MONOMIAL IDEALS AND THE GORENSTEIN LIAISON CLASS OF A COMPLETE INTERSECTION

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ABSTRACT. In an earlier work the authors described a mechanism for lifting monomial ideals to reduced unions of linear varieties. When the monomial ideal is Cohen-Macaulay (including Artinian), the corresponding union of linear varieties is arithmetically Cohen-Macaulay. The first main result of this paper is that if the monomial ideal is Artinian then the corresponding union is in the Gorenstein linkage class of a complete intersection (glicci). This technique has some interesting consequences. For instance, given any $(d+1)$-times differentiable O-sequence $\underline{H}$, there is a non-degenerate arithmetically Cohen-Macaulay reduced union of linear varieties with Hilbert function $\underline{H}$ which is glicci. In other words, any Hilbert function that occurs for arithmetically Cohen-Macaulay schemes in fact occurs among the glicci schemes. This is not true for licci schemes. Modifying our technique, the second main result is that any Cohen-Macaulay Borel-fixed monomial ideal is glicci. As a consequence, all arithmetically Cohen-Macaulay subschemes of projective space are glicci up to flat deformation.

CONTENTS

1. INTRODUCTION

Liaison theory has reached a very satisfying state in codimension two, but in higher codimension there are still many open problems. Much of the theory has been built around linking with complete intersections, called CI-liaison theory. However, it has long been known that more generally it is also possible to link using arithmetically Gorenstein schemes (cf. [17] for example). Indeed, a development of G-liaison theory is possible (cf. [10], [13], [16]). In practice, however, this has been studied less because it is not so easy to find arithmetically Gorenstein schemes other than complete intersections, especially

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containing a given scheme. Note that in codimension two all arithmetically Gorenstein schemes are complete intersections, so both theories include the codimension two case.

Nevertheless, there has been recent work in the direction of G-liaison theory, most notably in [10] where a very geometric approach is taken and where this theory is compared and contrasted with the more classical CI-liaison theory. See also [13] for extensive background and comparisons.

In the codimension two case, one of the first important results was the theorem of Gaeta that every arithmetically Cohen-Macaulay, codimension two scheme is in the liaison class of a complete intersection (i.e. is *licci*, a term introduced in [9]). In [10] the authors introduced the notion of *glicci* schemes, i.e. those which are in the Gorenstein liaison class of a complete intersection. They generalized Gaeta's theorem by showing that every scheme which arises as the maximal minors of a homogeneous matrix (and which have the right codimension depending on the size of the matrix) is glicci. (Note that arithmetically Cohen-Macaulay schemes of codimension two satisfy this property, thanks to the Hilbert-Burch theorem.) However, the authors of [10] asked if a more general result might hold:

Question 1.1 ([10]). *Is it true that all arithmetically Cohen-Macaulay subschemes of* $\mathbb{P}^n$ *are glicci?*

Some evidence of this was provided by showing that on a smooth rational arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$ all arithmetically Cohen-Macaulay curves (i.e. divisors) are glicci. Casanellas and Miró-Roig [3] extended this by finding a large class of smooth surfaces in $\mathbb{P}^4$ where the same conclusion holds, and in a more recent paper [4] they extend this to a large class of smooth schemes of any codimension.

In this paper we make some further progress in this direction. We prove the glicciness of two different kinds of Cohen-Macaulay ideals. First we recall that if J is any Artinian monomial ideal then it is shown in [14] how to produce the ideal I of a non-degenerate arithmetically Cohen-Macaulay reduced union of linear varieties of any dimension whose Artinian reduction is precisely J. The first main result of this paper is that any such I is glicci. As a corollary we get that given any numerical function which occurs as the Hilbert function of some non-degenerate arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n$ of any codimension, there is a reduced, glicci subscheme with precisely that Hilbert function. Example 3.3 shows that this is not true if we replace Gorenstein links by complete intersection links.

Our second main result is that any Cohen-Macaulay Borel-fixed monomial ideal (Artinian or not) is glicci. This result is of a rather general nature. Indeed, it is well-known that every generic initial ideal of an arithmetically Cohen-Macaulay subscheme is a Cohen-Macaulay Borel-fixed ideal which defines a deformation of the original scheme. Thus our result says that every arithmetically Cohen-Macaulay subscheme admits a flat deformation which is glicci. In other words, we have found an affirmative answer to Question 1.1 "up to flat deformation."

2. Preliminaries

Let K be an infinite field and let $S = K[x_1, \dots, x_n]$ and $R = K[x_1, \dots, x_n, u_1, \dots, u_t]$, $t \geq 0$. We first recall the set-up and one of the main results of [14].

Definition 2.1. Let $I \subset R$ and $J \subset S$ be homogeneous ideals. Then we say I is a *t-lifting* of J to R (or when R is understood, simply a *t-lifting* of J) if $(u_1, \ldots, u_t)$ is a regular sequence on R/I and $(I, u_1, \ldots, u_t)/(u_1, \ldots, u_t) \cong J$.

The definition of a t-lifting can be extended to modules, but Definition 2.1 suffices for our purposes. Consider now a matrix of linear forms

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \ldots \\ L_{2,1} & L_{2,2} & L_{2,3} & \ldots \\ \vdots & \vdots & \vdots & \\ L_{n,1} & L_{n,2} & L_{n,3} & \ldots \end{bmatrix}$$

where the $L_{j,i}$ are in R. A will be called the *lifting matrix*, for reasons that will be apparent shortly. For now we assume that there are infinitely many columns, but in practice when we have a specific ideal $J \subset S$ that we want to lift we can assume that the number of columns is finite, for instance equal to the regularity of J (or less). Assume that the polynomials $F_j = \prod_{i=1}^{N} L_{j,i}$, $1 \leq j \leq n$, define a complete intersection, X. Note that F_j is the product of the entries of the j-th row, and that the height of the complete intersection is n, the number of variables in S.

Let $m = \prod_{j=1}^{n} x_j^{a_j} \subset S$ be a monomial. We associate to m the homogeneous polynomial

$$\bar{m} = \prod_{j=1}^{n} \left(\prod_{i=1}^{a_j} L_{j,i} \right) \in R.$$

Let $J = (m_1, \ldots, m_r) \subset S$ be a monomial ideal. Associated to J we define the ideal $I = (\bar{m}_1, \ldots, \bar{m}_r) \subset R$.

Theorem 2.2 ([14])**.** (i) *The ideal I is saturated.*

(ii) *S/J is Cohen-Macaulay (including the case where it is Artinian) if and only if R/I is Cohen-Macaulay. In fact, I (as an R-module) and J (as an S-module) have the same graded Betti numbers.*

(iii) *If, for each j and i, we have $L_{j,i} \in K[x_j, u_1, \ldots, u_t]$ then I is a t-lifting of J. Otherwise we say that I is a* pseudo-lifting *of J.*

If the entries of A are chosen sufficiently generally then I in fact defines a reduced union of linear varieties with good intersection properties. Now we recall some results from [10].

We now recall the notion of *Basic Double G-linkage* introduced in [10], so called because of part (iv) and the notion of Basic Double Linkage ([11], [2], [8]).

Theorem 2.3 ([10] Lemma 4.8, Remark 4.10 and Proposition 5.10)**.** *Let $J \supset I$ be homogeneous ideals of $R' = K[x_0, \ldots, x_n]$, defining schemes $W \subset V \subset \mathbb{P}^n$ such that* $\operatorname{codim} W = \operatorname{codim} V + 1$. *We also allow the possibility that J is Artinian and V is a zeroscheme. Let $A \in R'$ be an element of degree d such that $I : A = I$. Then we have*

(i) $\deg(I + A \cdot J) = d \cdot \deg I + \deg J$.

(ii) *If I is perfect and J is unmixed then $I + A \cdot J$ is unmixed.*

(iii) *$J/I \cong [(I + A \cdot J)/I](d)$.*

(iv) *If V is arithmetically Cohen-Macaulay and generically Gorenstein and J is unmixed then J and $I + A \cdot J$ are linked in two steps using Gorenstein ideals.*

(v) *The Hilbert functions are related by*

$$\begin{aligned} h_{R'/(I+A\cdot J)}(t) &= h_{R'/(I+(A))}(t) + h_{R'/J}(t-d) \\ &= h_{R'/I}(t) - h_{R'/I}(t-d) + h_{R'/J}(t-d) \end{aligned}$$

Theorem 2.3 should be interpreted as viewing the scheme W defined by J as a divisor on the scheme V defined by I, and adding to it a hypersurface section H_A of V defined by the polynomial A. Note that $I_{H_A} = I_V + (A)$. If V and W are arithmetically Cohen-Macaulay then the divisor $W + H_A$ is again arithmetically Cohen-Macaulay (by step 4). As an immediate application we have the following by successively applying Theorem 2.3.

Corollary 2.4 ([15]). *Let $V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{P}^n$ be arithmetically Cohen-Macaulay schemes of the same dimension, all generically Gorenstein. Let $H_1, \dots, H_r$ be hypersurfaces, defined by forms $F_1, \dots, F_r$, such that for each i, H_i contains no component of V_j for any $j \leq i$. Let W_i be the arithmetically Cohen-Macaulay schemes defined by the corresponding hypersurface sections: $I_{W_i} = I_{V_i} + (F_i)$. Then we have the following.*

(i) *Viewed as divisors on V_r, the sum Z of the W_i (which is just the union if the hypersurfaces are general enough) is in the same Gorenstein liaison class as W_1.*
(ii) *In particular, Z is arithmetically Cohen-Macaulay.*
(iii) *As ideals we have*

$$I_Z = I_{V_r} + F_r \cdot I_{V_{r-1}} + F_r F_{r-1} I_{V_{r-2}} + \cdots + F_r F_{r-1} \cdots F_2 I_{V_1} + (F_r F_{r-1} \cdots F_1).$$

(iv) *Let $d_i = \deg F_i$. The Hilbert functions are related by the formula*

$$\begin{aligned} h_Z(t) &= h_{W_r}(t) + h_{W_{r-1}}(t-d_r) + h_{W_{r-2}}(t-d_r-d_{r-1}) + \dots \\ &\quad + h_{W_1}(t - d_r - d_{r-1} - \cdots - d_2). \end{aligned}$$

Corollary 2.5. *We keep the notation of Corollary 2.4. If V_1 is glicci then so is Z.*

Proof. This follows from part (iv) of Theorem 2.3, and from the fact that Gorenstein liaison is preserved under hypersurface sections ([13] Proposition 5.2.17). Note that the reverse direction is not necessarily true (or in any case is not known to be true): if W_1 is glicci, it does not necessarily (or at least immediately) hold that V_1 is. See [13] Example 5.2.26 for some discussion. $\square$

We now discuss the decomposition of a monomial ideal, which we will use in the remaining sections.

Definition 2.6. Let $>$ denote the degree-lexicographic order on monomial ideals, i.e. $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ if the first nonzero coordinate of the vector

$$\left(\sum_{i=1}^{n} (a_i - b_i), a_1 - b_1, \dots, a_n - b_n \right)$$

is positive. Let J be a monomial ideal. Let m_1, m_2 be monomials in S of the same degree such that $m_1 > m_2$. Then J is a *lex-segment ideal* if $m_2 \in J$ implies $m_1 \in J$. When $\operatorname{char}(K) = 0$, we say that J is a *Borel-fixed ideal* if

$$m = x_1^{a_1} \cdots x_n^{a_n} \in J, \ a_i > 0, \ \text{implies } \frac{x_j}{x_i} \cdot m \in J$$

for all $1 \leq j < i \leq n$.

Remark 2.7. Definition 2.6 says that if J is Borel-fixed and $m \in J$ is a monomial then one can reduce any power of a variable occurring in m by one and increase the power of a larger variable by one, and the result is again in J. Note that this is not the same as lex-segment. For example, in the ring $K[x_1, x_2, x_3]$ consider the ideal

$$J = \langle x_1^3, x_1^2 x_2, x_1 x_2^2 \rangle.$$

This is Borel-fixed but not lex-segment, since $x_1^2 x_3 \notin J$. The two notions are not even equivalent in the Artinian case, as the same example shows if we adjoin to J all monomials of degree 4. However, a lex-segment ideal is always Borel-fixed.

Lemma 2.8. *Let $J \subset S = K[x_1, \dots, x_n]$ be a monomial ideal. Let α be the highest power of x_1 occurring in a minimal generator of J. Then there is a uniquely determined decomposition*

$$J = \sum_{j=0}^{\alpha} x_1^j \cdot (I_j \cdot S)$$

where $I_0 \subset I_1 \subset \dots \subset I_{\alpha-1} \subset I_\alpha$ are monomial ideals in $T = K[x_2, \dots, x_n]$. Furthermore,

(i) $I_j = (J : x_1^j) \cap K[x_2, \dots, x_n]$.
(ii) *If J is Artinian then so is each I_j and $I_\alpha = (1)$.*
(iii) *Assume char$(K) = 0$. If J is a Borel-fixed ideal (e.g. a lex-segment ideal) then α is the initial degree of J, $I_\alpha = (1)$, and each I_j is again Borel-fixed.*

Proof. The case of Artinian lex-segment ideals was observed in [15].

The existence of the decomposition is clear if we choose the ideals I_j as described in (i). Conversely, if we have the decomposition, then we get in case $0 \le j \le \alpha$:

$$\begin{aligned} J : x_1^j &= \sum_{k=0}^{j} I_k \cdot S + \sum_{k=j+1}^{\alpha} x_1^{k-j} \cdot (I_k \cdot S) \\ &= I_j \cdot S + \sum_{k=j+1}^{\alpha} x_1^{k-j} \cdot (I_k \cdot S), \end{aligned}$$

thus

$$(J : x_1^j) \cap T = (I_j \cdot S) \cap T = I_j$$

proving (i) and the uniqueness of the decomposition.

For (ii), since J is Artinian then it contains pure powers of $x_2, \dots, x_n$, so these are automatically in I_0, making I_0 Artinian. Then the inclusions imply that the other I_j are also Artinian. Furthermore, x_1^α is a minimal generator of J, so $I_\alpha = (1)$ as claimed. For part (iii), the hypothesis implies that x_1^α is a minimal generator of J. The fact that I_j is Borel-fixed follows immediately from the definition of Borel-fixed and the description of I_j in the statement of the Lemma. □

Lemma 2.9. *Keeping the notation of Lemma 2.8, for any $s \ge 0$, we have*

$$h_{S/J}(s) = \sum_{j=0}^{\alpha-1} h_{T/I_j}(s-j) + h_{S/I_\alpha \cdot S}(s-\alpha).$$

Proof. If $\alpha = 0$ then $J = I_0 \cdot S$ and the claim is clear. If $\alpha > 0$ then multiplication by x_1 provides the exact sequence

$$0 \to S/\sum_{j=1}^{\alpha} x_1^{j-1}(I_j \cdot S)(-1) \xrightarrow{\cdot x_1} S/J \to T/I_0 \to 0.$$

Hence the claim follows by induction on α. □

Remark 2.10. If J is Artinian or Borel-fixed then the Hilbert function formula of Lemma 2.9 simplifies to

$$h_{S/J}(s) = \sum_{j=0}^{\alpha-1} h_{T/I_j}(s-j)$$

since in either of these cases $I_\alpha = (1)$ by Lemma 2.8.

3. Glicci Ideals

Let J be an Artinian monomial ideal in $S = K[x_1, \dots, x_n]$. Let A be a lifting matrix for J and assume that the entries of A are sufficiently general so that the lifted ideal is a reduced union of linear varieties.. The number of columns of A only has to be as large as the largest degree of a minimal generator of J; if J is lex-segment then this degree is $=\operatorname{reg}(J)$. Applying the pseudo-lifting procedure described in Section 2, we get an ideal $I \subset R = K[x_1, \dots, x_n, u_1, \dots, u_t]$ which, by Theorem 2.2, is the saturated ideal of an arithmetically Cohen-Macaulay subscheme Z of $\mathbb{P}^{n+t-1}$ of codimension n.

Theorem 3.1. *Z is glicci.*

Proof. The proof is by induction on n, the codimension. For codimension two it is known that any arithmetically Cohen-Macaulay subscheme of projective space is licci, so there is nothing to prove. Hence we assume $n \geq 3$.

By Lemma 2.8 we have

$$J = \sum_{j=0}^{\alpha} x_1^j \cdot I_j \tag{3.1}$$

where $I_0 \subset I_1 \subset \cdots \subset I_{\alpha-1} \subsetneq I_\alpha = S$ and for each j, I_j is an Artinian ideal in $K[x_2, \dots, x_n]$. Notice that the lifting matrix A has n rows, and if we remove the first row then the remaining matrix A' can be used to lift the ideals I_j.

Let $\bar{I}_j$ be the ideal obtained by lifting I_j using A'. Let Y_j be the arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^{n+t-1}$ defined by $\bar{I}_j$. Note that Y_j has codimension $n-1$, but the projective space does not change since the linear forms which are the entries of A were taken from the ring R. Note also that $Y_{\alpha-1} \subset \cdots \subset Y_1 \subset Y_0$ are arithmetically Cohen-Macaulay schemes of the same dimension. Furthermore, they are generically Gorenstein since they are reduced, thanks to the generality condition on the matrix A (hence also A') imposed at the beginning of this section. Hence the first sentence of the hypothesis of Corollary 2.4 holds in this case. Furthermore, since the polynomials $F_j = \prod_{i=1}^{N} L_{j,i}$, $1 \leq j \leq n$, define a complete intersection, it follows that any $L_{1,i}$ cuts any Y_j properly.

Hence the second sentence of the hypothesis of Corollary 2.4 holds. Therefore we can apply Corollary 2.4.

Thanks to (3.1) we have

$$I = \bar{I}_0 + L_{1,1} \cdot \bar{I}_1 + L_{1,1}L_{1,2} \cdot \bar{I}_2 + \cdots + L_{1,1}L_{1,2} \cdots L_{1,\alpha-1} \cdot \bar{I}_{\alpha-1} + (L_{1,1}L_{1,2} \cdots L_{1,\alpha-1}L_{1,\alpha})$$

Hence by Corollary 2.4 (iii), the scheme Z obtained from lifting is in fact also obtained by taking the union of the successive hypersurface sections of the Y_j. By induction, $Y_{\alpha-1}$ is glicci. By Corollary 2.5, then, Z is also glicci. □

As a corollary of Theorem 3.1 we would like to show that given "any" Hilbert function we can find a glicci subscheme with that Hilbert function. Recall from [7] that the Hilbert functions which can occur for arithmetically Cohen-Macaulay subschemes of a given dimension d have been completely characterized. Indeed, for a function $f : \mathbb{Z} \to \mathbb{Z}$ we define the first difference Δf by $\Delta f(n) = f(n) - f(n-1)$ and the k-th difference $\Delta^k f$ by iteration. An O-sequence is one that satisfies Macaulay's growth condition [12]. A k-times differentiable O-sequence is one for which also the first k differences are O-sequences. Then a function $f : \mathbb{N} \to \mathbb{N}$ occurs as the Hilbert function of some d-dimensional arithmetically Cohen-Macaulay scheme (in fact it can always be chosen reduced) if and only if f is a $(d+1)$-times differentiable O-sequence.

We immediately get the following somewhat surprising conclusion.

Corollary 3.2. *Let $\underline{H}$ be any $(d+1)$-times differentiable O-sequence. Then $\underline{H}$ occurs as the Hilbert function of some non-degenerate glicci subscheme of projective space.*

Proof. Let $\underline{h}$ be the $(d+1)$-st difference of $\underline{H}$. Let J be the Artinian lex-segment ideal with Hilbert function $\underline{h}$. If $J \subset K[x_1, \dots, x_n]$ then choose a lifting matrix with entries $L_{j,i} \in K[x_j, u_1, \dots, u_{d+1}]$. The lifted ideal I defines a glicci subscheme of $\mathbb{P}^{n+d-1}$ by Theorem 3.1, and it has Hilbert function $\underline{H}$ since it is a $(d+1)$-lifting. The non-degenerate property comes directly from the lifting, cf. [14]. □

Example 3.3. We remark that Corollary 3.2 is false for complete intersection liaison. Indeed, the h-vector $(1, 3)$ cannot occur for any codimension 3 licci subscheme of projective space. To see this, note that the minimal free resolution of any arithmetically Cohen-Macaulay subscheme with this h-vector is linear, and [9], Corollary 5.13, then guarantees that it is not licci. (Note that degenerate subschemes of projective space, of codimension > 3, could also have this h-vector, and we do not know if the "extra room" makes a difference in the non-licciness.)

From the proof of Corollary 3.2 one would be very tempted to conclude that we have proved that any Artinian monomial ideal is glicci, since liaison is preserved under general hyperplane sections, even for the Artinian reduction (cf. [13] Remark 5.2.18). However, the proofs above use *bilinks*, so even if a variable u_i is a non zero-divisor for the scheme Z (and hence any of its components), it is not necessarily true that the same is true for the linked schemes. However, we can obtain an important case of this result, and in fact more, by modifying the above approach slightly.

From now on we assume $\mathrm{char}(K) = 0$. We begin with a lemma.

Lemma 3.4. *Let J be a Borel-fixed monomial ideal of codimension c. The following are equivalent.*

(i) *J is Cohen-Macaulay.*
(ii) *J is unmixed.*
(iii) *J contains a pure power of x_c, and the variables $x_{c+1}, \dots, x_n$ do not occur in any of the minimal generators.*
(iv) *J is a cone over an Artinian Borel-fixed ideal in $K[x_1, \dots, x_c]$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are always true. Note that condition (iii) implies that J in fact contains pure powers of each of the variables $x_1, \dots, x_c$, by the Borel-fixed property. Then the implication (iii) $\Rightarrow$ (iv) is immediate, since Borel-fixed is already assumed.

So we have only to prove (ii) $\Rightarrow$ (iii). Since J has codimension c, it contains a regular sequence of length c. By the Borel-fixed property we may take this regular sequence to consist of pure powers of variables, and again by the Borel-fixed property we can take it to be powers of $x_1, \dots, x_c$. Suppose that one of the other variables, say x_{c+1}, occurs in one of the minimal generators of J to some power $a \geq 1$. By a standard trick on monomial ideals (cf. for instance [5] Exercise 3.8) we can then decompose J as $J = A \cap (J + (x_{c+1}^a))$ where A is again a monomial ideal. But this shows that the primary decomposition of J has at least one component of height $c+1$, contradicting the hypothesis that J is equidimensional. □

Theorem 3.5. *Any Cohen-Macaulay Borel-fixed monomial ideal is glicci.*

Proof. Let J be a Cohen-Macaulay Borel-fixed monomial ideal in $S = K[x_1, \dots, x_n]$ of height c. By Lemma 3.4, we may view J as a cone over an Artinian Borel-fixed ideal in $K[x_1, \dots, x_c]$. By Lemma 2.8,

$$\begin{aligned} J &= I_0 + x_1 I_1 + x_1^2 I_2 + \dots + x_1^\alpha I_\alpha \\ &= I_0 + x_1 I' \end{aligned}$$

where $I' = I_1 + x_1 I_2 + \dots + x_1^{\alpha-1} I_\alpha$, α is the initial degree of J, the I_j are cones over Artinian Borel-fixed ideals in $K[x_2, \dots, x_c]$ satisfying $I_0 \subset I_1 \subset \dots$ and $I_\alpha = (1)$. In particular, I_0 is a cone over an Artinian Borel-fixed ideal in $K[x_2, \dots, x_c] \subset S$, and I' is a Borel-fixed monomial ideal in S whose initial degree is one less than that of J.

Following Theorem 2.2, we can lift I_0 to an ideal $\bar{I}_0$ in $K[x_1, \dots, x_c] \cap S$; that is, we choose a lifting matrix A whose entries are linear forms $L_{j,i} \in K[x_j, x_1]$, $2 \leq j \leq n$. For example, take

$$A = \begin{bmatrix} x_2 & x_2 + x_1 & x_2 + 2x_1 & x_2 + 3x_1 \dots \\ x_3 & x_3 + x_1 & x_3 + 2x_1 & x_3 + 3x_1 \dots \\ \vdots & \vdots & \vdots & \vdots \\ x_n & x_n + x_1 & x_n + 2x_1 & x_n + 3x_1 \dots \end{bmatrix}.$$

We now make some observations.

(1) I' is also Cohen-Macaulay by Lemma 3.4, and it has the same height c as J since it contains a complete intersection consisting of powers of $x_1, \dots, x_c$.
(2) $I_0 \subset I'$, since $I_0 \subset I_1 \subset I'$ from above.
(3) $\bar{I}_0 \subset I'$. This follows immediately. For instance, suppose that $x_2^3 x_3^4 \in I_0$. Then

$$x_2(x_2 + x_1)(x_2 + 2x_1)(x_3)(x_3 + x_1)(x_3 + 2x_1)(x_3 + 3x_1) \in \bar{I}_0.$$

By the Borel-fixed property of J and the fact that $I_0 \subset I'$, it follows immediately that each term of this polynomial is in I'.

(4) I_0 and $\bar{I}_0$ are both Cohen-Macaulay, and $\mathrm{ht}(\bar{I}_0) = \mathrm{ht}(I_0) = c - 1$. This follows from the fact that I_0 is Cohen-Macaulay by Lemma 3.4 and that the Cohen-Macaulay property and the codimension are preserved under lifting.

Let $\bar{J} = \bar{I}_0 + x_1 \cdot I'$. An analysis similar to observation (3) above shows quickly that $\bar{J} \subset J$. But both are Cohen-Macaulay of the same height in S, and they have the same Hilbert function (since the Hilbert function of $K[x_2, \dots, x_c]/I_0$ is the first difference of that of $K[x_1, \dots, x_c]/\bar{I}_0$). Hence we obtain that $\bar{J} = J$.

Although I_0 is not necessarily generically Gorenstein, the lifting results guarantee that $\bar{I}_0$ is, and we have noted that $\bar{I}_0$ is Cohen-Macaulay. Hence Theorem 2.3 (iv) says that $J = \bar{J}$ is G-bilinked to I', and in particular I' is Cohen-Macaulay. We have noted that the initial degree of I' is one less than that of J. Hence in a finite (even) number of steps we obtain that $J = \bar{J}$ is linked to the hyperplane section $\bar{I}_0 + (x_1) = I_0 + (x_1)$. Thus it is enough to show that I_0 is glicci. Let J_0 denote the ideal $I_0 \cap T$ in $T := K[x_2, ..., x_n]$. Then I_0 is just a cone over J_0. By induction on the height, J_0 is glicci in T. Then taking cones we get that also I_0 is glicci. Hence we have shown that $J = \bar{J}$ is glicci, as claimed. □

Remark 3.6. Theorem 3.5 is of a rather general nature. It is well-known that every generic initial ideal of an arithmetically Cohen-Macaulay subscheme is a Cohen-Macaulay Borel-fixed ideal which defines a deformation of the original scheme. Indeed, the fact that it is Borel-fixed is due to Galligo [6]; that it gives a flat deformation is due to Bayer [1]; that it is again Cohen-Macaulay follows from a result of Bayer and Stillman (cf. [5] Theorem 15.13). Thus our result says that every arithmetically Cohen-Macaulay subscheme admits a flat deformation which is glicci. In other words, we have found an affirmative answer to Question 1.1 "up to flat deformation."

Example 3.7. We illustrate the above ideas by finding a glicci subscheme $Z \subset \mathbb{P}^3$ with h-vector

$$\underline{h} = (1, 3, 6, 10, 4, 2).$$

Note that using complete intersections it does not seem promising that a licci subscheme with this h-vector can be found since the smallest complete intersection containing it would be the complete intersection of three quartics, and the residual would have even larger degree and will not lie in a smaller complete intersection.

Instead we consider the ring $S = K[x_1, x_2, x_3]$ and let J be the Artinian lex-segment ideal with Hilbert function $\underline{h}$. We have the decomposition

$$J = I_0 + x_1 \cdot I_1 + x_1^2 \cdot I_2 + x_1^3 \cdot I_3 + (x_1^4),$$

where the I_j are Artinian lex-segment ideals in $T = K[x_2, x_3]$ whose Hilbert functions are given as follows (note the shifting for I_1, I_2 and I_3 to apply Lemma 2.9):

	degree:						
	0	1	2	3	4	5	6
I_0	1	2	3	4	4	2	
I_1		1	2	3			
I_2			1	2			
I_3				1			
	1	3	6	10	4	2	

If

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \dots \\ L_{2,1} & L_{2,2} & L_{2,3} & \dots \\ L_{3,1} & L_{3,2} & L_{3,3} & \dots \end{bmatrix}$$

is a lifting matrix with 3 rows and at least 6 columns then the lifted ideal I is the saturated ideal of a zeroscheme Z in $\mathbb{P}^3$ which

(i) is reduced if A is sufficiently general,
(ii) is glicci, by Theorem 3.1, and
(iii) has h-vector $\underline{h}$.

The proof of Theorem 3.1 shows that Z can in fact be obtained as the union of successive hyperplane sections (denoting hyperplanes with the same notation as the corresponding linear forms)

$$Z = (L_{1,1} \cap V_0) \cup (L_{1,2} \cap V_1) \cup (L_{1,3} \cap V_2) \cup (L_{1,4} \cap V_3)$$

where

$$V_3 \subset V_2 \subset V_1 \subset V_0$$

are reduced arithmetically Cohen-Macaulay configurations of lines in $\mathbb{P}^3$ obtained by lifting $I_0, \dots, I_3$ using the submatrix

$$A' = \begin{bmatrix} L_{2,1} & L_{2,2} & L_{2,3} & \dots \\ L_{3,1} & L_{3,2} & L_{3,3} & \dots \end{bmatrix}$$

and the h-vectors of the V_j are given by the rows of the table above.

4. Further Comments

We end with some comments and questions raised by this paper. The results in this paper, as well as those in [10], [3] and [4], suggest strongly to us that the answer to Question 1.1 is "yes." The following ideas may help to ultimately give a final answer to this question.

1. We have seen that Cohen-Macaulay Borel-fixed monomial ideals are glicci. Is it in fact true that *every* Cohen-Macaulay monomial ideal is glicci? Or is it at least true that every Artinian monomial ideal is glicci?
2. Given a Hilbert function, our lifting gives the "worst" arithmetically Cohen-Macaulay scheme with that Hilbert function. As a result, this scheme should be the most difficult to find "good" arithmetically Gorenstein schemes containing it. Since we can find suitable ones for this "worst case," can this suggest how to link a different

arithmetically Cohen-Macaulay scheme with that same Hilbert function down to a complete intersection?

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