WHEN DOES A PROJECTIVE CURVE CONTAIN A PLANAR SUBCURVE?

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ABSTRACT. If a curve contains a planar subcurve of degree r then its general hyperplane section contains a subscheme of degree r spanning a line. Here we study to which extent the converse is true. If r is small we describe counterexamples. However, we give an affirmative answer if r is large with respect to the degree of the curve. This result is achieved by studying curves which admit an irreducible 2-dimensional family of r-secants. We show that such curves are forced to contain a planar subcurve of degree r unless they contain certain multiple lines. We describe the exceptional curves and give examples of them.

1. INTRODUCTION

It is a classical method to study curves using general hyperplane sections. For example, this is exactly the strategy of the so-called Castelnuovo method for bounding the genus of a curve. Of course, the basic problem of the method is to lift information about the hyperplane section to the curve itself. In this paper we study a particular case. Suppose that the general hyperplane section contains a (large) subscheme which is contained in a line. Does this force the existence of a planar subcurve having the collinear subscheme as general hyperplane section?

The motivation for this question stems from the observation that the existence of planar subcurves explains often interesting properties of the original curve. For example, a non-degenerate space curve of degree $d \geq 5$ has a Hartshorne-Rao module of maximal length if and only if it contains a planar subcurve of degree $d-1$ (cf. [2] and [8]). This point of view will be pursued in a subsequent paper.

In this paper we attack the lifting problem above from the point of view of multisecants. Let $C \subset \mathbb{P}^n$ be a curve (i.e. a pure 1-dimensional closed locally Cohen-Macaulay subscheme) over the algebraically closed field k. Let L denote a general hyperplane and let $\Gamma := C \cap L$ be the general hyperplane section of C. If C contains a planar subcurve C' of degree r then there is a subscheme $\Gamma' \subset \Gamma$ of degree r which is contained on a line. Here we ask to which extent the converse is true. This is already an interesting question in the special case when $r = d$ where d denotes the degree of C. Indeed, Hartshorne [5] has shown that then there is an affirmative answer for space curves, i.e., C must be in fact a planar curve, provided $d \geq 3$ and the characteristic of the ground field k is zero. Hartshorne also classified the exceptions which occur in positive characteristic.

If $r < d$ it is clear that we cannot hope that it is always possible to conclude the existence of a planar subcurve $C' \subset C$ of degree r. However, if we assume that the general hyperplane section Γ contains a unique collinear subscheme Γ' of degree $r \geq 3$, and no collinear subschemes of larger degree, then it is shown in [1] that this forces C to

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admit a 2-dimensional irreducible family of r-secant lines. This is the starting point for this paper. Our main result implies:

Theorem 1.1. *Let $C \subset \mathbb{P}^n$, $n \geq 3$, be a non-degenerate curve admitting an irreducible 2-dimensional family of r-secant lines of C such that $r \geq 3$ and every general hyperplane contains a unique r-secant of the family. Then C contains a planar subcurve of degree r, unless it contains one multiple line or a pair of skew multiple lines and the general r-secant of the family meets these lines but no other components of C.*

A more detailed description of the exceptional curves is given in Theorem 3.1 and Proposition 3.6. In particular, it turns out that exceptional curves are non-reduced. We also give examples that all the exceptional cases do really exist. However, if r is large with respect to d, we can exclude the occurrence of the exceptional curves. We get the following lifting result:

Theorem 1.2. *Let $C \subset \mathbb{P}^n$ be a non-degenerate curve of degree d. Suppose that* $\operatorname{char} k = 0$ *and that the general hyperplane section of C contains a subscheme of degree $r \geq \frac{d+3}{2}$ spanning a line, and contains no collinear subschemes of larger degree. Then C contains a unique planar subcurve of degree r.*

In the paper [1] we showed that the set of non-degenerate curves in $\mathbb{P}^n$ of degree d and genus g with non-degenerate general hyperplane section contains curves whose components of the Hartshorne-Rao module have maximal dimension in every degree among the curves of the set. We call these curves extremal curves. As a consequence of the lifting result above we show that an extremal curve $C \subset \mathbb{P}^n$ contains a planar subcurve of degree $d - n + 2$.

The paper is organized as follows. In section 2 we introduce the concept of generic embedding dimension of an irreducible subscheme. It turns out that it is the minimal embedding dimension at a closed point of the given subscheme. Furthermore, the generic embedding dimension of a non-reduced curve is at least two. As preparation for the subsequent sections, we study multiple lines of general embedding dimension two.

Section 3 is devoted to the proof of our main result, Theorem 3.1. Moreover, we describe examples of exceptional curves and show a stronger result provided that $\operatorname{char} k = 0$.

In the final section we give some applications. Here we prove Theorem 1.2. We also prove a lemma which allows to apply Theorem 1.2 by looking at the Hilbert function of the general hyperplane section of C.

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2. Curves with small generic embedding dimension

We work over an algebraically closed field k of arbitrary characteristic, and we denote by $\mathbb{P}^n$ the n-dimensional projective space over k.

By a *curve* we mean a pure one-dimensional closed subscheme $C \subseteq \mathbb{P}^n$ (in particular C is locally Cohen-Macaulay).

If $C \subseteq \mathbb{P}^n$ is a curve and $\ell \subseteq \mathbb{P}^n$ is a line, we say that ℓ is an *r-secant* of C if the scheme-theoretic intersection $C \cap \ell$ is a zero-dimensional scheme of degree r. A *secant line* is an r-secant line, where r is some integer ≥ 2.

If $Z \subseteq \mathbb{P}^n$ is any subscheme we denote by $\langle Z \rangle$ the span of Z, namely the least linear space containing Z as a subscheme.

We shall almost always consider only closed points of the schemes which occur. So we write "point" instead of "closed point", and we say explicitly when we consider a possibly non-closed point.

We give now some definitions and basic results.

Definition 2.1. (i) If $(A, \mathfrak{m}, K)$ is a local ring, the *embedding dimension of A* is the integer $\operatorname{emdim}(A) := \dim_K(\mathfrak{m}/\mathfrak{m}^2)$, i.e. the least number of generators of $\mathfrak{m}$.

(ii) If X is a scheme and x is a point of X, the *embedding dimension of X at x* is the integer $\operatorname{emdim}_x(X) := \operatorname{emdim}(\mathcal{O}_{X,x})$. So $\operatorname{emdim}_x(X)$ is the dimension of the Zariski tangent space of X at x.

(iii) If X is an irreducible (not necessarily reduced) scheme with generic point η, the *generic embedding dimension of X* is the integer $\operatorname{ged}(X) := \operatorname{emdim}_\eta(X) + \dim(X)$.

The following Lemma gives a motivation for the definition of the generic embedding dimension given above.

Lemma 2.2. *Let X be an irreducible algebraic k-scheme. Then:*

(i) *there is a non-empty open set $U \subseteq X$ such that* $\operatorname{ged}(X) = \operatorname{emdim}_x(X)$ *for all closed points $x \in U$;*

(ii) $\operatorname{ged}(X) = \min\{\operatorname{emdim}_x(X),\ x \in X,\ x \text{ closed}\}$.

Proof. If $(A, \mathfrak{m}, K)$ is a local k-algebra such that K/k is separably generated, there is an exact sequence of K-vector spaces

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

(see [4], p. 187, ex. 8.1 (a)). Moreover $\dim_K(\Omega_{K/k}) = \operatorname{t.d.}(K/k)$ (see [4], p.174, th. 8.6 A), whence

$$(*) \qquad \operatorname{emdim}(A) = \dim_K(\Omega_{A/k} \otimes_A K) - \operatorname{t.d.}(K/k)$$

Consider now the $\mathcal{O}_X$-module $\Omega_{X/k}$. Since X is of finite type over k, $\Omega_{X/k}$ is coherent, whence there is a non-empty open set $U \subseteq X$ such that $\dim_{\kappa(y)}(\Omega_{X/k} \otimes_{\mathcal{O}_{X,y}} \kappa(y))$ has a constant value a for all (not necessarily closed) $y \in U$. If $y = \eta$ is the generic point of X, by $(*)$ we have: $\operatorname{emdim}_\eta(X) = \operatorname{emdim}\mathcal{O}_{X,\eta} = a - \operatorname{t.d.}(\kappa(\eta)/k) = a - \dim(X)$. If $y = x \in U$ is a closed point, by $(*)$ we have $\operatorname{emdim}_x(X) = a$ and (i) follows.

Moreover, since $\Omega_{X/k}$ is coherent, the function $\dim_{\kappa(x)}(\Omega_{X/k} \otimes_{\mathcal{O}_{X,x}} \kappa(x)), x \in X$, is upper-semicontinuous, whence (ii). $\square$

Remark 2.3. (i) Let C be an irreducible curve, L a general hyperplane and $x \in L \cap C$: then $\operatorname{emdim}_x(C) = 1 + \operatorname{emdim}_x(L \cap C)$.

(ii) Let C be a non-reduced irreducible curve, L a hyperplane not containing C_{red} and suppose that there exists a point $x \in L \cap C$ such that $\operatorname{emdim}_x(L \cap C) = 1$: then $\operatorname{ged}(C) = 2$. Indeed we have $\operatorname{emdim}_x(C) = 2$ and $\operatorname{ged}(C) \geq 2$, being C non-reduced, whence the conclusion follows by Lemma 2.2 (ii).

Definition 2.4. Let C be an irreducible curve and $x \in C$ a general point. The *multiplicity* of C, denoted by $e(C)$, is the multiplicity $e_x(C)$ of C at x (i.e. the multiplicity of the local ring $\mathcal{O}_{C,x}$).

Remark 2.5. Let C be an irreducible curve. Then

(i) by general facts on multiplicities it follows that if L is a general hyperplane and $x \in L \cap C$ is a general point we have $e_x(C) = \lambda(\mathcal{O}_{L\cap C,x})$, where λ denotes length, hence $\deg(C) = e(C)\deg(C_{red})$;

(ii) moreover we have $e(C) = \lambda(\mathcal{O}_{C,\eta})$, where η is the generic point of C (see [7], Theorem (23.5)).

We need the following Lemma, whose easy proof is omitted.

Lemma 2.6. *Let $Z \subseteq \mathbb{P}^n$ be a zero-dimensional scheme and let Z_1 and Z_2 be two closed subschemes of Z. Then:*

$$\deg(Z_1 \cap Z_2) \geq \deg(Z_1) + \deg(Z_2) - \deg(Z)$$

and equality holds if and only if $Z = Z_1 \cup Z_2$.

The following Lemmas give some properties of multiple lines with generic embedding dimension 2, which will be useful later on.

Lemma 2.7. *Let C be a multiple line with* $\operatorname{ged}(C) = 2$. *Then:*

(i) *for each integer m with $1 \leq m \leq \deg(C)$, there is a unique subcurve $C_m \subseteq C$ of degree m;*

(ii) *$C_m \subseteq C_{m'}$ if and only if $m \leq m'$;*

(iii) *let L be a hyperplane not containing C_{red} with the following property: if $x := L\cap C_{red}$, then* $\operatorname{emdim}_x(C) = 2$. *Then $C_m \subseteq C_{m'}$ if and only if $L \cap C_m \subseteq L \cap C_{m'}$;*

(iv) *if L is as in (iii), then the map $C_m \mapsto L \cap C_m$ is a bijection between the set of subcurves of C and the set of subschemes of $L \cap C$.*

Proof. If η is the generic point of C and $A := \mathcal{O}_{C,\eta}$, then A is an artinian local ring with $\operatorname{emdim}(A) = 1$ and $\lambda(A) = \deg(C)$ (see 2.5). Therefore, the ideals of A are totally ordered by inclusion and they correspond bijectively to the subcurves of C, whence (i) and (ii). Moreover $L \cap C$ is supported at one point and has embedding dimension 1, hence its subschemes are totally ordered by their degrees, whence (iii) and (iv) follow easily. □

Lemma 2.8. *Let C be a multiple line. Then* $\operatorname{ged}(C) = 2$ *if and only if any general hyperplane contains exactly one line ℓ which is secant to C.*

Proof. Let L be a general hyperplane and let $x := L\cap C_{red}$. We may assume that x is the origin in an affine chart with coordinates $x_1, \ldots, x_n$, and that L is the hyperplane $x_n = 0$. Set $A := \mathcal{O}_{\mathbb{P}^n,x}$, $\mathfrak{m} := \mathfrak{m}_x = (x_1, \ldots, x_n)A$, $\Gamma := L\cap C$ and $B := \mathcal{O}_{\Gamma,x} = A/\mathfrak{a}$ (for a suitable ideal $\mathfrak{a}$). We may assume that the maximal ideal $\overline{\mathfrak{m}} := \mathfrak{m}/\mathfrak{a}$ of B is minimally generated by the images $\overline{x}_1, \ldots, \overline{x}_s$ of $x_1, \ldots, x_s$. Put $B' := B/\overline{\mathfrak{m}}^2 = A/\mathfrak{a} + \mathfrak{m}^2$. By Nakayama's Lemma it follows that the maximal ideal $\mathfrak{m}B'$ of B' is minimally generated by the images of $x_1, \ldots, x_s$, whence $\operatorname{emdim} B' = \operatorname{emdim} B = s$. We may also assume, after a linear change of coordinates, that $x_{s+1}, \ldots, x_n \in \mathfrak{a} + \mathfrak{m}^2$, i.e. that $(x_{s+1}, \ldots, x_n)B' = (0)$.

Assume now that $\operatorname{ged} C = 2$. Since L is general it follows by Lemma 2.2 and Remark 2.3 that $s = 1$. Consider the line ℓ having equations $x_2 = x_3 = \cdots = x_n = 0$.

We have $\ell \cap C = \ell \cap \Gamma$, whence $\mathcal{O}_{\ell\cap C,x} = B/(x_2,\ldots,x_n)B$. It follows $\deg(\ell \cap C) \geq \lambda(B'/(x_2,\ldots,x_n)B') = \lambda(B') > 1$, whence ℓ is a secant line to C.

Observe now that, being $s = 1$, the ideals of B are totally ordered by inclusion, whence the subschemes of Γ are totally ordered by inclusion.

Thus, if ℓ' is a secant line such that $x \in \ell' \subseteq L$, we have either $\ell' \cap \Gamma \subseteq \ell \cap \Gamma$, or $\ell \cap \Gamma \subseteq \ell' \cap \Gamma$.

Assume that $\ell' \cap \Gamma \subseteq \ell \cap \Gamma$. Then ℓ and ℓ' contain $\ell' \cap \Gamma$ and since $\deg(\ell' \cap \Gamma) \geq 2$ we have $\ell = \ell'$. We get the same conclusion if $\ell \cap \Gamma \subseteq \ell' \cap \Gamma$.

Therefore if $s = 1$ L contains a unique secant line, namely ℓ.

Assume now that $\operatorname{ged} C \neq 1$. Then applying again 2.2 and 2.3 we have $s \neq 1$.

If $s = 0$, then C is a reduced line and has no secants.

Assume $s \geq 2$. Let ℓ_1, ℓ_2 be the lines having equations $x_1 = x_3 = \cdots = x_n = 0, x_2 = x_3 = \cdots = x_n = 0$, respectively, and let $\mathfrak{a}_i \subseteq B$ be the corresponding ideals.

Since $\mathfrak{a}_i B' \supseteq x_i B' \neq (0)$ we have $\lambda(B/\mathfrak{a}_i) \geq \lambda(B'/\mathfrak{a}_i B') > 1$ for $i = 1, 2$ and this means that ℓ_1, ℓ_2 are two different secant lines with $x \in \ell_i \subseteq L$. □

The following Theorem is a straightforward generalization to $\mathbb{P}^n$ of Hartshorne's Restriction Theorem (see [5]); it gives a criterion for a curve to be planar, in terms of the "degeneracy" of its general hyperplane section.

Theorem 2.9. *Let $C \subset \mathbb{P}^n$ be a curve with collinear general hyperplane section and* $\deg(C) \geq 3$. *Assume that either* $\deg(C_{red}) \geq 2$, *or char$(k) = 0$. Then C is planar.*

Proof. After replacing $\mathbb{P}^n$ by $\langle C \rangle$ we may assume that C is non-degenerate, and we have to show that $n \leq 2$. By Hartshorne's Restriction Theorem [5] we cannot have $n = 3$. We assume then that $n > 3$ and we show by induction that C is degenerate, getting a contradiction. Let $P \in \mathbb{P}^n$ be a general point and let $C' \subseteq \mathbb{P}^{n-1}$ be the curve obtained by projecting C from P. Since $\operatorname{ged}(C) = 2$ (see 2.8), we have $\deg(C') = \deg(C)$, thus the general hyperplane section of C' is collinear. Moreover $\deg(C_{red}) = \deg(C'_{red})$, whence C' is degenerate by induction. If $L \subseteq \mathbb{P}^{n-1}$ is the hyperplane containing C', then $\langle P, L \rangle$ is a hyperplane of $\mathbb{P}^n$ containing C. □

3. Main Results

In this section we deal with curves having an irreducible 2-dimensional family of r-secants ($r \geq 3$) such that every general hyperplane contains exactly one such secant. In Theorem 3.1 we completely describe such curves; in particular we show that they must contain a planar subcurve, unless they contain lines. A description of the exceptional situations is given.

To complement the result of Theorem 3.1, in Proposition 3.6 we study the case of multiple lines having generic embedding dimension 2, in characteristic zero.

Theorem 3.1. *Let $C \subset \mathbb{P}^n, n \geq 3$, be a non-degenerate curve and let $X \to T$ be an irreducible 2-dimensional family of r-secant lines of C ($r \geq 3$). Assume that for every general hyperplane $L \subseteq \mathbb{P}^n$ there exists a unique $t \in T$ such that $X_t \subseteq L$.*

Then one and only one of the following conditions holds:

(i) *C contains a planar subcurve E of degree r, and $X_t \subseteq \langle E \rangle$ for every $t \in T$;*
(ii) *C_{red} contains two skew lines ℓ_1 and ℓ_2 such that for every general $t \in T$ the line X_t meets ℓ_1 and ℓ_2 and no other components of C_{red};*

(iii) *C contains a non-planar double line C' such that for any general $t \in T$ the line X_t is a secant line of C' and meets no other components of C;*
(iv) *C is not as in (iii), but it contains a multiple line C'' with $\deg(C'') \geq 3$ and $\operatorname{ged}(C'') \geq 3$, such that for any general $t \in T$ the line X_t is a secant line of C'' and meets no other components of C. Moreover C'' is contained in the first neighborhood of C''_{red}.*

For reduced curves we have the following stronger result which has been proved for $n = 3$ in [14]:

Corollary 3.2. *Let C be a curve as in Theorem 3.1 and suppose furthermore that C is reduced. Then C contains a planar subcurve of degree r.*

Proof. Since C is reduced cases (ii) and (iii) of Theorem 3.1 cannot occur. Case (ii) cannot occur either because $r \geq 3$. □

Now we prove our main result.

Proof. The proof proceeds in several steps that include some Lemmas. The starting idea (borrowed from [14]) is to construct a certain subcurve $D \subseteq C$ which encodes all the relevant information concerning the behavior of the family $X \to T$ with respect to C, and then to study D_{red}.

Step 1. Let G be the Grassmannian of lines in $\mathbb{P}^n$. By the universal property of G the obvious map $\rho : T \to G$ is a morphism and it is not restrictive to assume that ρ is a locally closed embedding. Then we may assume that T is a locally closed subscheme of G.

Step 2. Now let $Hilb^2_{\mathbb{P}^n}$ be the Hilbert scheme parameterizing the zero-dimensional subschemes of degree 2. The following Lemma seems to be obvious, but we feel that it needs a proof.

Lemma 3.3. *Let $Z := Hilb^2_{\mathbb{P}^n}$ and let G be the Grassmannian of lines in $\mathbb{P}^n$. Then there is a morphism $\phi : Z \to G$ such that, for every $z \in Z$, $\phi(z)$ is the line spanned by the closed subscheme corresponding to z.*

Proof. Let $\mathcal{Z} \subseteq Z \times \mathbb{P}^n$ and $\mathcal{G} \subseteq G \times \mathbb{P}^n$ be the tautological families and let $\alpha : \mathcal{Z} \to Z$ and $\beta : \mathcal{G} \to G$ be the projections. We have canonically a diagram

$$\begin{array}{ccc} \mathcal{Z} \times_{\mathbb{P}^n} \mathcal{G} & \subset & (Z \times G) \times \mathbb{P}^n \\ {\scriptstyle\tau}\downarrow & & \\ Z \times G & & \end{array}$$

where $\tau := \alpha \times_{\mathbb{P}^n} \beta$ and the projection of $\tau^{-1}(z, g)$ in $\mathbb{P}^n$ is the intersection between the line corresponding to g and the subscheme of $\mathbb{P}^n$ corresponding to z. Moreover τ is proper and its fibers are either empty, or have dimension 0 and degree either 1 or 2. Therefore the set

$$S := \{(z, g) \mid \deg \tau^{-1}(z, g) = 2\}$$

is closed in $Z \times G$ and $\mathcal{S} := \tau^{-1}(S)$ is closed in $\mathcal{Z} \times \mathcal{G}$. Hence we have the commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \subset & \mathcal{Z} \times_{\mathbb{P}^n} \mathcal{G} \\ {\scriptstyle\tau_{|\mathcal{S}}}\downarrow & & \downarrow{\scriptstyle\tau} \\ S & \subset & Z \times G \end{array}$$

Consider in $\mathcal{S}$ and S the reduced scheme structures induced by $\mathcal{Z} \times \mathcal{G}$ and $Z \times G$ respectively. The projection $\pi : S \to Z$ is injective and has reduced fibers, whence it is an isomorphism. It follows that S is the graph of the morphism $\phi : Z \to G$ defined by the composition $Z \to \pi^{-1}(Z) \to G$. □

Step 3. First of all we need the following Lemma, whose easy proof is omitted.

Lemma 3.4. *Let Γ be a 0-dimensional subscheme of a smooth curve C and let $0 < e \leq \deg \Gamma$. Then Γ contains only a finite number of subschemes of degree e.*

Now we develop some preliminaries which will be used in the construction of the subcurve D.

Let $H := Hilb^2_C$ be the relative Hilbert scheme of the 0-dimensional closed subschemes of degree 2 of C (see e.g. [11]). H has the natural structure of a closed subscheme of $Hilb^2_{\mathbb{P}^n}$ and we denote again by ϕ the restriction to H of the morphism $\phi : Hilb^2_{\mathbb{P}^n} \to G$ (see Lemma 3.3). We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H} & \subset & C \times H \\ {\scriptstyle p}\downarrow & \searrow^{q} & \\ H & & C \end{array}$$

where $\mathcal{H}$ is the tautological family, p and q are the canonical projections. If $h \in H$, then $q(p^{-1}(h))$ is a 0-dimensional closed subscheme of C of degree 2. Set $K := \phi^{-1}(T)$ and $\mathcal{K} := p^{-1}(K) \subset K \times C$. Observe that if $k \in K$ and $t := \phi(k) \in T$ then $q(p^{-1}(k))$ is a 0-dimensional subscheme of degree 2 of C which is contained in the line X_t.

By generic flatness we may assume, after replacing T with a suitable non-empty open subscheme of T, that the composite morphism $\phi \circ p : \mathcal{K} \to T$ is flat.

Claim. *Let the notation be as above. Then:*

(i) *the morphism $\phi \circ p$ has zero-dimensional fibers;*
(ii) *every irreducible component of $\mathcal{K}$ has dimension 2;*
(iii) *every irreducible component of $\mathcal{K}_{red}$ dominates an irreducible component of C_{red}.*

Proof. Let's remark first of all that the morphism $\phi : K \to T$ has 0-dimensional fibers by Lemma 3.4. Moreover it is clear that the fibers of p are zero-dimensional, and (i) follows. (ii) follows easily from (i) and the flatness of $\phi \circ p$. If (iii) is false, then there is an irreducible component of $\mathcal{K}_{red}$ mapping to a point of C and it is easy to see that this contradicts the assumptions on the uniqueness of the secant line in a general hyperplane.

Step 4. Let's denote again by q the restriction to $\mathcal{K}$ of the morphism $q : \mathcal{H} \to C$ considered in Step 3 and let $D' := q(\mathcal{K})$. By construction of K we have that $\dim(D') = 1$ and by the Claim in Step 3 it follows that $\dim q^{-1}(x) = 1$ for all points $x \in D'$.

Now we take off the zero-dimensional components (if any) of the scheme-theoretical closure of D' and denote by D the resulting curve.

It is easy to see that D has the following properties:

(i) D is a closed subcurve of C and D' is a dense open subscheme of D;
(ii) for every $x \in D'$ the set $\gamma_x := \{t \in T \mid x \in X_t\}$ is a locally closed 1-dimensional subset of T and of G (that is: the lines X_t passing through x form a 1-dimensional family);
(iii) D is the smallest subcurve of C containing all the subschemes of degree 2 of $X_t \cap C$, for $t \in T$;
(iv) for general $t \in T$, the line X_t meets all the irreducible components of D_{red} and no other components of C_{red};

(v) D is contained in the first neighborhood of D_{red}.

Step 5. In this step we prove the following:

Claim. *If L is a general hyperplane, then the points of $L \cap D_{red}$ lie on the unique line X_t contained in L.*

Proof. For x general in D, let $V_x := q(\phi^{-1}(\gamma_x))$ (V_x is the locally closed cone with vertex x, spanned by the lines X_t passing through x). Put $\Gamma_x := \overline{V_x} \setminus V_x$. If $\overline{\gamma_x}$ is the closure of γ_x in G we have $\overline{V_x} = q(\phi^{-1}(\overline{\gamma_x}))$ whence Γ_x consists of a finite number of lines. When x varies in D, these lines fill up a finite number of 1-dimensional families and it follows that the hyperplanes containing any of these lines depend on $1 + n - 2 = n - 1$ parameters; hence a general hyperplane L does not contain any of them. Now put $L \cap D_{red} = \{P_1, \dots, P_u\}$. We have that $L \cap V_{P_i}$ is not just P_i; indeed $L \cap \overline{V_{P_i}}$ is a curve C_{P_i} which is not contained in Γ_{P_i} because L contains no lines of Γ_{P_i}, whence $L \cap V_{P_i}$ is also a curve. It follows that L contains a line X_{t_i} through P_i ($i = 1, \dots, u$); but $X_{t_1} = X_{t_2} = \cdots = X_{t_u} := X_t$, whence $L \cap D_{red}$ is contained in the line X_t. $\square$

Step 6. As an immediate consequence of Step 5 and Theorem 2.9 we get: if $\deg(D_{red}) \geq 3$, then D_{red} is planar.

Step 7. We prove the Theorem when $\deg(D_{red}) \geq 2$.
Let's first show the following:

Lemma 3.5. *Assume that C contains a planar subcurve C' of degree ≥ 2. If every line X_t is a secant line to C', then C contains a planar curve E of degree r and $X_t \subseteq \langle E \rangle$ for every $t \in T$.*

Proof. Let $E := \pi \cap C$ (without embedded points), where $\pi = \langle C' \rangle$. If L is a general hyperplane and X_t is the line of the family contained in L, we have $\deg(X_t \cap C') \geq 2$, whence $X_t \subseteq \pi$. It follows that $L \cap E = X_t$, whence $X_t \cap E = X_t \cap \pi \cap C = X_t \cap C$. Hence $L \cap \pi = X_t \cap E = X_t \cap \pi \cap C = X_t \cap C$. Thus $\deg(E) = r$, and E is planar by Theorem 2.9 . $\square$

From the previous Lemma we get the proof of the Theorem when D_{red} contains a planar curve of degree ≥ 2 (e.g. $\deg(D_{red}) \geq 3$, cf. Step 6).

Suppose now that $\deg(D_{red}) = 2$ and D_{red} is not planar.

In this case D_{red} is the union of two skew lines ℓ_1, ℓ_2 and for any general $t \in T$ the line X_t meets both ℓ_1 and ℓ_2, by the construction of D. Moreover if F is an irreducible component of C_{red} different from ℓ_1 and ℓ_2 we have $X_t \cap F = \emptyset$, for otherwise $F \subseteq D_{red}$ by the construction of D. This shows that we are in case (ii) of the Theorem.

Step 8. Finally we prove the Theorem when $\deg(D_{red}) = 1$.

If $\deg(D) = 2$ and D is planar we are in case (i) (see Lemma 3.5).

If D is non-planar we are in case (iii).

Let now $\deg(D) \geq 3$; assume that $\operatorname{ged}(D) = 2$ and let $\tilde{D} \subseteq D$ be the unique subcurve of degree 2 (see Lemma 2.7).

If L is a general hyperplane, by Lemma 2.8 there is only one line $\ell \subseteq L$ which is a secant line to $\tilde{D}$; this line is also a secant line to D, whence, being $\operatorname{ged}(D) = 2$, we must have $\ell = X_t$ for a unique $t \in T$. Then, as above, we are either in case (i) or in case (ii).

So the remaining case is $\deg(D) \geq 3$ with $\operatorname{ged}(D) \geq 3$ and we are in case (iv), with $C'' := D$.

This concludes the proof of the Theorem. $\square$

If $\mathrm{char}(k) = 0$ the following Proposition shows that case (iii) of Theorem 3.1 cannot occur.

This Proposition is proved in two ways, both needing the characteristic zero assumption. The first proof is based on Hartshorne's Restriction Theorem as stated in Theorem 2.9. The second one, based on Bertini's theorem, is independent of Hartshorne's Restriction Theorem and implies such theorem for the case of multiple lines. We feel that this proof can throw some extra light on the exceptions to it (the so-called Hartshorne's lines in positive characteristic).

Proposition 3.6. *Let the assumptions be as in Theorem 3.1, and assume further that* $\mathrm{char}(k) = 0$ *and that* C *is a multiple line with* $\mathrm{ged}(C) = 2$. *Then* C *contains a planar subcurve* E *of degree* r *and* $X_t \subseteq \langle E\rangle$ *for every* $t \in T$.

Proof. 1st Version. Let C' be the unique subcurve of C having degree r (see Lemma 2.7). Let L be a general hyperplane and let $\Gamma := L \cap C$. Since $\mathrm{emdim}\,\Gamma = 1$ there is a unique subscheme of Γ of degree r. Now, if r_L denotes the unique secant line in L we have: $r_L \cap C$ and $L \cap C'$ are subschemes of Γ having degree r and hence they coincide (see Lemma 2.7). This implies that the general hyperplane section of C' is collinear, and since $\mathrm{char}(k) = 0$ and $\deg(C') = r \geq 3$ we have that C' is planar (Theorem 2.9).

2nd Version. First of all observe that since $\mathrm{ged}(C) = 2$, there exists a surface $S \supseteq C$ such that S is generically smooth along C (see e.g. [3], Proposition 3.7 with $s = n - 2$). The proof will follow from the:

Claim. *Let* $Q \in C \cap S_{reg}$ *and let* π_Q *be the tangent plane to* S *at* Q. *Then* π_Q *is constant when* Q *varies in* $C \cap S_{reg}$.

Indeed, taking the claim for granted, let π be the tangent plane to S along C and set $E := \pi \cap C$ (with zero-dimensional components removed). As in the proof of Lemma 3.5 we see that $\deg(E) = r$ and E is the curve we were looking for.

Proof of the Claim. We argue by contradiction, assuming that π_Q varies as Q varies in $C \cap S_{reg}$.

First of all notice that there cannot be only a finite number of distinct such planes. Indeed they must contain the 2-dimensional family of r-secants lines and these lines, having intersection multiplicity at least 2 with S along C, are tangent to S, that is they are contained in a pencil in each π_Q. Hence, if the planes π_Q vary with Q, they must be ∞^1. Now let's denote by Φ the linear system of hyperplanes through C_{red}: every hyperplane of Φ contains at least one π_Q, hence it is tangent to S. Moreover for every $Q \in C_{red} \cap S_{reg}$ the general hyperplane of Φ does not contain π_Q, hence it is not tangent to S at Q. Let's denote by Σ' the linear system of curves cut out on S by Φ, and by Σ the linear system obtained from Σ' by removing C_{red}. We observe that a point $Q \in S_{reg} \cap C$ is singular for the curve $\sigma' = M \cap S$ where $M \in \Phi$ if and only if $\pi_Q \subset M$. Hence a general $Q \in S_{reg} \cap C$ is singular for some $\sigma' \in \Sigma$. Now let $P \in C_{red}$ be a general point; since the general hyperplane $M \in \Phi$ is not tangent to S at P, then the corresponding curve in Σ' is smooth at P, whence the residual curve in Σ does not contain P. It follows that P is not a base point for Σ, hence the general curve $\sigma \in \Sigma$ is smooth at all points except at most at $C \cap S_{sing}$ by Bertini's theorem. It follows that $\sigma' := \sigma \cup C_{red} = M \cap S$ has multiplicity exactly 2 at the points $Q \in C \cap S_{reg}$ where L is tangent to S. But this is impossible, because if M is a general hyperplane and $P := M \cap D_{red}$ we have $L \cap \sigma \cap S \supseteq r_L \cap C$, whence $\deg(L \cap \sigma \cap S) \geq r \geq 3$, a contradiction. Hence π_Q is constant, as claimed. □

Remark 3.7. Let C be a curve as in Theorem 3.1 which does not contain a planar subcurve. Then the "secancy" of the family of lines is necessarily either on two skew lines contained in C or in a multiple line contained in C.

In the last case when $\mathrm{char}(k) = 0$ the multiple line has generic embedding dimension at least 3.

We end up this section with some remarks and examples, to illustrate the role of our assumptions in the previous statements and to show that all cases described in Theorem 3.1 can really occur.

Remark 3.8. The assumption on the characteristic is necessary in Proposition 3.6. Indeed Hartshorne has shown in [5] that in positive characteristic there are non-degenerate multiple lines in $\mathbb{P}^3$ of any given degree having degenerate general hyperplane section.

Remark 3.9. The assumption on the uniqueness of the r-secant line X_t in every general hyperplane L in Theorem 3.1 cannot be dropped. For example let $D \subseteq \mathbb{P}^n$ $(n \geq 3)$ be a non-degenerate smooth connected curve with ideal sheaf $\mathcal{I}$, and let C be the curve corresponding to the ideal $\mathcal{I}^2$. The 2-secant lines of D form a 2-dimensional family of 4-secant lines of C, but C cannot contain any planar subcurve, nor any curve as in (ii),(iii) and (iv) in Theorem 3.1. On the other hand a general hyperplane L contains $d := deg(D)$ distinct points of D, no three on a line, whence the above mentioned assumption is not satisfied, because every line joining two such points is a 4-secant to C and lies in L.

Example 3.10. We show that cases (ii), (iii) and (iv) in Theorem 3.1 can actually occur.

1) Let (a, b, c, d) be homogeneous coordinates in $\mathbb{P}^3$ and let C be the curve corresponding to the homogeneous ideal $I := (a, b) \cap (c, d)^2$. Then $\deg(C) = 4$ and C_{red} is the union of the two skew lines ℓ_1 and ℓ_2 corresponding respectively to the homogeneous ideals (a, b) and (c, d). It is easy to see that every line spanned by a point in ℓ_1 and a point in ℓ_2 is a trisecant line for C. Then C admits a 2-dimensional irreducible family of 3-secant lines, but obviously it does not contain any plane curve of degree 3. The assumptions of 3.1 being verified, we are in fact in case (ii) of 3.1.

2) Let C' be a non-planar double line. Then by Lemma 2.8 and [1] (proof of Step 3, p. 703), the secant lines of C' form an irreducible 2-dimensional family. It is clear that there are many curves C containing C', with $C_{red} = C'_{red}$, such that every secant line to C' is an r-secant line to C for $r \geq 3$. The easiest examples are the locally Cohen-Macaulay curves corresponding to powers of the ideal sheaf $\mathcal{I}_{C'}$ (by removing the embedded points). These curves are examples for case (iii).

3) Let $\mathrm{char}(k) = 0$. Consider in $\mathbb{P}^n$ two skew lines ℓ, ℓ' and let C be the second neighborhood of ℓ (i.e. $I_C := I_\ell^3$). Let $X \to T$ be the family of lines joining ℓ and ℓ'. Then X_t is a 3-secant line of C. On the other hand if C' is a subcurve of C of degree 2, the general line X_t is not a secant line of C', for otherwise the curve $C' \cup \ell'$ had degree 3 and collinear general hyperplane section, whence it were a plane curve by Theorem 2.9, a contradiction. Then we are in case (iv) of Theorem 3.1. Notice that the lines X_t are secant lines to the curve $\tilde{C} = \ell \cup \ell'$ of degree 2, and $\tilde{C} \not\subseteq C$ (hence we are not in case (ii) of Theorem 3.1).

The following Remark describes the family $\{X_t\}$ in cases (iii) and (iv) of Theorem 3.1.

Remark 3.11. Let C be a curve satisfying (iii) (resp.(iv)) of Theorem 3.1, and let $\ell := C'_{red}$ (resp. $\ell := C''_{red}$).

Then there are a surface scroll $\pi : S \to B$, where B is an open set of $\mathbb{P}^1$, and a rational map $\phi : \ell \to B$ with the following property: every line X_t is the span of a point $P \in \ell$ and a point $Q \in \pi^{-1}(\phi(P))$.

Indeed the lines of the family $\{X_t\}$ passing through a fixed general point $P \in \ell$ span a plane π_P which moves in a 1-dimensional family (for otherwise we are in case (i)).

The planes π_P are the fibers of a (not necessarily complete) 3-dimensional rational scroll Y. Let $S := H \cap Y$, where H is a general hyperplane. Then S is a surface scroll as required (we leave the details to the reader).

4. Applications

In this section we give criteria which ensure that a curve C contains a planar subcurve.

The general idea is that if C has a 2-dimensional family of r-secants and r is sufficiently large, then (ii), (iii) and (iv) of Theorem 3.1 cannot occur.

We begin with two technical lemmas on multiple lines, which will be needed later on.

Lemma 4.1. *Let C be a multiple line with* $\mathrm{ged}(C) = 2$. *Let $\ell \neq C_{red}$ be a line meeting C at a point x such that* $\mathrm{emdim}_x(C) = 2$ *and put* $q := \deg(\ell \cap C)$. *If $C' \subseteq C$ is a subcurve of degree d', then* $\deg(\ell \cap C') = \min\{q, d'\}$.

Proof. Let L be a general hyperplane through ℓ and let $E \subseteq C$ be the subcurve corresponding to $C' \cap L$ (see Lemma 2.7): we have $\ell \cap C' \subseteq \ell \cap C = \ell \cap E = L \cap E$. If $d' \leq q$ we have $C' \subseteq E$ by Lemma 2.7, hence $\ell \cap C' = \ell \cap E \cap C' = L \cap E \cap C' = L \cap C'$ whence $\deg \ell \cap C' = d'$. If $d' > q$ then $E \subseteq C'$. But $\ell \cap C' \subseteq \ell \cap C = \ell \cap E = L \cap E$ and $\ell \cap C \supseteq \ell \cap C' \supseteq \ell \cap E = \ell \cap C$. It follows $\ell \cap C = \ell \cap C'$ and $\deg \ell \cap C' = q$. □

Lemma 4.2. *Let $C \subseteq \mathbb{P}^n$ be a multiple line of degree d, and let $X \to T$ be a family of r-secant lines with the property that for every general hyperplane $L \subseteq \mathbb{P}^n$ there exists a unique $t \in T$ such that $X_t \subseteq L$. Let q be a positive integer and assume that $2r \geq d + q$. Then C contains a subcurve C' of degree q with* $\mathrm{ged}(C') = 2$, *such that* $\deg(X_t \cap C') \geq \min\{r, q\}$ *for general $t \in T$.*

Proof. For a general hyperplane L let r_L be the unique line in the family $X \to T$ contained in L. Let Φ be a general pencil of hyperplanes and let $C' \subseteq C$ be the unique subcurve such that $r_L \cap C = L \cap C'$ for L general in ϕ (see [1], [12]). Then $\deg(C') = r$ and $\mathrm{ged}(C') = 2$, because C' has a hyperplane section with embedding dimension 1. Now let L be a general hyperplane. Since $2r \geq d + q$, by Lemma 2.6 we have

$$\begin{aligned} \deg(r_L \cap C') &= \deg[(r_L \cap C) \cap (L \cap C')] \\ &\geq \deg(r_L \cap C) + \deg(L \cap C') - \deg(L \cap C) \\ &= r + r - d \geq q \end{aligned}$$

The conclusion follows by Lemma 4.1 applied to C', with $\ell = r_L$. □

Proposition 4.3. *Assume $char(k) = 0$. Let $C \subseteq \mathbb{P}^n$ be a curve of degree $d \geq 3$, and let $X \to T$ be an irreducible 2-dimensional family of r-secant lines with $r \geq \frac{d+3}{2}$. Then C contains a unique planar subcurve E of degree r and $X_t \subseteq \langle E \rangle$ for every $t \in T$.*

Proof. We show first that a general hyperplane $L \subseteq \mathbb{P}^n$ contains at most one line of the family. Indeed let ℓ_1 and ℓ_2 be two lines in the family contained in L. Put $\Gamma_i := \ell_i \cap C \subseteq L \cap C$ for $i = 1, 2$. By Lemma 2.6 we have

$$\deg(\Gamma_1 \cap \Gamma_2) \geq \deg(\Gamma_1) + \deg(\Gamma_2) - \deg(L \cap C) = 2r - d \geq 3$$

It follows that $\ell_1 \cap \ell_2$ contains a subscheme of length ≥ 3, hence $\ell_1 = \ell_2$.

Let now $R \subseteq T \times (\mathbb{P}^n)^*$ be the reduced closed subscheme whose closed points are $\{(t, L) \mid X_t \subseteq L\}$, and let $p : R \to T$ and $q : R \to (\mathbb{P}^n)^*$ be the projections. Since the fibers of p have dimension $n-2$ we have $\dim R = n$, hence q is dominant because its fibers have non-positive dimension, as we have seen above. Since $q(R)$ is constructible by Chevalley's Theorem, it contains a non-empty open set. Then a general hyperplane contains exactly one of the lines X_t, whence it is sufficient to show that none of the cases (ii),(iii),(iv) of Theorem 3.1 can occur.

Assume first that we are in case (ii). Then there are two skew lines ℓ_1 and ℓ_2, contained in C, such that for general t the line X_t meets ℓ_1 and ℓ_2 and no other component of C_{red}. We may assume that $C_{red} = \ell_1 \cup \ell_2$ and we seek for a contradiction. We have $C = C_1 \cup C_2$, where the support of C_i is ℓ_i. Set $d_i := \deg(C_i)$ and $r_i := \deg(X_t \cap C_i)$ for general $t \in T$. If $2r_1 \geq d_1 + 2$ by Lemma 4.2 applied to C_1, there exists a subcurve $E \subseteq C_1$ of degree 2 such that X_t is a secant line to E for general $t \in T$. But this implies that the general hyperplane section of the degree 3 curve $E \cup \ell_2$ is collinear, whence such a curve is planar by Theorem 2.9, a contradiction. The same argument works if $2r_2 \geq d_2 + 2$, hence we must have $2r_1 \leq d_1 + 1$ and $2r_2 \leq d_2 + 1$, whence $2r = 2(r_1 + r_2) \leq d_1 + d_2 + 2 = d + 2$, a contradiction.

Now assume that one of the remaining cases occurs. As above we may assume that $C_{red} = \ell$ is a line. Then by Lemma 4.2 there is a subcurve $C' \subseteq C$ with $\deg(C') = r$ and $\mathrm{ged}(C') = 2$ such that $\deg(X_t \cap C') \geq \min\{r, 3\} = 3$. Then by Proposition 3.6, C' contains a planar subcurve E of degree ≥ 3 and all the lines X_t lie in the plane $\langle E \rangle$, a contradiction.

Hence, the required planar subcurve exists by Theorem 3.1, and is unique by an obvious degree argument. $\square$

The next statement is Theorem 1.2 of the introduction.

Corollary 4.4. *Assume* $\mathrm{char}(k) = 0$. *Let $C \subseteq \mathbb{P}^n$ be a non degenerate curve of degree d. Assume that for every general hyperplane L the section $L \cap C$ contains a subscheme of degree $r \geq \frac{d+3}{2}$ spanning a line, and contains no collinear subschemes of larger degree. Then C contains a unique planar subcurve E of degree r.*

Proof. By assumption a general hyperplane L contains an r secant line, which is unique by Lemma 2.6 (see proof of 4.3). Denote this r-secant by ℓ_L. By the argument used in [1] (see Step 3 in the proof of 1.3, p. 703) the lines ℓ_L move in an irreducible 2-dimensional family, hence we can apply Proposition 4.3. The conclusion follows. $\square$

Remark 4.5. 1) In Corollary 4.4 the assumption $r \geq \frac{d+3}{2}$ is necessary, as follows from Example 3.10. 1), with $r = 3$ and $d = 4$. Consider also the case $r = 4, d = 6$ and the curve corresponding to the homogeneous ideal $(a, b)^2 \cap (c, d)^2$.

We don't know if there are counter-examples for every n, d and r.

2) In Corollary 4.4 the assumption on the characteristic is also necessary; indeed consider in $\mathbb{P}^3$ the union of a Hartshorne line of degree $d - 1$ and a reduced line, skew to it.

The next result will allow to apply Corollary 4.4 by looking at the Hilbert function of the general hyperplane section of C. We denote the first difference of the Hilbert function of a subscheme X by Δh_X.

Lemma 4.6. *Let $Z \subset \mathbb{P}^{n-1}$ $(n \geq 3)$ be a zero-dimensional subscheme. Suppose there is an integer r such that*
$$\Delta h_Z(r-2) = \Delta h_Z(r-1) = 1.$$
Then Z contains a subscheme Z' of degree $\geq r$ spanning a line. Moreover, the degree of Z' equals r if and only if $\Delta h_Z(r) = 0$.

Proof. The proof uses the same arguments as the proof of Proposition 3.3 in [1]. Thus we give here an outline only and refer to [1] for the details.

Let J be the ideal generated by the forms of degree $\leq r-2$ contained in the homogeneous ideal I_Z of Z. The assumption on Δh_Z implies that Δh_Z has maximal growth in degree $r-2$. It follows that J defines a subscheme, which is the union of a line l and a 0-dimensional scheme, and that there is an integer $s \geq r$ such that
$$\Delta h_Z(j) = 1 \text{ if } r-2 \leq j < s \quad \text{and} \quad \Delta h_Z(s) = 0.$$
This growth condition implies that I_Z does not have a minimal generator of degree j with $r-1 \leq j < s$ but does have a minimal generator of degree s. The intersection of l and a general hypersurface of degree s containing Z gives the required 0-dimensional subscheme of Z. □

Corollary 4.7. *Assume* $\operatorname{char}(k) = 0$. *Let $C \subseteq \mathbb{P}^n$ be a curve of degree d with general hyperplane section Γ. If there is an integer $r \geq \frac{d+3}{2}$ such that $\Delta h_\Gamma(r-2) = \Delta h_\Gamma(r-1) = 1$ and $\Delta h_\Gamma(r) = 0$, then C contains a unique planar subcurve of degree r.*

Proof. Combine the two previous statements. □

Remark 4.8. Let $C \subset \mathbb{P}^n$ be a curve of degree d and genus g with non-degenerate general hyperplane section. For such a curve upper bounds for the Rao function $\rho_C(j) := h^1(\mathcal{J}_C(j))$ have been shown in [1], Theorem 2.1. These bounds are sharp. In fact, there are curves where the bound it attained for all integers j. Such curves are called *extremal* curves. They have to admit planar subcurves of large degree.

Corollary 4.9. *Assume* $\operatorname{char}(k) = 0$. *Let $C \subset \mathbb{P}^n$ be an extremal curve of degree $d \geq 2n-1$. Then C contains a unique planar subcurve of degree $d-n+2$.*

Proof. The proof of Proposition 2.1 in [1] shows that the Hilbert function of the general hyperplane section Γ satisfies
$$1 = \Delta h_\Gamma(d-n) = \Delta h_\Gamma(d-n+1) = \Delta h_\Gamma(d-n+2) + 1.$$
Thus we conclude by Corollary 4.7. □

Remark 4.10. 1) In [10] it is shown that the conclusion of Corollary 4.9 holds under weaker assumptions, namely $d \geq n+1$ and the Rao function is extremal in non-negative degrees.

On the other hand there are non-degenerate curves of degree 5 in $\mathbb{P}^4$ whose general hyperplane section contains a collinear subscheme of degree 3 but not containing planar subcurves of degree 3 (see Remark 4.14). According to [10] these curves cannot have extremal Rao function in non-negative degrees. This discussion leads to believe that the existence of planar subcurves can depend on the cohomology and not only on the general hyperplane section. We feel that this point of view should be further investigated.

2) Ellia [2] (cf. also [8]) has shown that the converse of Corollary 4.9 is true for space curves. However, it follows from the work in [10] that the converse is not true if $n \geq 4$.

3) Several authors have proved optimal bounds for the Rao functions of space curves which contain a planar subcurve of large degree r: if $r = d - 1$ (the case of extremal curves) this is done in [6], if $r = d - 2$ the bounds are given in [8] and the general case is studied in [9].

The next statement focuses on the case $r = d - 1$.

Proposition 4.11. *Assume* $\operatorname{char}(k) = 0$. *Let $C \subseteq \mathbb{P}^n$ ($n \geq 4$) be a non-degenerate curve of degree $d \geq 4$ such that for every general hyperplane L the section $L \cap C$ contains a subscheme of degree $d-1$ spanning a line. Then C contains a planar subcurve E of degree $d-1$, with the following exception: C is a multiple line with* $\deg(C) = 4$, $\operatorname{ged}(C) = 3$.

This result allows us to fill a gap in the proof of the Restriction Theorem 1.3 in [1]. In fact the proof uses [1], Lemma 1.9. which is false. The next corollary is a corrected version of Theorem 1.3 in [1]. Its proof can be obtained by combining the old proof and Proposition 4.11.

Corollary 4.12. *Let $C \subset \mathbb{P}^4$ be a non-degenerate curve of degree d over a field of characteristic zero. Assume that either $d \geq 5$ or $d = 4$ and C is not a multiple line with* $\operatorname{ged}(C) = 3$. *Then the general hyperplane section of Γ is non-degenerate if and only if C does not contain a planar subcurve of degree $d - 1$.*

Now we prove Proposition 4.11.

Proof. We distinguish 3 cases:

Case 1. $d \geq 5$.
The proof follows from Corollary 4.4, with $r = d - 1$.

Case 2. $d = 4$ and $\deg C_{red} > 1$.
We apply Theorem 3.1. Assume we are in case (ii) of Theorem 3.1 and let H be the hyperplane spanned by the two skew lines. Set $C' := C \cap H$. Then every line X_t is contained in H, whence $\deg(C') \geq (X_t \cap C') = 3$, for general t. On the other hand $C' \neq C$, because C is non-degenerate, whence $\deg(C') = 3$. Moreover, if L is a general hyperplane, we have $L \cap C' = r_L \cap C'$, where r_L is the unique r-secant line contained in L. Therefore C' is planar by Theorem 2.9. This is a contradiction.
Assume we are in case (iii) or (iv) of Theorem 3.1. Then $C = C_1 \cup C_2$, where C_1 is a multiple line and X_t meets C_1 but not C_2 for general $t \in T$. It follows that C_1 has degree 3 and its general hyperplane section is collinear, whence C_1 is planar by Theorem 2.9, again a contradiction.

Case 3. $d = 4$ and $\deg C_{red} = 1$.
If L is a general hyperplane, then the 0-dimensional scheme $L \cap C$ has a subscheme of length 3 spanning a line; therefore $\langle L \cap C \rangle$ is a plane, whence for $x := L \cap C_{red}$ we have $\operatorname{emdim}_x(L \cap C) \leq 2$.
It follows $\operatorname{ged}(C) \leq 3$. If $\operatorname{ged}(C) = 2$ the conclusion follows from by Proposition 3.6. Then if the conclusion is false we are in the exceptional case of our claim. □

Remark 4.13. We want to show that for $n = 4$ the exceptional case of Proposition 4.11 can actually occur.
We fix an affine chart with coordinates x, y, z, t and we identify a hypersurface with one of the polynomials defining it. Let $\ell \subseteq \mathbb{P}^4$ be the line having affine equations $x = y =$

$z = 0$, and let

$$\begin{aligned} F(x,y,z,t) &:= xf_1(t) + yf_2(t) + zf_3(t) \\ G(x,y,z,t) &:= xf_1'(t) + yf_2'(t) + zf_3'(t) \end{aligned}$$

where $f_1(t), f_2(t), f_3(t)$ are polynomials in t only (to be chosen as described below) and ' means derivative.

Note that F and G are two hypersurfaces containing ℓ, whence $S := F \cap G$ is a surface containing ℓ.

We choose $f_1(t), f_2(t), f_3(t)$ so that S is smooth at a general point of ℓ and moreover the tangent planes to S at general points of ℓ span $\mathbb{P}^4$.

Note that the tangent plane of S at the point $(0,0,0,t_0)$, if it exists, has equations $F(x,y,z,t_0) = G(x,y,z,t_0) = 0$.

Consider now the ideals $I_1 := (x,y,z)^2 + (F)$ and $I_2 := (x,y,z)^3 + (F,G)$ and let $X_i \subseteq \mathbb{P}^4$ be the subscheme defined by I_i $(i = 1,2)$, which are multiple structures on ℓ plus some embedded points. Taking a general hyperplane section (and working in affine coordinates) it is easy to see that $\deg(X_1) = \deg(X_2) = 3$. The scheme $X := X_1 \cup X_2$ is determined by the ideal $I := I_1 \cap I_2 = (x,y,z)^3 + (F, xG, yG, zG)$, and the scheme $Y := X_1 \cap X_2$ is defined by the ideal $J := I_1 + I_2 = (x,y,z)^2 + (F,G)$. Hence it follows $\deg(Y) = 2$ and, by Lemma 2.6 applied to a general hyperplane section, we have that $\deg(X) = 4$.

Now we denote by C_1, C_2, C and D respectively the curves obtained from X_1, X_2, X and Y by removing the embedded points. By the above we have: $\deg(C) = 4$, $\deg(C_1) = \deg(C_2) = 3$, $\deg(D) = 2$, $\operatorname{ged}(C_1) = \operatorname{ged}(C) = 3$, $\operatorname{ged}(C_2) = \operatorname{ged}(D) = 2$. Moreover the secant lines to D at a general point $P \in \ell$ span the tangent plane π_P of S at P and then by our choice of the polynomials f_1, f_2, f_3 it follows that these secant lines span $\mathbb{P}^4$, whence D spans $\mathbb{P}^4$. It follows that the curves D, C_1, C_2, C are non-degenerate.

Now let $L = t - (ax + by + cz + d)$ be a general hyperplane. Then $P := L \cap \ell$ has coordinates $(0,0,0,d)$ and, since P is non-embedded in X, the scheme $\Gamma := L \cap C$ is defined by the ideal $I + (L) = (x,y,z)^3 + (F, xG, yG, zG, L)$. A straightforward calculation shows that $xf_1(d) + yf_2(d) + zf_3(d) \in I + (L)$, whence the general hyperplane section of C is degenerate. Then C admits a 2-dimensional family of 3-secant lines and every general hyperplane contains exactly one of them (see [1], 1.4 (iii)).

Every 3-secant line of C is also a secant line to D. Indeed by Lemma 2.6 we have $\deg(\ell \cap C_2) \geq 2$, hence ℓ is secant to C_2; now since $\operatorname{ged}(D) = 2$ by Lemma 2.8 it follows that if L is a general hyperplane, then L must contain a unique secant ℓ' to D. So we have that ℓ' is also secant to C_2, but since $\operatorname{ged}(C_2) = 2$ by Lemma 2.8 it follows that $\ell' = \ell$.

The considerations above show that the trisecant lines span $\mathbb{P}^4$ and that C cannot contain any planar subcurve of degree 4.

To get a concrete example take for instance $f_1 = t^3$, $f_2 = t^2$, $f_3 = t$. Direct calculations using MACAULAY give the following (in projective coordinates x, y, z, t, w, and denoting by I_C the saturated homogeneous ideal of the curve C):

(i)

$$\begin{aligned} I_C = \ & (y^2 - 4xz,\ 2xzt + yzw,\ yzt + 2z^2w,\ 2x^2t + xyw, \\ & xyt + 2xzw,\ xt^2 + ytw + zw^2,\ z^3,\ yz^2, \\ & xz^2,\ xyz,\ x^3,\ x^2y,\ x^2z) \end{aligned}$$

(ii) If $H = x + y + z - t + w$ then H contains the 3-secant line of C

$$r_H : x + y + z = t - w = y + 2z = 0$$

.

(iii) If $L = x - y + z - t + 2w$ then L contains the 3-secant line of C

$$r_L : 9x - t + 2w = 9y + 4t - 8w = 9z - 4t + 8w = 0$$

(iv) The lines r_H and r_L are skew.

Remark 4.14. 1) In example 3.10 1) we have constructed a curve $C \subseteq \mathbb{P}^3$ of degree 4, whose support is the union of two skew lines and whose general hyperplane contains a 3-secant line of C. Thus Proposition 4.11 is false for $n = 3$ and $d = 4$. Note that for $n \geq 4$ examples of this kind (i.e. with $\deg(C_{red}) > 1$) cannot occur.

2) If C is a general projection in $\mathbb{P}^3$ of one of the curves constructed in Remark 4.13, we have again that C is supported on a line and this provides another counterexample to Proposition 4.11 for $n = 3$ and $d = 4$.

3) If we add a general reduced line to one of the multiple lines constructed in Remark 4.13 we get a non-degenerate curve of degree 5 in $\mathbb{P}^4$ whose general hyperplane section contains a collinear subscheme of degree 3, but not containing any planar subcurve of degree 3. Compare with Corollary 4.4.

Remark 4.15. In [13] Strano studies a somewhat related problem. Let J be the saturated ideal of the general hyperplane section Γ of C. He supposes that there is a positive integer s such that J_s and J_{s+1} have a greatest common divisor F of degree $e > 0$. His main result is that then C contains a subcurve whose general hyperplane section is $\Gamma \cap V(F)$. This result implies, with $e = 1$, Corollary 4.7 in case $n = 3$. This follows from some standard calculations on the Hilbert function of the general hyperplane section. However Strano's result does not give Corollary 4.7 if $n > 3$.

Example 3.10, 1) and the example given in Remark 4.5 show that it is in general not sufficient to assume only the existence of a greatest common divisor of J_s, but is sufficient for reduced curves (cf. [13], Theorem 4 and also Corollary 3.2).

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