page 120) all this can be made (conjecturally) more explicit if $\mathfrak{c}$ is a general complete intersection. In order to state this conjecture recall that the Hilbert series of an algebra A is the formal power series

$$H_A(Z) := \sum_{j \geq 0} h_A(j) Z^j.$$

We define for a power series $\sum_{j\geq 0} a_j Z^j$ with real coefficients

$$\left|\sum_{j\geq 0} a_j Z^j\right| := \sum_{j\geq 0} b_j Z^j$$

where

$$b_j := \begin{cases} a_j & \text{if } a_i \geq 0 \text{ for all } i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 2.7 (Fröberg)**.** *Let $J \subset R$ be an ideal generated by general forms of degree $d_1, \ldots, d_r$. Then the Hilbert series of R/J is*

$$H_{R/J}(Z) = \left| \frac{\prod_{i=1}^r (1 - Z^{d_i})}{(1-Z)^n} \right|.$$

Note that it is easy to see that

$$H_{R/J}(Z) \geq \left| \frac{\prod_{i=1}^r (1 - Z^{d_i})}{(1-Z)^n} \right|$$

where the estimate compares the coefficients of same degree powers. Moreover, the conjecture was proved to be true for $n = 2$ in [7], for $n = 3$ in [1] and for arbitrary n if J is a complete intersection or if J is an almost complete intersection [23].

Now we specialize to the case where $\mathfrak{c} \subset R$ is a complete intersection of type $(d_1, \ldots, d_c)$. Then the Castelnuovo-Mumford regularity of $R/\mathfrak{c}$ is $e = d_1 + \ldots + d_c - c$.

Corollary 2.8. *Let $A = R/I$ be a relatively compressed algebra with respect to the complete intersection $\mathfrak{c}$ and with socle degrees $(s_1, \ldots, s_t)$. Put*

$$e' = (n-c)s + d_1 + \ldots + d_c - c.$$

Then

$$H_A(Z) \leq \left| \frac{\prod_{i=1}^c (1 - Z^{d_i}) \cdot (1 - Z^{s+1})^{n-c}}{(1-Z)^n} \right| - \left| \frac{\prod_{i=1}^c (1 - Z^{d_i}) \cdot (1 - Z^{s+1})^{n-c} \cdot \prod_{i=1}^t (1 - Z^{e'-s_i})}{(1-Z)^n} \right|.$$

Proof. Using the notation of Lemma 2.4 it is easy to see that

$$H_{R/J'}(Z) \geq \left| \frac{\prod_{i=1}^c (1 - Z^{d_i}) \cdot (1 - Z^{s+1})^{n-c} \cdot \prod_{i=1}^t (1 - Z^{e'-s_i})}{(1-Z)^n} \right|.$$

The claim follows. □

In fact, if $\mathfrak{c}$ is a *general* complete intersection then we often expect equality.

Conjecture 2.9. *Let $J \subset R$ be the ideal generated by general forms $G_1, \ldots, G_{c+t}$ of degree $d_1, \ldots, d_c, e - s_1, \ldots, e - s_t$ and general forms $F_1, \ldots, F_{n-c}$ of degree $s + 1$. Assume that $\{G_1, \ldots, G_{c+t}, F_1, \ldots, F_{n-c}\}$ is a minimal basis of J. Then the Hilbert series of a relatively compressed algebra A with respect to a* general *complete intersection of type $(d_1, \ldots, d_n)$ and with socle degrees $(s_1, \ldots, s_t)$ is*

$$H_A(Z) = \left| \frac{\prod_{i=1}^c (1 - Z^{d_i}) \cdot (1 - Z^{s+1})^{n-c}}{(1-Z)^n} \right| - \left| \frac{\prod_{i=1}^c (1 - Z^{d_i}) \cdot (1 - Z^{s+1})^{n-c} \cdot \prod_{i=1}^t (1 - Z^{e'-s_i})}{(1-Z)^n} \right|.$$

Remark 2.10. By Corollary 2.5 this conjecture is true if and only if Fröberg's conjecture is true in the corresponding case. Note that granting Fröberg's conjecture, the assumption of Conjecture 2.9 can be translated into a purely numerical condition involving only the numbers $d_1, \ldots, d_n$, $e - s_1, \ldots, e - s_t$ in every specific example though it seems difficult to make this explicit in general.

Remark 2.11. Potentially, the Hilbert function of a relatively compressed algebra with respect to a specific complete intersection could differ from the one with respect to a *general* complete intersection of the same type, and the result can change as the characteristic varies. While it might not be the case over fields of characteristic zero (it is an open question) that different complete intersections yield different Hilbert functions for relatively compressed algebras of the same type, this phenomenon does occur over fields of positive characteristic. The following example illustrates all these things.

Example 2.12. We illustrate the assertions of Remark 2.11. We will consider the case of three variables, $R = k[x_1, x_2, x_3]$ (leaving open for now the characteristic), and a complete intersection $\mathfrak{c} = (F_1, F_2, F_3)$ generated by three quartics. This complete intersection has Hilbert function

$$1\ 3\ 6\ 10\ 12\ 12\ 10\ 6\ 3\ 1.$$

We will be interested in relatively compressed Gorenstein algebras of socle degree 8. The expected Hilbert function for this algebra is

$$1\ 3\ 6\ 10\ 12\ 10\ 6\ 3\ 1,$$

and by [15] Theorem 4.16, this is achieved if k has characteristic zero and if, in addition, the complete intersection is either the monomial complete intersection (x_1^4, x_2^4, x_3^4), or general.

We now consider a field k of characteristic 2. A standard mapping cone argument (as used also elsewhere in this paper) gives that the relatively compressed Gorenstein algebra is linked via $\mathfrak{c}$ to an ideal I with four generators, and the fact that the desired socle degree of the Gorenstein algebra is 8 yields that $I = (L, F_1, F_2, F_3)$, where L is a linear form. The fact that the Gorenstein algebra has maximal Hilbert function tells us that the linear form is general, and that the Hilbert function of I is as small as possible among ideals with these generator degrees. Note also that in order to obtain the expected Hilbert function for the relatively compressed Gorenstein algebra, the Hilbert function of I must be

$$1\ 2\ 3\ 4\ 2.$$

We first consider the complete intersection $\mathfrak{c} = (x_1^4, x_2^4, x_3^4)$. If $L = ax_1 + bx_2 + cx_3$ is a general linear form, then (because of the characteristic) in fact I is a complete intersection, $I = (L, x_1^4, x_2^4)$. Its Hilbert function is

$$1\ 2\ 3\ 4\ 3\ 2\ 1.$$

This is not the required Hilbert function, so $R/\mathfrak{c}$ does not have a relatively compressed Gorenstein algebra with the predicted Hilbert function. In fact, the Hilbert function of a relatively compressed Gorenstein algebra with respect to this complete intersection is

$$1\ 3\ 6\ 9\ 10\ 9\ 6\ 3\ 1.$$

In fact, by studying the mapping cone for the link of I via this complete intersection, one sees that the relatively compressed Gorenstein algebra is in fact itself a complete intersection of type $(3, 4, 4)$.

Now we consider what happens (still in characteristic 2) if we change the generators. We have verified using `macaulay` that in characteristic 2, if we change the complete intersection to (F_1, F_2, F_3) where

$$\begin{aligned}
F_1 &= x_1^4 + x_1x_2^3 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_1x_3^3 + x_2x_3^3 \\
F_2 &= x_1^3x_2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_2^3x_3 + x_2^2x_3^2 + x_2x_3^3 + x_3^4 \\
F_3 &= x_1x_2^3 + x_1^3x_3 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_2^3x_3 + x_1^2x_3^2 + x_2^2x_3^2 + x_1x_3^3
\end{aligned}$$

and take $L = x_2$, then the Hilbert function we get is the same as that obtained in characteristic zero, the expected one, completing our verification of the assertions in Remark 2.11.

Notice that similar behavior occurs when we look for relatively compressed Gorenstein algebras with respect to any monomial complete intersection where all the generators have the same degree and this degree is a multiple of the characteristic.

The ideas in this example are directly related to the question of whether $R/\mathfrak{c}$ has the Weak Lefschetz Property – cf. [12] Remark 2.9 and Corollary 2.4.

We now consider relatively compressed *level* algebras. First recall (cf. [14], [8]) that a *compressed* (in the classical sense) level Artinian algebra A with socle degree s and socle dimension c has Hilbert function

$$h_A(t) = \min\{\dim R_t, c \cdot \dim R_{s-t}\}.$$

(The idea is that from the left and from the right the function grows as fast as it theoretically can; so it is not hard to see that this is an upper bound, but the hard part is to show that this bound is actually achieved, and even more there is an irreducible parameter space for which the bound is achieved on a Zariski-open subset.) For example, when $n = 3, s = 10$ and $c = 3$ we get the Hilbert function of the compressed level algebra to be 1 3 6 10 15 21 28 30 18 9 3.

We now want to consider relatively compressed level Artinian algebras, and in particular algebras that are relatively compressed in a complete intersection. So, letting $\mathfrak{c} \subset R$ be a complete intersection, we consider relatively compressed level quotients of $R/\mathfrak{c}$. Occasionally such algebras are compressed in the classical sense, but not usually. We are interested in both the Hilbert function and the minimal free resolution of such algebras.

We first determine another upper bound for the Hilbert function of a graded level Artinian algebra A that is relatively compressed with respect to a complete intersection $\mathfrak{c} \subset R$.

Lemma 2.13. *Let $A = R/I$ be a graded level Artinian algebra of socle dimension c, socle degree s and relatively compressed with respect to a complete intersection $\mathfrak{c} \subset R$. Then*

$$h_A(t) \leq \min\{\dim(R/\mathfrak{c})_t, c \cdot \dim(R/\mathfrak{c})_{s-t}\}.$$

Proof. We use the theory of inverse systems (cf. [9], [15]) and refer to those sources for the necessary background. The following short summary is taken from [10] Chapter 5. Let $S = k[y_1, \dots, y_n]$. We consider the action of R on S by differentiation: if $F \in S_j$ then $x_i \circ F = (\frac{\partial}{\partial y_i})F$. There is an order-reversing function from the ideals of R to the R-submodules of S defined by

$$\phi_1 : \{\text{ideals of } R\} \to \{R\text{-submodules of } S\}$$

defined by

$$\phi_1(I) = \{F \in S \mid G \circ F = 0 \text{ for all } G \in I\}.$$

This is a 1-1 correspondence, whose inverse ϕ_2 is given by $\phi_2(M) = \mathrm{Ann}_R(M)$. We denote $\phi_1(I)$ by I^{-1}, called the *inverse system* to I. The pairing

$$R_j \times S_j \to S_0 \cong k$$

is perfect. For a subspace V of R_j we write $V^\perp \subset S_j$ for the annihilator of V in this pairing. If F is an element of $[\mathfrak{c}_j]^\perp$ then the ideal $J = \mathrm{Ann}(F)$ is a Gorenstein ideal containing $\mathfrak{c}$, of socle degree j.

Now we proceed by induction on c. For $c = 1$, we take $F \in [\mathfrak{c}_s]^\perp$ and we consider the Gorenstein graded algebra $R/\operatorname{Ann}(F)$. Because $R/\operatorname{Ann}(F)$ is a quotient of $R/\mathfrak{c}$, which is Gorenstein, we clearly have

$$h_A(t) \le \min\{\dim(R/\mathfrak{c})_t, \dim(R/\mathfrak{c})_{s-t}\}. \tag{1}$$

Note that by [15], Theorem 4.16, if $\mathfrak{c}$ and F are both general (or if $\mathfrak{c}$ is a monomial complete intersection and F is general) and if k has characteristic zero (but see also Remark 3.6) then we have equality in (1) and $R/\operatorname{Ann}(F)$ is relatively compressed. To prove the general case we can choose independent elements $F_1, \dots, F_c \in [\mathfrak{c}_s]^\perp$. Summing up the Hilbert functions of the Gorenstein quotients $R/\operatorname{Ann}(F_1), \dots, R/\operatorname{Ann}(F_c)$ of $R/\mathfrak{c}$, we get

$$\dim(R/\operatorname{Ann}(F_1, \dots, F_c))_t \le \sum_{i=1}^{c} \dim(R/\operatorname{Ann}(F_i))_t$$

from which the result follows. □

It was noted in the proof above that when $c = 1$ and $\mathfrak{c}$ and F are general, and if the field has characteristic zero or satisfies a certain numerical condition (see Remark 3.6) then the Gorenstein quotient that we get is relatively compressed with respect to $\mathfrak{c}$, i.e. the inequality (1) is an equality. It is natural to ask if the same holds for $c \ge 2$. We now present some examples that show that the naive guess that the inequality in the statement of Lemma 2.13 is an equality even for "generic" choices, is not always correct.

Example 2.14. Consider the general complete intersection $\mathfrak{c}$ of type $(3, 3, 3)$ in 3 variables. Its Hilbert function is 1 3 6 7 6 3 1. Suppose we want a level algebra $A = R/I$ with $s = 5$ and $c = 2$ that is relatively compressed with respect to $R/\mathfrak{c}$. One would "expect" that its Hilbert function would be 1 3 6 7 6 2.

First, we note that there is an algebra with this Hilbert function, which is a quotient of that complete intersection, but it *must* have a ghost term making it not be level. The reason comes from liaison theory. Let J be the residual to I with respect to the complete intersection $\mathfrak{c}$. Then R/J would have Hilbert function 1 1 (cf. [17] Corollary 5.2.19), so its resolution begins

$$\cdots \to R(-2) \oplus R(-3)^2 \to R(-1)^2 \oplus R(-2) \to J \to 0.$$

Since the complete intersection $\mathfrak{c}$ does not contain a quadric, the mapping cone procedure (cf. [17] Proposition 5.2.10) does not split off any summands corresponding to generators of J, so the ideal I has resolution that ends

$$0 \to R(-8)^2 \oplus R(-7) \to R(-7) \oplus R(-6)^5 \to \dots$$

and so is not level.

Second, we show that there is a relatively compressed algebra that has Hilbert function 1 3 6 7 5 2. Indeed, it can be obtained as the residual of a complete intersection of type $(1, 1, 3)$ inside complete intersection of type $(3, 3, 3)$.

Finally, we observe that all this can even be done at the level of points in $\mathbb{P}^3$ by lifting the Artinian ideals to ideals of sets of points: simply start with a set of three points on a line in $\mathbb{P}^3$ and link using three cubics.

Example 2.15. Now we consider ideals in the polynomial ring R with 4 variables. Let $\mathfrak{c}$ be a general complete intersection of type (3,3,3,3). Its Hilbert function is

1 4 10 16 19 16 10 4 1.

Suppose we look for a relatively compressed level algebra with s = 7 and c = 2. A first guess could be that the correct Hilbert function should be 1 4 10 16 19 16 8 2. If this were true, the residual J in the complete intersection $(3,3,3,3)$ would have Hilbert function 1 2 2. But such J has (at least) generators of degrees 1, 1, 2, 3, 3, so the mapping cone procedure shows that R/I has Cohen-Macaulay type (at least) 3, hence it is not a level algebra. Again there is a ghost term. Note that generically we get Cohen-Macaulay type exactly 3.

On the other hand, we can again construct a relatively compressed level algebra with $s = 7$, $c = 2$, and Hilbert function 1 4 10 16 19 16 7 2. This is done by starting with an algebra with Hilbert function 1 2 3. Its generators will be of degree 1, 1, 3, 3, 3, 3 so the complete intersection will split off all the terms corresponding to the cubic generators, leaving a residual that is a level algebra.

Example 2.16. This time we will even compute the graded Betti numbers, not just the "surprising" Hilbert function. We work over a polynomial ring with 3 variables. Recall that in this case we know Fröberg's conjecture to hold, thanks to work of Anick [1], which gives the Hilbert function of an ideal generated by general forms of any prescribed degrees (as illustrated below).

Consider a level algebra with $s = 7$, $c = 2$ that is relatively compressed with respect to the general complete intersection $\mathfrak{c}$ of type $(4,4,4)$. It is constructed as follows. Adjoining two general forms of degree 2 to the general complete intersection $\mathfrak{c}$ of type $(4,4,4)$ we get (using Anick's result) successively the Hilbert functions

$$\begin{array}{cccccccccc} 1 & 3 & 6 & 10 & 12 & 12 & 10 & 6 & 3 & 1 \\ 1 & 3 & 5 & 7 & 6 & 2 \\ 1 & 3 & 4 & 4 & 1. \end{array}$$

The residual with respect to $\mathfrak{c}$ provides the desired algebra A. It has Hilbert function

$$1\ 3\ 6\ 10\ 12\ 11\ 6\ 2,$$

where one might have expected in degree 5 a 12 rather than 11. The reason that there is no level algebra with a 12 rather than an 11 in degree 5, that is a quotient of $R/\mathfrak{c}$, is precisely that such an algebra would be residual to an algebra with Hilbert function 1, 3, 4, 4, which has too many generators to allow the residual to be level! Note however that there is a level algebra R/I with Hilbert function $1,3,6,10,12,12,6,2$ if we do not require that I contains a regular sequence of type $(4,4,4)$. It can be constructed as Artinian quotient of the coordinate ring of 12 points in $\mathbb{P}^2$ using [10], Proposition 7.1.

In order to compute the graded Betti numbers of A we start with the set X of 3 general points in $\mathbb{P}^3$. Their resolution has the shape

$$0 \to R^2(-4) \to R^5(-3) \to R(-1) \oplus R^3(-2) \to I(X) \to 0.$$

Linking by a complete intersection of type $(2,2,4)$ we get a residual J whose Betti numbers read as

$$0 \to R(-7) \oplus R(-6) \to R^5(-5) \oplus R(-4) \to R^3(-4) \oplus R^2(-2) \to J \to 0$$

because the two generators of degree 2 split off while the Koszul ghost term does not split off.

Since J contains a complete intersection of type (2,2,4), it certainly contains one of type (4,4,4) as well. Linking again, this time by a complete intersection of type $(4,4,4)$, we get an algebra A as above. After splitting off the three quartics the mapping cone procedure provides that its minimal free resolution has the form

$$0 \to R^2(-10) \to R(-8) \oplus R^5(-7) \to R(-6) \oplus R(-5) \oplus R^3(-4) \to J \to 0.$$

3. Minimal free resolution of relatively compressed Gorenstein Artinian algebras

In the previous section we discussed what the "expected" behavior should be for the Hilbert function of a relatively compressed level algebra, and how this is sometimes not achieved. We now begin our study of the following problem:

Problem 3.1. *To determine the "generic" graded Betti numbers (or, equivalently, the minimal free R-resolution) of Artinian level graded algebras of embedding dimension n, socle degree s and socle dimension c and relatively compressed with respect to a general complete intersection $\mathfrak{a} \subset R$.*

In this section we consider the minimal free R-resolution of Gorenstein Artinian graded algebras of embedding dimension n that are relatively compressed with respect to the ideal $\mathfrak{a} \subset k[x_1, ..., x_n]$ of a general complete intersection of type $(d_1, ..., d_r)$, $r \leq n$. We consider the case of even socle degree and odd socle degree separately. For even socle degree we completely determine the minimal free R-resolution. We also show that all redundant ("ghost") terms that appear are due to Koszul syzygies (or are forced by duality from Koszul syzygies). For odd socle degree we do not have quite as clean a statement, but we show in Example 3.10 that this is the best we could hope for.

We begin with even socle degree. We will see that once we fix n, t and $(d_1, ..., d_r)$ *all* have the same graded Betti numbers (there is no need for a "generic" choice). This was known only for compressed level algebras of even socle degree:

Proposition 3.2. *Let A be a compressed Gorenstein Artinian graded algebra of embedding dimension n and socle degree $2t$. Then A has a minimal free R-resolution of the following type:*

$$0 \to R(-2t-n) \to R(-t-n+1)^{\alpha_{n-1}} \to ... \to R(-t-p)^{\alpha_p} \to$$

$$.... \to R(-t-2)^{\alpha_2} \to R(-t-1)^{\alpha_1} \to R \to A \to 0$$

where

$$\alpha_i = \binom{t+i-1}{i-1}\binom{t+n}{n-i} - \binom{t-1+n-i}{n-i}\binom{t-1+n}{i-1}$$

for $i = 1, ..., n-1$.

Proof. Because the socle degree is even, in fact A is a so-called *extremely compressed* Artin level algebra. Then the result follows from [4] Proposition 3.6 or [14] Proposition 4.1. □

Remark 3.3. The above result is also a special case of [20], Theorem 8.14. This latter result has the extra hypothesis that A has the Weak Lefschetz Property (i.e. that multiplication by a general linear form, from any component to the next, has maximal rank). However, it was noted in [18] Remark 3.6(c) that in the situation of compressed Gorenstein algebras of even socle degree, this property is automatically satisfied.

Note also that the formula for α_i given above is not presented in the same way as it is in [20] Theorem 8.14, but a calculation shows that they are equivalent.

From now on, when we say that A is a Gorenstein Artinian graded algebra of embedding dimension n, even socle degree $2t$ and relatively compressed with respect to a general complete intersection ideal $\mathfrak{a} \subset k[x_1, ..., x_n]$ of type $d_1 \leq \cdots \leq d_r$ we will assume without loss of generality that $d_r \leq t$; otherwise A is a Gorenstein Artinian graded algebra of embedding dimension n, even socle degree $2t$ and relatively compressed with respect to a general complete intersection ideal $\mathfrak{b} \subset k[x_1, ..., x_n]$ of type $d_1 \leq \cdots \leq d_j$ where $d_j = \max_{1 \leq i \leq r}\{d_i \leq t\}$.

We first fix some notation that we will use from now on.

Notation 3.4.

(1) Given a complete intersection ideal $\mathfrak{a} = (G_1, \ldots, G_r) \subset R = k[x_1, ..., x_n]$ with $r \leq n$ and $d_1 = deg(G_1) \leq \cdots \leq d_r = deg(G_r)$, we denote by $K_i(d_1, \cdots, d_r)$ (or, simply, $K_i(\underline{d})$ if $\underline{d} = (d_1, \cdots, d_r)$) the i-th module of syzygies of $R/\mathfrak{a}$. So, we have

$$K_i(d_1, \cdots, d_r) = K_i(\underline{d}) := \bigwedge^i \left(\bigoplus_{i=1}^r R(-d_i) \right)$$

and the minimal free R-resolution of $R/\mathfrak{a}$:

$$0 \to K_r(\underline{d}) \to K_{r-1}(\underline{d}) \to \cdots \to K_2(\underline{d}) \to K_1(\underline{d}) \to R \to R/\mathfrak{a} \to 0.$$

(2) For any free R-module $F = \oplus_{t \in \mathbb{Z}} R(-t)^{\alpha_t}$ and any integer $y \in \mathbb{Z}$, we set

$$F^{\leq y} := \oplus_{t \leq y} R(-t)^{\alpha_t}.$$

Theorem 3.5. *Let $A = R/I$ be a Gorenstein Artinian graded algebra of embedding dimension n and socle degree $2t$, where $R = k[x_1, \ldots, x_n]$ and k has characteristic zero. Assume that A is relatively compressed with respect to a general complete intersection ideal $\mathfrak{a} = (G_1, \cdots, G_r)$, $r \leq n$ and $\deg(G_1) = d_1, \ldots, \deg(G_r) = d_r$. Set $\underline{d} = (d_1, \cdots, d_r)$. Then, A has a minimal free R-resolution of the following type:*

$$\begin{gathered} 0 \to R(-2t-n) \to R(-t-n+1)^{\alpha_{n-1}(\underline{d},n,t)} \oplus K_{n-1}(\underline{d})^{\leq t+n-2} \oplus K_1(\underline{d})^\vee(-2t-n) \to \\ R(-t-n+2)^{\alpha_{n-2}(\underline{d},n,t)} \oplus K_{n-2}(\underline{d})^{\leq t+n-3} \oplus (K_2(\underline{d})^{\leq t+1})^\vee(-2t-n) \to \cdots \to \\ \to R(-t-2)^{\alpha_2(\underline{d},n,t)} \oplus K_2(\underline{d})^{\leq t+1} \oplus (K_{n-2}(\underline{d})^{\leq t+n-3})^\vee(-2t-n) \to \\ \oplus_{j=1}^r R(-d_j) \oplus R(-t-1)^{\alpha_1(\underline{d},n,t)} \oplus (K_{n-1}(\underline{d})^{\leq t+n-2})^\vee(-2t-n) \to R \to A \to 0 \end{gathered}$$

where

$$\alpha_i(\underline{d}, n, t) = \alpha_{n-i}(\underline{d}, n, t) \text{ for } i = 1, \cdots n-1,$$

and $\alpha_i(\underline{d}, n, t)$ is completely determined by the Hilbert function of A.

Proof. Since $\mathfrak{a} \subset I$ and A is relatively compressed with respect to $\mathfrak{a}$, we have $h_A(\nu) = \min\{\dim(R/\mathfrak{a})_\nu, \dim(R/\mathfrak{a})_{2t-\nu}\}$ (thanks to [15], Theorem 4.16). Thus $\mathfrak{a}_\nu = I_\nu$ for all $\nu \leq t$. We deduce that A has a minimal free R-resolution of the following type

$$0 \to R(-2t-n) \to F_{n-1} \to F_{n-2} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0 \tag{2}$$

where

$$F_i = \oplus_{m \geq t+i} R(-m)^{\alpha_{i,m}(\underline{d},n,t)} \oplus K_i(\underline{d})^{\leq t+i-1},$$

$\underline{d} = (d_1, \cdots, d_r)$, $K_i(\underline{d}) = K_i(d_1, \cdots, d_r)$ and $\alpha_{1,t+1}(\underline{d}, n, t)$ is completely determined by the Hilbert function of A. More precisely, $\alpha_{1,t+1}(\underline{d}, n, t) = \dim(R/\mathfrak{a})_{t+1} - \dim(A)_{t+1}$.

Recall that the minimal free R-resolution of a Gorenstein Artinian algebra is self-dual (up to shift). Dualizing (2) and twisting by $R(-2t-n)$, we get

$$\begin{gathered} 0 \to R(-2t-n) \to F_1^\vee(-2t-n) \to F_2^\vee(-2t-n) \to \cdots \\ F_{n-2}^\vee(-2t-n) \to F_{n-1}^\vee(-2t-n) \to R \to A = R/I \to 0. \end{gathered}$$

Therefore, for all $1 \le i \le n-1$, we have the isomorphism

$$\begin{aligned} F_i &= \oplus_{m \ge t+i} R(-m)^{\alpha_{i,m}(\underline{d},n,t)} \oplus K_i(\underline{d})^{\le t+i-1} \\ &\cong F^{\vee}_{n-i}(-2t-n) \\ &= \oplus_{m \ge t+n-i} R(m-2t-n)^{\alpha_{n-i,m}(\underline{d},n,t)} \oplus (K_{n-i}(\underline{d})^{\le t+n-i-1})^{\vee}(-2t-n). \end{aligned} \tag{3}$$

We first consider $i = 1$. We rewrite the third line of (3) as follows:

$$\left[R(-t-1)^{\alpha_{n-1,t+n-1}(\underline{d},n,t)} \oplus R(-t)^{\alpha_{n-1,t+n}(\underline{d},n,t)} \oplus \cdots\right] \oplus \\ \left[(K_{n-1}(\underline{d})^{\le t+n-2})^{\vee}(-2t-n)\right]$$

and observe that each summand on the first line is of the form $R(-i)$ for some $i \le t+1$, while each summand on the second line is of the form $R(-i)$ for some $i \ge t+2$. It follows that

$$\oplus_{m \ge t+n-1} R(m-2t-n)^{\alpha_{n-1,m}(\underline{d},n,t)} = K_1(\underline{d})^{\le t} \oplus R(-t-1)^{\alpha_{1,t+1}(\underline{d},n,t)}, \text{ and} \\ (K_{n-1}(\underline{d})^{\le t+n-2})^{\vee}(-2t-n) = \oplus_{m \ge t+2} R(-m)^{\alpha_{1,m}(\underline{d},n,t)}$$

and we conclude that

$$F_1 = K_1(\underline{d})^{\le t} \oplus R(-t-1)^{\alpha_1(\underline{d},n,t)} \oplus (K_{n-1}(\underline{d})^{\le t+n-2})^{\vee}(-2t-n); \text{ and}$$

$$F_{n-1} = (K_1(\underline{d})^{\le t})^{\vee}(-2t-n) \oplus R(-t-n+1)^{\alpha_1(\underline{d},n,t)} \oplus K_{n-1}(\underline{d})^{\le t+n-2}$$

where $\alpha_1(\underline{d},n,t) := \alpha_{1,t+1}(\underline{d},n,t)$.

Substituting F_1 and F_{n-1} in the exact sequence (2) and using again the Hilbert function of A, we determine $\alpha_{2,t+2}(\underline{d},n,t)$. Moreover, an analogous numerical analysis taking into account that $F_2 \cong F^{\vee}_{n-2}(-2t-n)$ gives us

$$F_2 = K_2(\underline{d})^{\le t+1} \oplus R(-t-2)^{\alpha_2(\underline{d},n,t)} \oplus (K_{n-2}(\underline{d})^{\le t+n-3})^{\vee}(-2t-n); \text{ and}$$

$$F_{n-2} = (K_2(\underline{d})^{\le t})^{\vee}(-2t-n) \oplus R(-t-n+2)^{\alpha_2(\underline{d},n,t)} \oplus K_{n-2}(\underline{d})^{\le t+n-3}$$

where $\alpha_2(\underline{d},n,t) := \alpha_{2,t+2}(\underline{d},n,t)$.

Going on and using the isomorphism $F_i \cong F^{\vee}_{n-i}(-2t-n)$ for all $i = 1, \cdots, n-1$, we obtain that

$$F_i = K_i(\underline{d})^{\le t+i-1} \oplus R(-t-i)^{\alpha_i(\underline{d},n,t)} \oplus (K_{n-i}(\underline{d})^{\le t+n-i-1})^{\vee}(-2t-n)$$

where $\alpha_i(\underline{d},n,t) = \alpha_{n-i}(\underline{d},n,t)$ for all $i = 1, \cdots, n-1$ and $\alpha_i(\underline{d},n,t)$ is determined by the Hilbert function of A. ☐

Remark 3.6. The equality in the first line of the proof of Theorem 3.5 follows from the assumption that the complete intersection is general, as well as the assumption on the characteristic. We remark that the assumptions can be weakened somewhat.

As we saw in Section 2, the fact that the Gorenstein algebra A has the expected Hilbert function (i.e. that it is relatively compressed with respect to $\mathfrak{a}$) is directly related to the Fröberg conjecture, in this case for $n+1$ forms. This in turn is equivalent to the so-called *Maximal Rank Property*, namely that $R/\mathfrak{a}$ have the property that for any d and any i, a general form F of degree d induces a map of maximal rank from $(R/\mathfrak{a})_i$ to $(R/\mathfrak{a})_{i+d}$. And this follows from the Strong Lefschetz Property. Now, it was shown in [15] (based on the proof of Fröberg's conjecture for $n+1$ forms in n variables in [23] and [25]) that all of these hold for a monomial complete intersection (hence for a general complete intersection) provided that either k has characteristic zero or that k has characteristic p, assuming that certain numerical conditions hold. More precisely, they assume that either $\mathrm{char}(k) = 0$ or else that $\mathrm{char}(k) > j$, $\mathfrak{a} = (f_1, \ldots, f_a), a \leq n$, $f_i = x_i^{d_i}$, and that there exist nonnegative integers t, u, v with $j = u + v$, and if $a = n$ then $j \leq \sum d_i - n$. Hence Theorem 3.5 is also true with these assumptions on the characteristic.

Note that in fact it is unknown if the Strong Lefschetz Property holds for *all* complete intersections, even in characteristic zero. However, it is true that over fields of characteristic zero all complete intersections in 3 variables have the Weak Lefschetz Property, due to [12].

Example 3.7. Let A be a general Gorenstein Artinian graded algebra of embedding dimension 4, socle degree 10 and relatively compressed with respect to the ideal $\mathfrak{a} = (F, G, H)$ of a complete intersection set of points $P \subset \mathbb{P}^3$ of type (3,3,4). The h-vector of A is

$$1\ 4\ 10\ 18\ 26\ 32\ 26\ 18\ 10\ 4\ 1$$

and the "expected" minimal free R-resolution is

$$0 \to R(-14) \to R(-8)^9 \oplus R(-11)^2 \oplus R(-10) \to R(-7)^{20} \oplus R(-8) \oplus R(-6)$$
$$\to R(-6)^9 \oplus R(-3)^2 \oplus R(-4) \to R \to A \to 0.$$

It is known that such algebras exist [15], and the precise resolution comes from Theorem 3.5. However, to illustrate our technique from the previous section we explicitly construct it. To this end, we consider a subset $X \subset P \subset \mathbb{P}^3$ of 32 points of P that truncate the Hilbert function. The h-vector of X is thus

$$1\ 3\ 6\ 8\ 8\ 6\ 0$$

and $I(X)$ has a minimal free R-resolution of the following type

$$0 \to R(-8)^6 \to R(-7)^{10} \oplus R(-6) \to R(-6)^3 \oplus R(-4) \oplus R(-3)^2 \to R \to R/I(X) \to 0$$

(as can be verified, for example, by linkage).

The canonical module ω_X of $R/I(X)$ can be embedded as an ideal $\omega_X(-10) \subset R/I(X)$ of initial degree 6 and we have a short exact sequence

$$0 \to \omega_X(-10) \to R/I(X) \to A \to 0$$

where A is a Gorenstein Artinian graded algebra of codimension 4, socle degree 10, h-vector

$$1\ 4\ 10\ 18\ 26\ 32\ 26\ 18\ 10\ 4\ 1.$$

So, it is relatively compressed with respect to $\mathfrak{a} = (F, G, H)$, $\deg(F) = \deg(G) = 3$ and $\deg(H) = 4$. Moreover, applying once more the mapping cone process we get that A has the following minimal free R-resolution

$$0 \to R(-14) \to R(-8)^9 \oplus R(-11)^2 \oplus R(-10) \to R(-7)^{20} \oplus R(-8) \oplus R(-6)$$
$$\to R(-6)^9 \oplus R(-3)^2 \oplus R(-4) \to R \to A \to 0.$$

So, it has the expected minimal free R-resolution in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function of such relatively compressed Gorenstein algebra.

We see that the summand $R(-6)$ does not split off because it is a Koszul relation among the two cubic generators. The summand $R(-8)$ does *not* correspond to a Koszul syzygy. However, it is forced by the self duality property (up to twist) of the minimal free

R-resolution of an Artinian Gorenstein graded algebra. So, in the minimal free resolution, the ghost terms are forced to be there (directly and then indirectly) by the Koszul relations among the generators.

Remark 3.8. Theorem 3.5 shows that the observation at the end of the last example holds in general: in the minimal free resolution of an Artinian Gorenstein algebra of even socle degree, relatively compressed with respect to a general complete intersection, the only ghost terms that appear are forced to be there by Koszul relations among the generators, or by duality because of such Koszul relations.

A more difficult situation is when the Artinian Gorenstein graded algebra has odd socle degree. The technique of the previous section only gives a partial result, not the precise resolution. Indeed, it is no longer true that the Hilbert function alone determines the graded Betti numbers. (For instance see [18] Example 3.12.)

Theorem 3.9. *Let $A = R/I$ be a Gorenstein Artinian graded algebra of embedding dimension n and socle degree $2t+1$, where $R = k[x_1, \dots, x_n]$ and k has characteristic zero. Assume that A is relatively compressed with respect to a general complete intersection ideal $\mathfrak{a} = (G_1, \cdots, G_r)$, $r \le n$ and $deg(G_1) = d_1, \cdots, deg(G_r) = d_r$. Set $\underline{d} = (d_1, \cdots, d_r)$. Then, A has a minimal free R-resolution of the following type:*

$$0 \to R(-2t-1-n) \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to A \to 0$$

where $F_i \cong F^{\vee}_{n-i}(-2t-1-n)$ for all $i = 1, \cdots, n-1$. Moreover, if n is even (say, $n = 2p$), then

$$F_i = R(-t-i-1)^{y_{i+1}} \oplus R(-t-i)^{\alpha_i(\underline{d},n,t)+y_i} \oplus \\ K_i(\underline{d})^{\le t+i-1} \oplus (K_{n-i}(\underline{d})^{\le t+n-i-1})^{\vee}(-2t-1-n) \text{ for } i = 2, \cdots, p-1;$$

$$F_1 = \oplus_{j=1}^{r} R(-d_j) \oplus (K_{n-1}(\underline{d})^{\le t+n-2})^{\vee}(-2t-1-n) \oplus R(-t-1)^{\alpha_1(\underline{d},n,t)} \oplus R(-t-2)^{y_2}; \text{ and}$$

$$F_p = R(-t-p-1)^{\alpha_p(\underline{d},n,t)+y_p} \oplus R(-t-p)^{\alpha_p(\underline{d},n,t)+y_p} \oplus K_p(\underline{d})^{\le t+p-1} \oplus (K_p(\underline{d})^{\le t+p-1})^{\vee}(-2t-1-n)$$

where $\alpha_i(\underline{d}, n, t)$, $i = 1, \cdots p$, is completely determined by the Hilbert function of A.

If n is odd (say, $n = 2p+1$), then

$$F_1 = \oplus_{j=1}^{r} R(-d_j) \oplus (K_{n-1}(\underline{d})^{\le t+n-2})^{\vee}(-2t-1-n) \oplus R(-t-1)^{\alpha_1(\underline{d},n,t)} \oplus R(-t-2)^{y_2}; \text{ and}$$

$$F_i = R(-t-i-1)^{y_{i+1}} \oplus R(-t-i)^{\alpha_i(\underline{d},n,t)+y_i} \oplus \\ K_i(\underline{d})^{\le t+i-1} \oplus (K_{n-i}(\underline{d})^{\le t+n-i-1})^{\vee}(-2t-1-n) \text{ for } i = 2, \cdots, p$$

where $\alpha_i(\underline{d}, n, t)$, $i = 1, \cdots p$, is completely determined by the Hilbert function of A.

Proof. Analogous to the proof of Theorem 3.5. See also Remark 3.6 concerning the assumption on the characteristic. □

Example 3.10. This example shows that in fact a relatively compressed Gorenstein algebra of odd socle degree can have ghost terms that are not forced by Koszul relations among the generators, even taking duality into account. It was verified by `macaulay`, but the calculations can be done by hand as well. Let I be an ideal of general forms of degrees 4,4,4,4,11 in four variables. Link I using a general complete intersection of type

(4,4,4,11). The residual is a Gorenstein ideal G, which is relatively compressed since I is an ideal of general forms. The Hilbert functions are

deg	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
CI	1	4	10	20	32	44	54	60	63	64	64	63	60	54	44	32	20	10	4	1
G	1	4	10	20	32	44	54	60	60	54	44	32	20	10	4	1				

The Betti diagram of G is

```
total:      1     7    12     7     1
--------------------------------------
      0:     1     -     -     -     -
      1:     -     -     -     -     -
      2:     -     -     -     -     -
      3:     -     3     -     -     -
      4:     -     -     -     -     -
      5:     -     -     -     -     -
      6:     -     -     3     -     -
      7:     -     3     3     1     -
      8:     -     1     3     3     -
      9:     -     -     3     -     -
     10:     -     -     -     -     -
     11:     -     -     -     -     -
     12:     -     -     -     3     -
     13:     -     -     -     -     -
     14:     -     -     -     -     -
     15:     -     -     -     -     1
```

The three copies of $R(-8)$ in the second free module represent Koszul syzygies, hence the corresponding "ghost" terms are forced by Koszul relations. However, the copy of $R(-9)$ in the second free module does not come from Koszul relations among generators. It illustrates that things are very different in the case of odd socle degree.

Remark 3.11. The two preceding examples illustrate the difference between the case of even socle degree and odd socle degree. In fact, Theorem 3.5 shows that there are no "ghost" terms in the minimal free resolution of a relatively compressed Gorenstein Artinian algebra of even socle degree, relatively compressed with respect to a general complete intersection, apart from those corresponding to Koszul relations or forced from the Koszul ones by duality.

Now we introduce a new technique, which gives the precise resolution at least in a special case.

Example 3.12. Let A be a general Gorenstein Artinian graded algebra of embedding dimension 4, socle degree 9 and relatively compressed with respect to the ideal $\mathfrak{a} = (F)$ of a smooth quadric $Q = V(F) \subset \mathbb{P}^3$. The h-vector of A is

$$1\ 4\ 9\ 16\ 25\ 25\ 16\ 9\ 4\ 1$$

and the expected minimal free R-resolution is

$$0 \to R(-13) \to R(-11) \oplus R(-8)^{11} \to R(-6)^{11} \oplus R(-7)^{11} \to R(-2) \oplus R(-5)^{11} \to R \to A \to 0.$$

Let us explicitly construct such an algebra. To this end, we consider 30 general points $X \subset \mathbb{P}^3$ on the quadric $Q \subset \mathbb{P}^3$. The h-vector of X is 1 3 5 7 9 5 and $I(X)$ has a minimal free R-resolution of the following type [11]:

$$0 \to R(-8)^5 \to R(-6)^5 \oplus R(-7)^6 \to R(-2) \oplus R(-5)^6 \to R \to R/I(X) \to 0.$$

By [3]; Theorem 3.2, the canonical module ω_X of $R/I(X)$ can be embedded as an ideal $\omega_X(-9) \subset R/I(X)$ of initial degree 5 and we have a short exact sequence

$$0 \to \omega_X(-9) \to R/I(X) \to A \to 0$$

where A is a Gorenstein Artinian graded algebra of embedding dimension 4, socle degree 9, h-vector

$$1\ 4\ 9\ 16\ 25\ 25\ 16\ 9\ 4\ 1$$

and relatively compressed with respect to $\mathfrak{a} = (F)$. Moreover applying the mapping cone process we get that A has the following minimal free R-resolution

$$0 \to R(-13) \to R(-11) \oplus R(-8)^{11} \to R(-6)^{11} \oplus R(-7)^{11} \to R(-2) \oplus R(-5)^{11} \to R \to A \to 0.$$

The following result from [11] is crucial if we want to generalize the above example.

Proposition 3.13. *Let X be a set of N general points on a smooth quadric $Q \subset \mathbb{P}^3$. Write $N = i^2 + h$ with $0 < h \le 2i+1$. Then, the h-vector of X is*

$$1\ 3\ 5\ 7\ \cdots\ 2i-1\ h\ 0$$

and $I(X)$ has a minimal free R-resolution of the following type:

$$0 \to R(-i-2)^{max(0,-\delta_{i+2})} \oplus R(-i-3)^h \to R(-i-1)^{max(0,\delta_{i+1})} \oplus R(-i-2)^{max(0,\delta_{i+2})}$$
$$\to R(-i)^{2i+1-h} \oplus R(-i-1)^{max(0,-\delta_{i+1})} \oplus R(-2) \to R \to R/I(X) \to 0$$

where $\delta_n := \Delta^4 h_X(n)$ (the fourth difference of the Hilbert function of X).

Proof. See [11]; §4. □

Proposition 3.14. *Let A be a general Gorenstein Artinian graded algebra of embedding dimension 4 and socle degree $2t+1$. Assume that A is relatively compressed with respect to $\mathfrak{a} = (F)$ with $\deg(F) = 2$ and F general. If $2 \le t$, then A has a minimal free R-resolution of the following type:*

$$0 \to R(-2t-5) \to R(-t-4)^{2t+3} \oplus R(-2t-3) \to R(-t-2)^{2t+3} \oplus R(-t-3)^{2t+3}$$
$$\to R(-t-1)^{2t+3} \oplus R(-2) \to R \to A \to 0.$$

Proof. We will explicitly construct a Gorenstein Artinian graded algebra A of embedding dimension 4, socle degree $2t+1$, relatively compressed with respect to $\mathfrak{a} = (F)$, $\deg(F) = 2$, and with the expected graded Betti numbers. To this end, we consider a set of $(t+1)^2 + (t+1)$ general points $X \subset \mathbb{P}^3$ lying on a smooth quadric $Q = V(F) \subset \mathbb{P}^3$. The h-vector of X is

$$1\ 3\ 5\ \cdots\ 2t+1\ \ t+1\ 0$$

and $I(X)$ has a minimal free R-resolution of the following type

$$0 \to R(-t-4)^{t+1} \to \begin{array}{c} R(-t-3)^{t+2} \\ \oplus \\ R(-t-2)^{t+1} \end{array} \to \begin{array}{c} R(-t-1)^{t+2} \\ \oplus \\ R(-2) \end{array} \to R \to R/I(X) \to 0.$$

By Proposition 3.13 such a set of points X exists.

By [3]; Theorem 3.2, the canonical module ω_X of $R/I(X)$ can be embedded as an ideal $\omega_X(-2t-1) \subset R/I(X)$ of initial degree $t+1$ and we have a short exact sequence

$$0 \to \omega_X(-2t-1) \to R/I(X) \to A \to 0$$

where A is a Gorenstein Artinian graded algebra of codimension 4, socle degree $2t+1$, h-vector

$$1\ 4\ 9\ 25\ \cdots\ (t+1)^2\ (t+1)^2\ \cdots\ 25\ 9\ 4\ 1.$$

So, it is relatively compressed with respect to $\mathfrak{a} = (F)$, $\deg(F) = 2$. Moreover, applying the mapping cone process we get that A has the following minimal free R-resolution

$$0 \to R(-2t-5) \to R(-t-4)^{2t+3} \oplus R(-2t-3) \to R(-t-2)^{2t+3} \oplus R(-t-3)^{2t+3}$$
$$\to R(-t-1)^{2t+3} \oplus R(-2) \to R \to A \to 0.$$

So it has the expected minimal free R-resolution in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function of such relatively compressed Gorenstein graded algebra.

□

The approach used in Proposition 3.14 suggests that there is a close relation between the Minimal Resolution Conjecture (MRC) for points on a projective variety due to Mustaţă (see [21] page 64) and the existence of relatively compressed Gorenstein algebras A of odd socle degree and with the "expected resolution" in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function of A. We will end this section by writing down this relation.

Definition 3.15. Let $X \subset \mathbb{P}^n$ be a projective variety of $\dim(X) \geq 1$. A set Γ of δ distinct points on X is in *general position* if

$$h_{R/I(\Gamma)}(t) = \min\{h_{R/I(X)}(t), \delta\}.$$

If $X = \mathbb{P}^n$ then the Minimal Resolution Conjecture predicts the graded Betti numbers of points in general position. It has been proved if the number of points is large compared to n by Simpson and Hirschowitz [13], but may fail for a small number of points as shown by Eisenbud and Popescu [5]. If $X \neq \mathbb{P}^n$ one has to modify the "expectations." In [21] pg. 64, Mustaţă states the Minimal Resolution Conjecture (MRC) for points on a projective variety. Let us recall it.

Conjecture 3.16. *Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\operatorname{reg}(X) = m$ and with Hilbert polynomial P_X. Let δ be an integer with $P_X(r-1) \leq \delta < P_X(r)$ for some $r \geq m+1$ and let Γ be a set of δ points on X in general position. Let*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0$$

be a minimal free R-resolution of $R/I(X)$. Then $R/I(\Gamma)$ has a minimal free R-resolution of the following type

$$0 \to F_n \oplus R(-r-n+1)^{a^n_{r+n-1}} \oplus R(-r-n)^{a^n_{r+n}} \to F_{n-1} \oplus R(-r-n+2)^{a^{n-1}_{r+n-2}} \oplus R(-r-n+1)^{a^{n-1}_{r+n-1}} \to$$
$$\cdots \to F_2 \oplus R(-r-1)^{a^2_{r+1}} \oplus R(-r-2)^{a^2_{r+2}} \to F_1 \oplus R(-r)^{a^1_r} \oplus R(-r-1)^{a^1_{r+1}} \to R \to R/I(\Gamma) \to 0$$

with $a^i_{r+i} a^{i+1}_{r+i} = 0$ for $i = 1, \cdots, n-1$.

Example 3.17. The MRC holds for $\delta \gg 0$ points on a general smooth rational quintic curve $C \subset \mathbb{P}^3$ [21].

For the purposes of this paper, it would be enough to know that the MRC holds on a complete intersection variety.

Proposition 3.18. *Let $A = R/I$ be a general Gorenstein Artinian graded algebra of embedding dimension n and socle degree $2t+1$. Assume that A is relatively compressed with respect to a general complete intersection ideal $\mathfrak{a} = (G_1, \cdots, G_r) \subset k[x_1, \cdots, x_n]$, $r \leq n-2$, and $deg(G_1) = d_1, \cdots, deg(G_r) = d_r$. Set $m = \operatorname{reg}(X)$ where $X = V(G_1, \cdots, G_r) \subset \mathbb{P}^{n-1}$ and $\underline{d} = (d_1, \cdots, d_r)$. If $t \geq m$ and MRC holds for points on complete intersection projective varieties, then A has a minimal free R-resolution of the following type:*

$$0 \to R(-2t-1-n) \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to A \to 0$$

where $F_i \cong F_{n-1}^{\vee}(-2t-1-n)$ for all $i = 1, \cdots, n-1$. Moreover, if n is even (say, $n = 2p$), then

$$F_i = R(-t-i)^{\alpha_i(\underline{d},n,t)} \oplus K_i(\underline{d}) \oplus K_{n-i}(\underline{d})^{\vee}(-2t-1-n) \text{ for } i = 1, \cdots, p-1; \text{ and}$$

$$F_p = R(-t-p-1)^{\alpha_p(\underline{d},n,t)} \oplus R(-t-p)^{\alpha_p(\underline{d},n,t)} \oplus K_p(\underline{d}) \oplus K_p(\underline{d})^{\vee}(-2t-1-n)$$

where $\alpha_i(\underline{d}, n, t)$, $i = 1, \cdots p$, is completely determined by the Hilbert function of A.

If n is odd (say, $n = 2p+1$), then

$$F_i = R(-t-i)^{\alpha_i(\underline{d},n,t)} \oplus K_i(\underline{d}) \oplus K_{n-i}(\underline{d})^{\vee}(-2t-1-n) \text{ for } i = 1, \cdots, p$$

where $\alpha_i(\underline{d}, n, t)$, $i = 1, \cdots p$, is completely determined by the Hilbert function of A.

Proof. We will only prove the case n even and we leave to the reader the case n odd. To this end, we will explicitly construct a Gorenstein Artinian graded algebra A of embedding dimension n, socle degree $2t+1$, relatively compressed with respect to $\mathfrak{a} = (G_1, \cdots, G_r)$ and with the expected graded Betti numbers.

By hypothesis the complete intersection projective variety $X = V(G_1, \cdots, G_r) \subset \mathbb{P}^{n-1}$ satisfies MRC for any set of δ points on X in general position with $P_X(s-1) \leq \delta < P_X(s)$ for some $s \geq m+1$. Hence, since $t \geq m$, for a suitable δ with $P_X(t) \leq \delta < P_X(t+1)$, a set Γ of δ general points on X has a minimal free R-resolution of the following type:

$$(4) \qquad 0 \to K_{n-1}(\underline{d}) \oplus R(-t-n)^{a_{t+n}^{n-1}} \to K_{n-2}(\underline{d}) \oplus R(-t-n+1)^{a_{t+n-1}^{n-2}}$$

$$\to \cdots \to K_{n/2}(\underline{d}) \oplus R(-t-\frac{n}{2})^{a_{t+\frac{n}{2}}^{n/2}} \oplus R(-t-\frac{n}{2}-1)^{a_{t+\frac{n}{2}+1}^{n/2}} \to \cdots$$

$$\to K_2(\underline{d}) \oplus R(-t-2)^{a_{t+2}^{2}} \oplus \to K_1(\underline{d}) \oplus R(-t-1)^{a_{t+1}^{1}} \to R \to R/I(\Gamma) \to 0.$$

Dualizing and twisting the exact sequence (4), we get a minimal free R-resolution of the canonical module $\omega_\Gamma(-2t-1)$ of Γ:

$$0 \to R(-2t-1-n) \to R(-t-n)^{a^1_{t+1}} \oplus K_1(\underline{d})^\vee(-2t-1-n) \to$$
$$R(-t-n+1)^{a^2_{t+2}} \oplus K_2(\underline{d}^\vee(-2t-1-n) \to \cdots \to$$
$$R(-t-1-\tfrac{n}{2})^{a^{n/2}_{t+\frac{n}{2}}} \oplus R(-t-\tfrac{n}{2})^{a^{n/2}_{t+\frac{n}{2}+1}} \oplus K_{n/2}(\underline{d})^\vee(-2t-1-n) \to \cdots \to \tag{5}$$
$$R(-t-2)^{a^{n-2}_{t+n-1}} \oplus K_{n-2}(\underline{d})^\vee(-2t-1-n) \to$$
$$R(-t-1)^{a^{n-1}_{t+n}} \oplus K_{n-1}(\underline{d})^\vee(-2t-1-n) \to \omega_\Gamma(-2t-1) \to 0.$$

By [3]; Theorem 3.2, the canonical module ω_Γ of $R/I(\Gamma)$ can be embedded as an ideal $\omega_\Gamma(-2t-1) \subset R/I(\Gamma)$ of initial degree $t+1$ and we have a short exact sequence

$$0 \to \omega_\Gamma(-2t-1) \to R/I(\Gamma) \to A \to 0 \tag{6}$$

where A is a Gorenstein Artinian graded algebra of codimension n, socle degree $2t+1$. A straightforward computation using the exact sequences (4), (5) and (6) gives us $h_A(\lambda) = \min\{h_{R/\mathfrak{a}}(\lambda), h_{R/\mathfrak{a}}(2t+1-\lambda)\}$. Therefore, A is relatively compressed with respect to $\mathfrak{a} = (G_1, \cdots, G_r)$. Moreover, using the exact sequences (4), (5) and (6) and applying the mapping cone process, we get that A has the following minimal free R-resolution:

$$0 \to R(-2t-1-n) \to R(-t-n)^{a^1_{t+1}+a^{n-1}_{t+n}} \oplus K_{n-1}(\underline{d}) \oplus K_1(\underline{d})^\vee(-2t-1-n) \to$$
$$R(-t-n-1)^{a^2_{t+2}+a^{n-2}_{t+n-1}} \oplus K_{n-2}(\underline{d}) \oplus K_2(\underline{d})^\vee(-2t-1-n) \to \cdots \to$$
$$R(-t-\frac{n}{2})^{a^{n/2}_{t+\frac{n}{2}}+a^{n/2}_{t+\frac{n}{2}+1}} \oplus R(-t-\frac{n}{2}-1)^{a^{n/2}_{t+\frac{n}{2}}+a^{n/2}_{t+\frac{n}{2}+1}} \oplus K_{n/2}(\underline{d}) \oplus K_{n/2}(\underline{d})^\vee(-2t-1-n)$$
$$\to \cdots \to R(-t-2)^{a^2_{t+2}+a^{n-2}_{t+n-1}} \oplus K_2(\underline{d}) \oplus K_{n-2}(\underline{d})^\vee(-2t-1-n) \to$$
$$R(-t-1)^{a^1_{t+1}+a^{n-1}_{t+n}} \oplus K_1(\underline{d}) \oplus K_{n-1}(\underline{d})^\vee(-2t-1-n) \to R \to A \to 0.$$

So A has the expected minimal free R-resolution in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function of such relatively compressed Gorenstein algebra and there are no non-Koszul ghost terms. $\square$

4. Unexpected behavior for the minimal free resolution of relatively compressed level algebras of socle dimension $c \geq 2$: "ghost" terms

We saw in section 2 that the Hilbert function of a relatively compressed algebra can fail to be the "expected" one. In this section we consider the question of when (if at all) the minimal free resolution of a relatively compressed level algebra must have redundant (or "ghost") terms that are not computable only from the Hilbert function.

We first give two examples of level $c \geq 2$ algebras relatively compressed with respect to a complete intersection $\mathfrak{c}$, with fixed socle degree s, embedding dimension 3 and with the "expected" minimal free R-resolution.

The first example involves a `macaulay` computation, while the second does not.

Example 4.1. We set $R = k[x, y, z]$. Let A be a level 2 graded algebra of embedding dimension 3, socle degree 15 and relatively compressed with respect to a complete intersection ideal $\mathfrak{a} = (F_1, F_2, F_3)$, $\deg(F_i) = 9$ for $i = 1, 2, 3$. Note that the Hilbert function of $R/\mathfrak{a}$ is

1 3 6 10 15 21 28 36 45 52 57 60 61 60 57 52 45 36 28 21 15 10 6 3 1.

The expected h-vector of A (according to the discussion preceding Example 2.14) is

$$1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ 52\ 42\ 30\ 20\ 12\ 6\ 2; \tag{7}$$

and the expected minimal free R-resolution is

$$0 \to R(-18)^2 \to R(-11)^{15} \oplus R(-12)^4 \to R(-9)^3 \oplus R(-10)^{15} \to R \to A \to 0.$$

Let us explicitly construct it. To this end, we consider the ideal $J = \mathfrak{a} + (F_4, F_5)$ where F_4 and F_5 are two general forms of degree 9. We know the Hilbert function of R/J thanks to [1] (using a calculation similar to that in Example 2.16):

$$1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ 50\ 51\ 48\ 41\ 30\ 15.$$

Using `macaulay` we compute a minimal free R-resolution of J and we get

$$0 \to R(-17)^{15} \to R(-16)^{15} \oplus R(-15)^4 \to R(-9)^5 \to R \to R/J \to 0.$$

The ideal $\mathfrak{a}$ links J to an ideal $J' := [\mathfrak{a} : J]$ and it is easily seen that $A = R/J'$ is a level 2 graded algebra of embedding dimension 3 and socle degree 15. One quickly checks that its Hilbert function is the one predicted above in (7), so A is relatively compressed with respect to a complete intersection ideal $\mathfrak{a} = (F_1, F_2, F_3)$, $deg(F_i) = 9$ for $i = 1, 2, 3$. Using the standard mapping cone construction we get that A has the following minimal free R-resolution:

$$0 \to R(-18)^2 \to R(-11)^{15} \oplus R(-12)^4 \to R(-9)^3 \oplus R(-10)^{15} \to R \to A \to 0.$$

So, it has the expected minimal free R-resolution in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function of such relatively compressed level graded algebras of socle dimension 2. Note that the only place where we needed the aid of the computer was to confirm the expected minimal free resolution of an ideal of five general forms of degree 9.

Example 4.2. We set $R = k[x, y, z]$. Let A be a level graded algebra of socle dimension 4 and embedding dimension 3, socle degree 5 and relatively compressed with respect to an ideal $\mathfrak{a} = (F)$, $\deg(F) = 2$. The h-vector of A is 1 3 5 7 9 4; and the expected minimal free R-resolution is

$$0 \to R(-8)^4 \to R(-6)^8 \oplus R(-7)^3 \to R(-2) \oplus R(-5)^7 \to R \to A \to 0.$$

Let us explicitly construct it. To this end, we consider a set of 29 general points $X \subset \mathbb{P}^3$ lying on a smooth quadric surface $Q \subset \mathbb{P}^3$. The h-vector of X is 1 3 5 7 9 4; and $I(X) \subset S := k[x_0, x_1, x_2, x_3]$ has a minimal free S-resolution of the following type (cf. [11])

$$0 \to S(-8)^4 \to S(-6)^8 \oplus S(-7)^3 \to S(-2) \oplus S(-5)^7 \to S \to S/I(X) \to 0.$$

The Artinian reduction of $S/I(X)$ is a level graded Artinian algebra A of socle dimension 4 and embedding dimension 3, socle degree 5 and relatively compressed with respect to an ideal $\mathfrak{a} = (F) \subset k[x, y, z]$, $\deg(F) = 2$. The h-vector of A is 1 3 5 7 9 4; and the minimal free R-resolution of A is

$$0 \to R(-8)^4 \to R(-6)^8 \oplus R(-7)^3 \to R(-2) \oplus R(-5)^7 \to R \to A \to 0.$$

Again, it has the expected minimal free R-resolution in the sense that the graded Betti numbers are the smallest consistent with the Hilbert function of such relatively compressed level graded algebra with socle dimension 2.

In the next example we show that sometimes there must be redundant terms in the minimal free resolution of a relatively compressed level algebra.

Example 4.3. We work in four variables, $R = k[x_1, x_2, x_3, x_4]$. Let I_1 be a general ideal with h-vector 1 2, so its minimal free resolution is

$$0 \to R(-5)^2 \to R(-4)^7 \to R(-3)^8 \oplus R(-2) \to R(-1)^2 \oplus R(-2)^3 \to I_1 \to 0.$$

Linking with a complete intersection of type $(3,3,3,3)$ we get an ideal I_2 with h-vector

1 4 10 16 19 16 10 2

and minimal free resolution

$$0 \to \begin{array}{c} R(-11)^2 \\ \oplus \\ R(-10)^3 \end{array} \to \begin{array}{c} R(-9)^{12} \\ \oplus \\ R(-10) \end{array} \to \begin{array}{c} R(-8)^7 \\ \oplus \\ R(-6)^6 \end{array} \to \begin{array}{c} R(-7)^2 \\ \oplus \\ R(-3)^4 \end{array} \to I_2 \to 0$$

(that you can compute from that of I_1 with the mapping cone). I_2 is an ideal of general forms of degrees 3,3,3,3,7,7, and its minimal free resolution has a non-Koszul ghost term $R(-10)$. This approach for finding ideals of general forms with non-Koszul ghost terms was developed in [19].

Now we link I_2 with a complete intersection $\mathfrak{c}$ of type $(3,3,7,7)$. The Hilbert function of $\mathfrak{c}$ is

1 4 10 18 27 36 45 52 55 52 45 36 27 18 10 4 1.

Letting I_3 be the residual, its Hilbert function is

1 4 10 18 27 36 45 52 55 50 35 20 8 2.

Note first that I_3 does not "look" relatively compressed in $\mathfrak{c}$:

deg	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathfrak{c}$	1	4	10	18	27	36	45	52	55	52	45	36	27	18	10	4	1
I_3	1	4	10	18	27	36	45	52	55	50	35	20	8	2			

The 35 "should" be a 36, and the 50 "should" be 52. But given any relatively compressed level algebra of socle dimension 2 and socle degree 13 inside a complete intersection of type $(3,3,7,7)$, the residual is an ideal generated by six forms, of degrees 3,3,3,3,7,7. The smallest Hilbert function of such an ideal is given by our I_2 (general forms), so the biggest (i.e. relatively compressed) for the level algebra is the one above.

But now we consider minimal free resolutions. First note that in the resolution of I_2 above, all the copies of $R(-6)$ are Koszul. We know the resolution of $\mathfrak{c}$ (the Koszul resolution). The mapping cone gives the following. Note that we split off not only generators of degrees 3,3,7,7, but also one first syzygy $R(-6)$, namely the Koszul one. Studying the mapping cone carefully, we see that there is no other possible splitting. We get:

$$0 \to R(-17)^2 \to \begin{array}{c} R(-12)^7 \\ \oplus \\ R(-14)^5 \end{array} \to \begin{array}{c} R(-11)^{12} \\ \oplus \\ R(-10)^5 \\ \oplus \\ R(-6) \end{array} \to \begin{array}{c} R(-10)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-7)^2 \\ \oplus \\ R(-3)^2 \end{array} \to I_3 \to 0.$$

Note that four of the copies of $R(-10)$ in the second free module are Koszul, but the fifth one is not. So we have a relatively compressed level algebra with a non-Koszul ghost term at the beginning of the resolution.

The following result generalizes the last example. It is based on [19] Theorem 3.3.

Proposition 4.4. *Let $R = k[x_1, \dots, x_n]$ and let $\mathfrak{c}' = (F_1, \dots, F_n)$ be a complete intersection of forms of degree $d_1, \dots, d_n$, respectively. We do not assume that $d_1 \leq \dots \leq d_n$, but we do assume that each $d_i > 2$. Suppose that $d_1 = \dots = d_p = a$ (say) for some $1 \leq p \leq n-2$. Let $d = d_1 + \dots + d_n$ and choose general forms $F_{n+1}, \dots, F_{n+p}$ all of degree $d-n-1$. Let $\mathfrak{c}$ be the complete intersection $(F_{p+1}, \dots, F_n, F_{n+1}, \dots, F_{n+p})$ and let $J = (F_1, \dots, F_{n+p})$. Let $e = \sum_{i=p+1}^{n} d_i + p(d-n-1)$. Then the residual ideal $I = \mathfrak{c} : J$ is a level algebra of socle dimension p and socle degree $e-n-a$, and R/I is relatively compressed in $R/\mathfrak{c}$. Furthermore, the minimal free resolution of R/I*

$$0 \to \mathbb{F}_n \to \cdots \to \mathbb{F}_2 \to \mathbb{F}_1 \to R \to R/I \to 0$$

has ghost terms between $\mathbb{F}_j$ and $\mathbb{F}_{j+1}$ for $1 \leq j \leq n-p-1$.

Proof. Clearly R/J has a minimal free resolution

$$0 \to \mathbb{G}_n \to \cdots \to \mathbb{G}_{p+2} \to \mathbb{G}_{p+1} \to \cdots \to \mathbb{G}_1 \to R \to R/J \to 0$$

where $\mathbb{G}_1 = R(-a)^p \oplus \bigoplus_{i=p+1}^{n} R(-d_i) \oplus R(-d+n+1)^p$. Furthermore, by [19] Theorem 3.3, for $j = p+1, \dots, n-1$ there is a ghost term $R(-d+n+1-j)$ between $\mathbb{G}_j$ and $\mathbb{G}_{j+1}$ that does not arise from Koszul syzygies. Note also that since the socle degree of R/J is $d-n-1$, the largest twist of any $\mathbb{G}_j$ (including $\mathbb{G}_n$) is

$$\begin{aligned} R(-d+1-n+j) &= R(-d_1 - \cdots - d_n + 1 - n + j) \\ &= R(-pa - d_{p+1} - \cdots - d_n + 1 - n + j). \end{aligned}$$

Consider the minimal free (Koszul) resolution of $\mathfrak{c}$

$$\begin{array}{ccccccccccccc} & & & & & & & & & & 0 & & \\ & & & & & & & & & & \downarrow & & \\ 0 & \to & \mathbb{K}_n & \to \cdots \to & \mathbb{K}_{p+2} & \to & \mathbb{K}_{p+1} & \to \cdots \to & \mathbb{K}_1 & \to & \mathfrak{c} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{G}_n & \to \cdots \to & \mathbb{G}_{p+2} & \to & \mathbb{G}_{p+1} & \to \cdots \to & \mathbb{G}_1 & \to & J & \to & 0 \end{array}$$

Using the mapping cone procedure, we obtain a free resolution for R/I. We include the obvious splitting of the terms $R(-a)^p$ between $\mathbb{K}_1$ and $\mathbb{G}_1$.

$$0 \to R(-e+a)^p \to \mathbb{G}_2^\vee(-e) \to \cdots \to \mathbb{G}_n^\vee(-e) \oplus \mathbb{K}_{n-1}^\vee \to R \to R/I \to 0.$$

It is clear that no further terms split from $R(-e+a)^p$ (by the minimality of the resolution of J). Since the ghost terms of the minimal free resolution of R/J are not Koszul and since the generators of $\mathfrak{c}$ are taken from the generators of J, it is clear that no splitting will remove these terms after taking the mapping cone, so they remain (suitably twisted) in the minimal free resolution of R/I. □

Example 4.5. In Proposition 4.4 we concluded that under certain hypotheses we can find a relatively compressed level algebra with non-Koszul ghost terms at the beginning of the resolution. Following [19] Corollary 3.13, we can even arrange some splitting at the beginning of the resolution, leaving ghost terms only in the middle. Note that (according to the observation following [19] Corollary 3.13) this will only work for $n = 4, 5, 6$. Here is one example.

Let $n = 5$ and start with a quotient of R with Hilbert function 1 1 (it is a complete intersection of type $(1,1,1,1,2)$). We link using a complete intersection of type $(2,4,4,4,4)$

to obtain an ideal with generators of degrees 2,4,4,4,4,12, with "expected" Hilbert function. We link this in turn using generators of degrees 2,4,4,4,12 to obtain a Gorenstein ideal J with Hilbert function

$$1 \quad 5 \quad 14 \quad 30 \quad 52 \quad 76 \quad 98 \quad 114 \quad 123 \quad 123 \quad 114 \quad 98\ 76 \quad 52 \quad 30 \quad 14 \quad 5 \quad 1.$$

The minimal free resolution of J can be computed (although it is very tedious). It is

$$\begin{array}{ccccccccccc}
 & & & & R(-16)^3 & & R(-12)^3 & & & & \\
 & & & & \oplus & & \oplus & & & & \\
 & & R(-20) & & R(-14)^3 & & R(-11)^4 & & R(-9)^4 & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 \to R(-22) & \to & R(-18)^3 & \to & R(-12)^6 & \to & R(-10)^6 & \to & R(-4)^3 & \to J \to 0. & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & R(-13)^4 & & R(-11)^4 & & R(-8)^3 & & R(-2) & & \\
 & & & & \oplus & & \oplus & & & & \\
 & & & & R(-10)^3 & & R(-6)^3 & & & &
\end{array}$$

Then we point out first that there are no ghost terms, Koszul or otherwise, between the first free module and the second. Furthermore, there are some copies of $R(-10)$ between the second and third modules, but the three copies of $R(-10)$ in the third module are Koszul. But now consider the copies of $R(-12)$ between the second and third modules. None of the copies of $R(-12)$ in the second module come because of Koszul syzygies, and in the third module at most one copy of $R(-12)$ is Koszul. So even after accounting for that, we have non-Koszul ghost terms between the second and third modules in the resolution.

Example 4.6. When $n = 3$ we are not aware of any relatively compressed level algebras of socle dimension $c \geq 2$ that have non-Koszul ghost terms. However, for $c = 1$ these do exist. We heavily use the results of [18]. First note that if $I = (G_1, G_2, G_3, G_4)$ is an ideal of general forms in 3 variables, and if $\mathfrak{c}$ is a complete intersection defined by any three of the four generators, then the residual $\mathfrak{c} : I = G$ is a Gorenstein ideal. Furthermore, we claim that it is relatively compressed with respect to $\mathfrak{c}$. Indeed, any Gorenstein ideal containing a complete intersection of the same degrees, and having the same socle degree, will be linked by that complete intersection to an ideal of four forms of the same degrees as G_1, G_2, G_3, G_4. Since R/I has the smallest possible Hilbert function among all such ideals, by linkage R/G has the largest Hilbert function among all Gorenstein ideals with that socle degree, containing a complete intersection of that type.

In the case where $\deg G_1 \leq \cdots \leq \deg G_4$ and $\mathfrak{c} = (G_1, G_2, G_3)$, it was shown in [18] Proposition 4.1 that there appears a non-Koszul ghost term if and only if either (i) $d_2 + d_3 < d_1 + d_4 + 3$ and $d_1 + d_2 + d_3 + d_4$ is even, or (ii) $d_2 + d_3 \geq d_1 + d_4 + 3$ and $d_1 + d_2 + d_3 + d_4$ is even. In particular, such examples exist.

We believe that "most of the time," a relatively compressed level quotient of a complete intersection has only Koszul ghost terms. We believe, furthermore, that the only counterexamples arise through linkage, as special cases of Proposition 4.4. More precise conjectures are the following:

Conjecture 4.7. *Let $\mathfrak{c}$ be a general complete intersection of fixed generator degrees in a polynomial ring $R = k[x_1, \ldots, x_n]$, and let $A = R/I$ be a relatively compressed level quotient of $R/\mathfrak{c}$ of socle degree ≥ 2 and socle dimension $c \geq 1$. Assume that either $n = 3$ and $c \geq 2$ or else $n \geq 4$. If the minimal free resolution of R/I has non-Koszul ghost terms then I is linked in two steps, first by $\mathfrak{c}$ and then by a "predictable" complete intersection, to an ideal containing at least two independent linear forms.*

Note that the ideals in Example 4.3 and Proposition 4.4 have this property of being linked in two steps to an ideal containing at least two independent linear forms. Indeed, the Koszul relations of this latter ideal are what produce the non-Koszul relations of the final ideal.

Conjecture 4.8. *Assume that $\mathfrak{c}$ is a complete intersection generated by forms all of the same degree. Let A be a relatively compressed level Artinian quotient of $R/\mathfrak{c}$ of socle dimension c and socle degree ≥ 2,and assume that either $n = 3$ and $c \geq 2$, or $n \geq 4$. Then the minimal free resolution of A has no ghost terms (Koszul or otherwise).*

Remark 4.9. The assumptions in Conjecture 4.8 are necessary. In the case of socle degree 1, a counterexample would be any algebra with Hilbert function $1\ t\ 0$ for $t \leq n-2$. In the case of Gorenstein algebras of height 3, we have from [18] Proposition 4.1 that four general forms all of the same degree are *always* linked (using three of the four generators) to a Gorenstein ideal whose minimal free resolution has a non-Koszul ghost term. That this ideal is relatively compressed follows for instance from [18] Lemma 2.6.

Remark 4.10. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring where the Fröberg conjecture holds (equivalently, where Maximal Rank Property holds – cf. [19]). For instance, this is true for $n = 2, 3$ and conjecturally for all n. Let I be an ideal minimally generated by $r \geq n+1$ general forms of degrees $a_1 \leq a_2 \cdots \leq a_r$. In particular, we are assuming that none of these generators is redundant. Let J be the complete intersection defined by the first three generators. Then R/I is a quotient of R/J, and in particular (since $h_{R/J}(a_1 + a_2 + \cdots + a_n - n) = 1$), the socle degree, δ, of R/I is strictly less than that of R/J. That is,

$$\delta + n < a_1 + a_2 + \cdots + a_n.$$

It follows that if the minimal free resolution of R/I is

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to R \to R/I \to 0$$

then no summand of F_n corresponds to a Koszul $(n-1)$-st syzygy among any of the generators, since the highest twist of F_n is $R(-\delta - n)$.

Proposition 4.11. *Let $R = k[x_1, \dots, x_n]$ and assume that R satisfies the Fröberg conjecture (i.e. any ideal of general forms satisfies the Maximal Rank Property). Let A be a level quotient of R of socle dimension $c \geq 2$. (The case $c = 1$ is the Gorenstein case that we have already discussed in Example 4.6). Assume that A has socle degree s and is relatively compressed with respect to a general complete intersection $\mathfrak{c} = (F_1, F_2, \dots, F_n) \subset R$. Set $d_i = deg(F_i)$, and $d = d_1 + d_2 + \cdots + d_n$. Let I be the ideal generated by $\mathfrak{c}$ and c general forms of degree $d - s - n$, and set δ to be the socle degree of R/I. Let $\mathbb{G} = R(-d_1) \oplus R(-d_2) \oplus \cdots \oplus R(-d_n) \oplus R(s+3-d)^c$ and let $K_t = \wedge^t \mathbb{G}$. Finally, let $\mathbb{F} = R(-d_1) \oplus \cdots \oplus R(-d_n)$ and let $L_t = \wedge^t \mathbb{F}$.*

Then A has a free R-resolution of the following type:

$$0 \to R(-s-n)^c \to \begin{array}{c} R(\delta+1-d)^{z_2} \\ \oplus \\ R(\delta+2-d)^{w_2} \\ \oplus \\ ((K_2)^{\leq\delta})^\vee(-d) \end{array} \to \begin{array}{c} R(\delta+2-d)^{z_3} \\ \oplus \\ R(\delta+3-d)^{w_3} \\ \oplus \\ ((K_3)^{\leq\delta+1})^\vee(-d) \\ \oplus \\ (L_2)^\vee(-d) \end{array} \to \cdots \to \begin{array}{c} R(\delta+n-1-d)^{z_n} \\ \oplus \\ R(\delta+n-d)^{w_n} \\ \oplus \\ R(-d_1) \\ \oplus \\ \vdots \\ \oplus \\ R(-d_n) \end{array} \to R \to A \to 0$$

where w_n and z_2 are determined by the Hilbert Function of A.

Proof. We have $\mathfrak{c} = (F_1, F_2, \dots, F_n)$, $I = (\mathfrak{c}, G_1, \cdots, G_c)$ with $\deg(G_j) = d - s - n$ for all $j = 1, \cdots, c$. Note that the socle degree δ is determined by $d_1, d_2, \dots, d_n, c$ and $d - s - n$ because we have assumed that an ideal of general forms in R satisfies the Maximal Rank Property.)

By [19] Theorem 3.15 and Remark 4.10 above, I has a free R-resolution of the following type:

(8)

$$0 \to \begin{array}{c} R(-\delta-n+1)^{z_n} \\ \oplus \\ R(-\delta-n)^{w_n} \end{array} \to \begin{array}{c} R(-\delta-n+2)^{z_{n-1}} \\ \oplus \\ R(-\delta-n+1)^{w_{n-1}} \\ \oplus \\ (K_{n-1})^{\leq \delta+n-3} \end{array} \to \cdots \to \begin{array}{c} R(-\delta-1)^{z_2} \\ \oplus \\ R(-\delta-2)^{w_2} \\ \oplus \\ (K_2)^{\leq \delta} \end{array} \to \begin{array}{c} R(s+n-d)^{c} \\ \oplus \\ R(-d_1) \\ \oplus \\ \vdots \\ \oplus \\ R(-d_n) \end{array} \to I \to 0.$$

Note that $\mathfrak{c}$ has a minimal free (Koszul) resolution

$$0 \to L_n \to L_{n-1} \to \cdots \to L_1 \to \mathfrak{c} \to 0$$

where $L_n = R(-d)$. Applying the mapping cone process we get that $J = [\mathfrak{c} : I]$ has a free R-resolution of the following type:

$$0 \to R(-s-n)^c \to \begin{array}{c} R(\delta+1-d)^{z_2} \\ \oplus \\ R(\delta+2-d)^{w_2} \\ \oplus \\ ((K_2)^{\leq \delta})^{\vee}(-d) \end{array} \to \begin{array}{c} R(\delta+2-d)^{z_3} \\ \oplus \\ R(\delta+3-d)^{w_3} \\ \oplus \\ ((K_3)^{\leq \delta+1})^{\vee}(-d) \\ \oplus \\ (L_2)^{\vee}(-d) \end{array} \to \cdots \to \begin{array}{c} R(\delta+n-1-d)^{z_n} \\ \oplus \\ R(\delta+n-d)^{w_n} \\ \oplus \\ R(-d_1) \\ \oplus \\ \vdots \\ \oplus \\ R(-d_n) \end{array} \to J \to 0$$

as claimed. □

Remark 4.12. It is natural to wonder how close the resolution in Proposition 4.11 is to being minimal. The first consideration is whether (8) is minimal. The minimality of the first free module is completely determined by the Maximal Rank Property, since the forms are general. When the redundant generators are removed, it is then possible (in any given example) to determine how much splitting occurs in the mapping cone. So in fact, the only unknowns concern the values of the graded Betti numbers in redundant terms. We conjecture that when $n = 3$, one or the other must always be zero (i.e. $yz = 0$ in the following result), so there are no non-Koszul ghost terms. However, for $n \geq 4$ we have seen in Example 4.3 that this is not true.

Corollary 4.13. *Let $R = k[x, y, z]$ and let A be a level quotient of R of socle dimension $c \geq 2$. (The case $c = 1$ is the Gorenstein case that we have already discussed in Example 4.6). Assume that A has socle degree s and is relatively compressed with respect to a general complete intersection $\mathfrak{c} = (F_1, F_2, F_3) \subset R$. Set $d_i = deg(F_i)$, and $d = d_1 + d_2 + d_3$. Let I be the ideal generated by $\mathfrak{c}$ and c general forms of degree $d - s - 3$, and set δ to be the socle degree of R/I. Let $\mathbb{F} = R(-d_1) \oplus R(-d_2) \oplus R(-d_3) \oplus R(s + 3 - d)^c$ and let $K_2 = \wedge^2 \mathbb{F}$.*

Then A has a free R-resolution of the following type:

$$0 \to R(-s-3)^c \to \begin{array}{c} R(\delta+1-d)^x \\ \oplus \\ R(\delta+2-d)^y \\ \oplus \\ (K_2)^{\leq\delta})^\vee(-d) \end{array} \to \begin{array}{c} R(\delta+2-d)^z \\ \oplus \\ R(\delta+3-d)^w \\ \oplus \\ R(-d_1) \\ \oplus \\ R(-d_2) \\ \oplus \\ R(-d_3) \end{array} \to R \to A \to 0$$

where w and x are determined by the Hilbert Function of A.

Proof. This follows from Proposition 4.11 since $k[x,y,z]$ satisfies Fröberg's conjecture (cf. [1]). $\square$

5. Applications to ideals of general forms

In this section, as an application of Theorem 3.5, we get new results about the generic graded Betti numbers of an almost complete intersection ideal. The idea (which is not new) is to link an Artinian Gorenstein graded algebra R/G, relatively compressed with respect to a complete intersection ideal $\mathfrak{a} = (G_1, \cdots, G_r) \subset k[x_1, \cdots, x_n]$, to an almost complete intersection ideal $(\mathfrak{a}, G_{r+1}, \cdots, G_n, G_{n+1})$ via a complete intersection $(\mathfrak{a}, G_{r+1}, \cdots, G_n)$ where $G_{r+1}, \cdots, G_n$ are suitably chosen.

Let $R = k[x_1, \ldots, x_n]$ and let $I = (G_1, \ldots, G_{n+1})$ be the ideal of $n+1$ general forms of degrees $d_1 = \deg(G_1) \leq \cdots \leq d_{n+1} = \deg(G_{n+1})$. The Hilbert function of R/I is well known (at least in characteristic zero – see Remark 3.6), coming from a result of R. Stanley [23] and of J. Watanabe [25] which implies that a general Artinian complete intersection has the Strong Lefschetz Property, and a very long-standing problem in Commutative Algebra is to determine the minimal free resolution of R/I. In [18], the first and second author gave the precise graded Betti numbers of R/I in the following cases:

- $n = 3$.
- $n = 4$ and $\sum_{i=1}^5 d_i$ is even.
- $n = 4$, $\sum_{i=1}^5 d_i$ is odd and $d_2 + d_3 + d_4 < d_1 + d_5 + 4$.
- n is even and all generators have the same degree, a, which is even.
- $(\sum_{i=1}^{n+1} d_i) - n$ is even and $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$.
- $(\sum_{i=1}^{n+1} d_i) - n$ is odd, $n \geq 6$ is even, $d_2 + \cdots + d_n < d_1 + d_{n+1} + n$ and $d_1 + \cdots + d_n - d_{n+1} - n \gg 0$.

As a nice application of Theorem 3.5, we will enlarge the above list. Since the calculations are somewhat complicated, we illustrate the method with an example before we proceed to the general statement.

Example 5.1. Let $n = 5$, $d_1 = 2$, $d_2 = d_3 = d_4 = 4$, $d_5 = 5$ and $d_6 = 6$. Consider a general Gorenstein Artinian graded algebra R/G of embedding dimension 5, socle degree 8 and relatively compressed with respect to a complete intersection ideal $\mathfrak{a} = (G_1, G_2, G_3, G_4)$ with $\deg(G_1) = 2$, $\deg(G_2) = 4$, $\deg(G_3) = 4$ and $\deg(G_4) = 4$. By Theorem 3.5 A has a minimal free R-resolution of the following type:

$$0 \to R(-13) \to R(-8)^{46} \oplus R(-11) \oplus R(-9)^3 \to R(-7)^{149} \to R(-6)^{149} \to$$
$$R(-5)^{46} \oplus R(-2) \oplus R(-4)^3 \to R \to R/G \to 0.$$

Hence, there exists a complete intersection $J \subset G$ with generators of degrees 2,4,4,4,5. By a standard mapping cone argument, the residual $I = [J : G]$ is an almost complete intersection of type (2,4,4,4,5,6) and with the following minimal free R-resolution

$$0 \to R(-14)^{45} \to R(-13)^{146} \to R(-12)^{150} \oplus R(-11)^3 \oplus R(-10)^3 \to \tag{9}$$
$$R(-6)^3 \oplus R(-7) \oplus R(-8)^4 \oplus R(-9)^3 \oplus R(-10)^3 \oplus R(-11)^{46} \to$$
$$R(-2) \oplus R(-4)^3 \oplus R(-5) \oplus R(-6) \to R \to R/I \to 0.$$

Since there are no non-Koszul ghost terms and the graded Betti numbers are the smallest consistent with the Hilbert function 1 5 14 30 52 75 92 95 79 45 0 of the general almost complete intersection of type (2,4,4,4,5,6), the exact sequence (9) gives us the minimal free R-resolution of the general almost complete intersection of type (2,4,4,4,5,6).

The idea behind Example 5.1 leads to the following result

Theorem 5.2. *Let $I = (G_1, \ldots, G_{n+1})$ be a general almost complete intersection in $R = k[x_1, \ldots, x_n]$, with $d_i = \deg G_i$, $2 \le d_1 \le d_2 \cdots \le d_n \le d_{n+1} \le (\sum_{i=1}^n d_i) - n$, (the latter condition only assures that I is not a complete intersection,) and $\sum_{i=1}^{n+1} d_i - n$ even. Then, R/I has a minimal free R-resolution of the form*

$$0 \to F_1^\vee(-d) \to \begin{matrix} K_1(\underline{\underline{d}})^\vee(-d) \\ \oplus \\ F_2^\vee(-d) \end{matrix} \to \begin{matrix} K_2(\underline{\underline{d}})^\vee(-d) \\ \oplus \\ F_3^\vee(-d) \end{matrix} \to \cdots$$

$$\to \begin{matrix} K_{n-2}(\underline{\underline{d}})^\vee(-d) \\ \oplus \\ F_{n-1}^\vee(-d) \end{matrix} \to \oplus_{i=1}^{n+1} R(-d_i) \to R \to R/I \to 0$$

where $\underline{\underline{d}} = (d_1, \cdots, d_n)$, $d := d_1 + \cdots + d_n$, $K_i(\underline{\underline{d}}) = K_i(d_1, \cdots, d_n)$ and

$$F_i = K_i(\underline{d})^{\le (c/2)+i-1} \oplus R(-(c/2)-i)^{\alpha_i(\underline{d},n,c/2)} \oplus (K_{n-i}(\underline{d})^{\le (c/2)+n-i-1})^\vee(-c-n)$$

with $c = (\sum_{i=1}^n d_i) - d_{n+1} - n$, $r = min(n, max\{i/d_i \le c/2\})$ $\underline{d} = (d_1, \cdots, d_r)$, $K_i(\underline{d}) = K_i(d_1, \cdots, d_r)$, $\alpha_i(\underline{d}, n, c/2) = \alpha_{n-i}(\underline{d}, n, c/2)$ and $\alpha_i(\underline{d}, n, c/2)$ determined by the Hilbert function of $R/(G_1, \cdots, G_r)$.

Proof. Consider a general Gorenstein Artinian graded algebra R/G of embedding dimension n, socle degree $c = (\sum_{i=1}^n d_i) - d_{n+1} - n$ and relatively compressed with respect to a complete intersection ideal $\mathfrak{a} = (G_1, \cdots, G_r)$ with $\deg(G_i) = d_i$ and $r = \min(n, \max\{i | d_i \le c/2\})$. So, $h_{R/G}(t) = \min\{h_{R/(G_1,\cdots,G_r)}(t), h_{R/(G_1,\cdots,G_r)}(c-t)\}$.

Since by hypothesis the socle degree c of R/G is even, we can apply Theorem 3.5 and we get that R/G has a minimal free R-resolution of the following type:

$$0 \to R(-c-n) \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R/G \to 0$$

where for all $i = 1, \cdots, n-1$, we have

$$F_i = K_i(\underline{d})^{\le (c/2)+i-1} \oplus R(-(c/2)-i)^{\alpha_i(\underline{d},n,c/2)} \oplus (K_{n-i}(\underline{d})^{\le (c/2)+n-i-1})^\vee(-c-n)$$

with $\alpha_i(\underline{d}, n, c/2) = \alpha_{n-i}(\underline{d}, n, c/2)$ and $\alpha_i(\underline{d}, n, c/2)$ is determined by the Hilbert function of R/G.

Hence, there exists a complete intersection $J \subset G$ with generators of degrees $d_1, \cdots, d_n$. The minimal free R-resolution of R/J is given by the Koszul resolution:

$$0 \to K_n(\underline{\underline{d}}) \to K_{n-1}(\underline{\underline{d}}) \to \cdots \to K_2(\underline{\underline{d}}) \to K_1(\underline{\underline{d}}) \to R \to R/J \to 0.$$

By a standard mapping cone argument applied to the diagram

$$\begin{array}{ccccccccccccccccc}
0 & \to & R(-d) & \to & K_{n-1}(\underline{\underline{d}}) & \to & \cdots & \to & K_2(\underline{\underline{d}}) & \to & K_1(\underline{\underline{d}}) & \to & R & \to & R/J & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R(-c-n) & \to & F_{n-1} & \to & \cdots & \to & F_2 & \to & F_1 & \to & R & \to & R/G & \to & 0
\end{array}$$

the residual $I = [J : G]$ is an almost complete intersection of type

$$(d_1, \cdots, d_n, -c - n + \sum_{i=1}^{n} d_i) = (d_1, \cdots, d_n, d_{n+1})$$

and with the following minimal free R-resolution

(10)

$$0 \to F_1^\vee(-d) \to \begin{array}{c} K_1(\underline{\underline{d}})^\vee(-d) \\ \oplus \\ F_2^\vee(-d) \end{array} \to \begin{array}{c} K_2(\underline{\underline{d}})^\vee(-d) \\ \oplus \\ F_3^\vee(-d) \end{array} \to \cdots \to \oplus_{i=1}^{n+1} R(-d_i) \to R \to R/I \to 0.$$

Since there are no non-Koszul ghost terms and the graded Betti numbers are the smallest consistent with the Hilbert function

$$h_{R/I'}(\ell) = \left[h_{R/J}\left((\sum_{i=1}^{n} d_i) - n - \ell \right) - h_{R/J}\left((\sum_{i=1}^{n} d_i) - n - \ell - d_{n+1} \right) \right]_+$$

(where $[x]_+$ denotes the maximum of x and 0) of the general almost complete intersection ideal $I' \subset k[x_1, \cdots, x_n]$ of type $(d_1, \cdots, d_n, d_{n+1})$, the exact sequence (10) gives us the minimal free R-resolution of the general almost complete intersection of type $(d_1, \cdots, d_n, d_{n+1})$. □

Remark 5.3. We point out that there are many new cases covered by Theorem 5.2, that were not known previously (and in particular not in the list on page 27). The most natural remaining open question is to determine the minimal free resolution of an ideal of $n + 1$ general forms of degree a in n variables, when either n is odd or n is even and a is odd.

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