SURFACES IN $\mathbb{P}^4$ WITH EXTREMAL GENERAL HYPERPLANE SECTION

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Abstract. Optimal upper bounds for the cohomology groups of space curves have been derived recently. Curves attaining all these bounds are called extremal curves. This note is a step to analyze the corresponding problems for surfaces. We state optimal upper bounds for the second and third cohomology groups of surfaces in $\mathbb{P}^4$ and show that surfaces attaining all these bounds exist and must have an extremal curve as general hyperplane section. Surprisingly, all the first cohomology groups of such surfaces vanish. It follows that an extremal curve does not lift to a locally Cohen-Macaulay surface unless the curve is arithmetically Cohen-Macaulay.

1. Introduction

When one studies the Hilbert scheme of subschemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial, it is interesting and useful to have upper bounds for the dimension of the cohomology groups of the subschemes it contains. In the case of curves good bounds are known (cf., for example, [13], [5]). The best information is available for space curves, i.e. for non-degenerate subschemes of $\mathbb{P}^3$ of pure dimension 1. The Hilbert scheme $H_{d,g}$ of such curves with fixed degree d and (arithmetic) genus g contains, if it is non-empty, a curve C such that the cohomological dimensions $h^1(\mathcal{I}_C(j))$ are maximal among all curves in $H_{d,g}$ for all $j \in \mathbb{Z}$. Every curve with this property is called an extremal curve. The extremal curves form a large part of their corresponding Hilbert scheme $H_{d,g}$ ([14]). One even wonders if every curve in $H_{d,g}$ can be deformed to an extremal curve. Results in this direction can be found, for example, in [10].

This note is a first step to explore the cohomological dimensions of non-degenerate surfaces in $\mathbb{P}^4$, where surface means a closed subscheme of pure dimension 2. Our results indicate that the situation for surfaces is much more complicated than for curves, as one might expect.

It is a classical method to derive upper bounds for the cohomology by using general hyperplane sections. Indeed, the optimal bounds for curves were obtained this way. Thus, it seems reasonable to expect that surfaces whose general hyperplane section is an extremal curve, have large cohomology.

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Of course, the first question is if such surfaces exist. They do, since we may take the cone over an extremal curve. However, the problem becomes more subtle if one requires that the surface should be locally Cohen-Macaulay. In general, it is a notoriously difficult problem to decide whether a given locally Cohen-Macaulay subscheme is the general hyperplane section of a locally Cohen-Macaulay subscheme. One of our main results is:

Theorem 1.1. *An extremal curve $C \subseteq \mathbb{P}^3$ is not the general hyperplane section of a locally Cohen-Macaulay surface in $\mathbb{P}^4$ unless C is arithmetically Cohen-Macaulay.*

Nevertheless, if one considers just surfaces, not necessarily locally Cohen-Macaulay, one might still guess that the surfaces with an extremal curve as general hyperplane section have large cohomology. Quite the contrary is true, at least for the first cohomology.

Theorem 1.2. *If $S \subseteq \mathbb{P}^4$ is a surface whose general hyperplane section is an extremal curve, then $h^1(\mathcal{I}_S(j)) = 0$ for all $j \in \mathbb{Z}$.*

The proof uses the precise information on extremal curves from [14] and combines various methods. One particular case requires a very careful analysis.

Using arguments from liaison theory, Theorem 1.2 implies quickly Theorem 1.1 (cf. Corollary 3.6).

The situation for the second and third cohomology of surfaces is different from the one for the first cohomology. Indeed, for surfaces with fixed degree and sectional genus we produce upper bounds for all the second and third cohomology groups. These bounds are optimal and the surfaces attaining the bounds for $h^2(\mathcal{I}_S(j))$ and $h^3(\mathcal{I}_S(j))$, respectively, for all $j \in \mathbb{Z}$ are called $H2$-extremal and $H3$-extremal, respectively. In accordance with the naive guess mentioned above, we show that $H2$-extremal as well as $H3$-extremal surfaces have extremal curves as general hyperplane section. Moreover, $H2$-extremal surfaces are also $H3$-extremal and the converse is also true if the degree is at least 5. Combining these results with Theorem 1.2 we obtain that there are surfaces such that all their second and third cohomology groups are maximal, but then all their first cohomology groups must vanish.

The paper is organized as follows. In Section 2 we fix notation, recall some results on extremal and sub-extremal curves as well as residual schemes and prove some observations on residual sequences. Then we define sectionally extremal and sectionally sub-extremal surfaces and construct examples of such surfaces.

The two theorems mentioned above are shown in Section 3. Here, we also discuss extensions of Theorem 1.2. It turns out that surfaces whose general hyperplane section is a sub-extremal curve have mostly vanishing first cohomology (cf. Proposition 3.10). But some exceptions occur and we describe some of their properties.

The final Section 4 is devoted to bounding the dimensions of the second and third cohomology groups. We also show that there are more $H2$-extremal and $H3$-extremal surfaces than just cones over extremal curves.

2. Notation and preliminary results

2.1. **Notation and conventions.**

We work over an algebraically closed field K of characteristic zero. We denote by R the graded polynomial ring $K[X_0, \ldots, X_n]$ and by $\mathbb{P}^n := \operatorname{Proj}(R)$ the n-dimensional projective space over K. If $X \subseteq \mathbb{P}^n$ is a closed subscheme we denote by $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^n}$ and $I_X \subseteq R$ the ideal sheaf and the saturated homogeneous ideal of X, respectively.

If M is a graded R-module we put $a(M) := \inf\{j \in \mathbb{Z} \mid M_j \neq 0\}$ and $e(M) = \sup\{j \in \mathbb{Z} \mid M_j \neq 0\}$.

Our standard reference will be [8]. We shall freely use some known results on liaison (see [15]) and duality (see [20], Chapter 0).

2.2. **Curves and Surfaces.**

By *curve* (resp. *surface*) in $\mathbb{P}^n$ we mean a closed subscheme of pure dimension 1 (resp. 2). In particular, curves and surfaces are not allowed to have embedded components. Note also that a curve is locally Cohen-Macaulay, while this property can fail for surfaces. However a surface, being equidimensional, contains at most finitely many non-Cohen-Macaulay points. It follows that a general hyperplane section of a surface is a curve.

Remark 2.1. We recall two well-known facts (see e.g. [4], 20.4.20).

(i) A closed subscheme $X \subseteq \mathbb{P}^n$ has no zero-dimensional components (embedded or not) if and only if $H^1(\mathcal{I}_X(j)) = 0$ for $j \ll 0$. Hence if X is either a curve or a surface we have $H^1(\mathcal{I}_X(j)) = 0$ for $j \ll 0$.

(ii) A surface X is locally Cohen-Macaulay if and only if $H^2(\mathcal{I}_X(j)) = 0$ for $j \ll 0$.

2.3. **Extremal curves.**

We recall some definitions and results concerning extremal curves in $\mathbb{P}^3$.

If $C \subseteq \mathbb{P}^n$ is a curve, the *Hartshorne-Rao module* of C is the graded R-module $M_C := H^1_*(\mathcal{I}_C) := \oplus_{j\in\mathbb{Z}} H^1(\mathcal{I}_C(j))$ and the *Rao function* ρ_C of C is the Hilbert function of M_C, i.e. $\rho_C(j) := h^1(\mathcal{I}_C(j))$ $(j \in \mathbb{Z})$.

Remark 2.2. (See [13]). If $C \subseteq \mathbb{P}^3$ is a non-degenerate curve of degree d and (arithmetic) genus g we have, for all $j \in \mathbb{Z}$, $\rho_C(j) \leq \rho^E_{d,g}(j)$ where $\rho^E_{d,g} : \mathbb{Z} \to \mathbb{Z}$ is the function defined as follows:

$$\rho^E_{d,g}(j) := \begin{cases} 0 & \text{if } \ j \leq -\binom{d-2}{2} + g \\ \binom{d-2}{2} - g + j & \text{if } \ -\binom{d-2}{2} + g \leq j \leq 0 \\ \binom{d-2}{2} - g & \text{if } \ 0 \leq j \leq d-2 \\ \binom{d-1}{2} - g - j & \text{if } \ d-2 \leq j \leq \binom{d-1}{2} - g \\ 0 & \text{if } \ \binom{d-1}{2} - g \leq j \end{cases}$$

Remark 2.3. The above bounds imply $g \leq \binom{d-2}{2}$ and if equality holds then C is arithmetically Cohen-Macaulay (see also [9]).

Definition 2.4. A non-degenerate curve $C \subseteq \mathbb{P}^3$ of degree d and genus g is said to be *extremal* if $\rho_C(j) = \rho^E_{d,g}(j)$ for every $j \in \mathbb{Z}$.

Note that an arithmetically Cohen-Macaulay curve is extremal if and only if it is a curve of maximal genus.

Remark 2.5. Let $C \subseteq \mathbb{P}^3$ be an extremal curve of degree $d \geq 2$ and genus g and put $a := \binom{d-2}{2} - g$, $\ell := d-2$. If $a > 0$ (i.e. if C is not arithmetically Cohen-Macaulay) the homogeneous ideal I_C is minimally generated by four homogeneous polynomials f_1, f_2, f_3, f_4 satisfying: $\deg(f_1) = \deg(f_2) = 2$, $\deg(f_3) = \ell + 2 = d$, $\deg(f_4) = a + \ell + 1 = a + d - 1$.

Moreover one can always assume that $f_1 = X_0^2$, $f_2 = X_0X_1$ and that f_1, f_3 form a regular sequence (see [14], Prop. 0.6).

Remark 2.6. The Hartshorne-Rao module $M := M_C$ of an extremal curve C is a Koszul module which (after a possible change of variables in the ring R) can be assumed to be of the form $M = (R/(X,Y,F,G))(a-1)$, where X, Y are linear forms, F, G are forms of degrees $a, a+\ell$ respectively and X, Y, F, G form a regular sequence (see [13] or [14]).

It follows that the socle of M is $M_{e(M)} = M_{a+\ell-1} = M_{a+d-3}$.

2.4. **Sub-extremal curves.**

We recall some results from [17]. Let $C \subseteq \mathbb{P}^3$ be a non-degenerate and non-extremal curve of degree d and genus g. Then $\rho_C(j) \leq \rho^{SE}_{d,g}$, where $\rho^{SE}_{d,g} : \mathbb{Z} \to \mathbb{Z}$ is the function defined by:

$$\rho^{SE}_{d,g}(j) := \begin{cases} 0 & \text{if } \ j < g - \binom{d-3}{2} \\ \binom{d-3}{2} - g + j & \text{if } \ g - \binom{d-3}{2} + 1 \leq j \leq 0 \\ \binom{d-3}{2} - g + 1 & \text{if } \ 1 \leq j \leq d-3 \\ \binom{d-2}{2} - g + 1 - j & \text{if } \ d-3 < j \leq \binom{d-2}{2} - g \\ 0 & \text{if } \ \binom{d-2}{2} - g + 1 \leq j \end{cases}$$

A curve satisfying $\rho_C(j) = \rho^{SE}_{d,g}$ for all $j \in \mathbb{Z}$ is called *sub-extremal*. Observe that a sub-extremal curve is arithmetically Cohen-Macaulay if and only if $g = \binom{d-3}{2} - 1$. A non-arithmetically Cohen-Macaulay curve C is sub-extremal if and only if it is a basic double link of height 1 on a quadric starting from an extremal curve. From this it follows that a sub-extremal curve which is not arithmetically Cohen-Macaulay has the following two properties:

(i) M_C is Koszul, and $\mathrm{soc}(M_C) = (M_C)_{e(M_C)} = (M_C)_{b+d-2}$, where $b := \binom{d-3}{2} - g + 1 > 0$;
(ii) the ideal I_C is minimally generated by 4 homogeneous polynomials $g_1, \ldots, g_4$ with $\deg(g_1) = 2$, $\deg(g_2) = 3$, $\deg(g_3) = d-1$ and $\deg(g_4) = b+d-3$, and where g_1, g_3 form a regular sequence.

2.5. **Residual schemes.**

We review and develop in higher dimension some basic facts on residual schemes which have been used, for example, to study extremal curves (see [7]).

Let $X \subseteq \mathbb{P}^n$ be a closed non-degenerate subscheme of pure codimension 2 and let H be a hyperplane such that $\dim(X \cap H) = \dim X$. Let X' be the residual scheme of X with respect to H, namely $\mathcal{I}_{X'} := \mathcal{I}_X : \mathcal{I}_H$. Then there is an exact sequence (called residual sequence):

$$0 \to \mathcal{I}_{X'}(-1) \to \mathcal{I}_X \to \mathcal{I}_{X \cap H, H} \to 0$$

where the map $\mathcal{I}_{X'}(-1) \to \mathcal{I}_X$ is induced by multiplication by an equation of H. (Note that this sequence is called Castelnuovo sequence in [1]).

Since $\dim(X \cap H) = \dim H - 1$ the scheme $X \cap H$ is the schematic union of a hypersurface $Y \subseteq H$ and, perhaps, some lower dimensional irreducible components (possibly embedded). Let $Z \subseteq H$ be the residual scheme of $X \cap H$ with respect to Y.

We have the following Lemma, whose proof is left to the reader.

Lemma 2.7. *With the notation and the assumptions above we have:*

(i) *X' is of pure codimension 2;*

(ii) $\deg X' + \deg(X \cap H) = \deg X$;
(iii) *let* $e := \deg(X \cap H) = \deg Y$ *and let* $f \in H^0(\mathcal{O}_H(e))$ *be an equation of* Y. *Then multiplication by* f *induces an isomorphism of* $\mathcal{O}_H$*-modules* $\mathcal{I}_{Z,H}(-e) \to \mathcal{I}_{X\cap H,H}$;
(iv) $\dim Z < \dim(X \cap H)$.

In particular the residual sequence can be rewritten as:

$$0 \to \mathcal{I}_{X'}(-1) \to \mathcal{I}_X \to \mathcal{I}_{Z,H}(-e) \to 0.$$

Lemma 2.8. *Let the notation and the assumptions be as above. Then* $Z \subseteq X' \cap H$.

Proof. Since the problem is local, we have to prove the following:

Claim. *Let* $A = k[X_1, \dots, X_n], h = X_i$ *for some* i *and let* I *be an ideal of* A *of pure height* 2. *Set* $\overline{A} = A/hA$ *and let* $f \in A$. *Assume that* $\overline{I} := I\overline{A} = (\overline{f}) \cap \overline{\mathfrak{q}}_1 \cap \cdots \cap \overline{\mathfrak{q}}_s$, *where* $\overline{\mathfrak{q}}_i := \mathfrak{q}_i/fA$ *is a primary ideal of* $\overline{A}$ *with* $\mathrm{ht}(\overline{\mathfrak{q}}_i) > 1$, *(*$\mathfrak{q}_i$ *being a primary ideal of* A *containing* f*). Then* $(I : hA)\overline{A} \subseteq \overline{I} : (\overline{f})$.

Proof of Claim. We have to show that if $x \in A$ and $hx \in I$, then $\overline{fx} \in \overline{I}$.

Let $\mathfrak{m}_j := \mathrm{rad}(\mathfrak{q}_j)$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the associated primes of I and $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_\ell$ the associated primes of (h, f). Then $\mathrm{ht}(\mathfrak{p}_i) = 2 < \mathrm{ht}(\mathfrak{q}_j) = \mathrm{ht}(\mathfrak{m}_j)$ for all i, j.

Hence by prime avoidance there is $u \in \cap(\mathfrak{m}_j)$, $u \notin \cup(\mathfrak{p}_i)$. Then $(f, h)A_u = (I + hA)A_u$, whence there are $a \in I$, $b \in A$ and a natural number t such that $u^t f = a + bh$. Since $hx \in I$ it follows that $u^t fx \in I$.

Let now $S := A \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_\ell)$. Then $u \in S$, whence $fx \in I(S^{-1}A) \cap A = I$ and the conclusion follows.

□

Corollary 2.9. *Let* $C \subseteq \mathbb{P}^3$ *be an extremal curve of degree* d *and genus* g. *Then there is a plane* H *containing a subcurve of* C *of degree* $d-1$ *and the corresponding residual sequence* (see Lemma 2.7) *is:*

$$0 \to \mathcal{I}_{C'}(-1) \to \mathcal{I}_C \to \mathcal{I}_{Z,H}(1-d) \to 0,$$

where C' *is a line and* Z *is a* 0*-dimensional scheme contained in a line. Finally* $\deg Z = a := \binom{d-2}{2} - g$.

Conversely: if $d \geq 5$, *then the existence of such a residual sequence implies that* C *is extremal.*

Proof. (See also [7]). The existence of H (hence of the residual sequence) follows from the explicit description of I_C given in [14] and recalled above. Then C' is a line by Lemma 2.7, and $Z \subseteq C' \cap H$ by Lemma 2.8, whence Z is contained in a line. The last assertion follows from this and the isomorphism $H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_Z)(1-d)$ obtained from the residual sequence.

The converse is true by [7]. □

Corollary 2.10. *Let* $C \subseteq \mathbb{P}^3$ *be a sub-extremal curve of degree* $d \geq 7$ *and genus* g. *Then there is a plane* H *containing a subcurve of* C *of degree* $d-2$ *and the corresponding residual sequence* (see Lemma 2.7) *is:*

$$0 \to \mathcal{I}_{C'}(-1) \to \mathcal{I}_C \to \mathcal{I}_{Z,H}(2-d) \to 0,$$

where C' *is a planar curve of degree* 2 *and* Z *is a zero-dimensional scheme of degree* $b := \binom{d-3}{2} - g + 1$ *contained in a line.*

Proof. Let Γ be a general hyperplane section of C. It can be shown, by using the hyperplane sequence, that the difference function of the Hilbert function of Γ satisfies $\Delta h_\Gamma(d-4) = \Delta h_\Gamma(d-3) = 1$ and $\Delta h_\Gamma(d-2) = 0$. Then, since $d \geq 7$, [6], Corollary 4.7, shows that C contains a planar subcurve of degree $d-2$ and the existence of H and of the residual sequence follows. Moreover C lies on a quadric containing H, whence C' lies on a plane by the residual sequence.

By Lemma 2.8 we have $\dim Z = 0$ and the last assertion follows from the isomorphism $H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_Z)(2-d)$ obtained from the residual sequence and the shape of the Rao function of a sub-extremal curve (see 2.4). $\square$

Remark 2.11. (i) The assumption on the degree in the statement above cannot be dropped. In fact, a complete intersection of a quadric and a cubic is sub-extremal, but does not necessarily contain planar subcurves. Curves of degree 5 linked to a line by such a complete intersection have the same property. The rational quartic shows that the statement above is also false for curves of degree 4.

(ii) The conclusion of Corollary 2.10 is also true for curves of degree ≥ 5 which are not arithmetically Cohen-Macaulay. This can easily proved by using liaison arguments and the precise knowledge of the generators of the homogeneous ideal.

2.6. **Sectionally extremal and sectionally sub-extremal surfaces.**

Let $S \subseteq \mathbb{P}^4$ be a non-degenerate surface, and let $C := S \cap L$ be a general hyperplane section of S. We put $d := \deg(S)$ and we denote by g the genus of C (also called *sectional genus* of S). Put $a := \binom{d-2}{2} - g$ and $b := \binom{d-3}{2} - g + 1$.

Remark 2.12. Let M be a hyperplane such that $D := M \cap S$ is a curve. Then D is non-planar. Indeed if D were planar it would be arithmetically Cohen Macaulay, and hence also S would be arithmetically Cohen Macaulay (e.g. by Remark 2.1 or [12], Proposition 2.1). But then $h^0(\mathcal{I}_S(1)) = h^0(\mathcal{I}_{D,M}(1)) \neq 0$, which is impossible since S is non-degenerate.

It is clear that D has degree d and genus g, and since it is a non-planar curve we have $a \geq 0$ by Remark 2.3. The same argument shows that if D is not extremal, then $b \geq 0$ by the bounds in 2.4.

Definition 2.13. We say that S is *sectionally extremal* (resp. *sectionally sub-extremal*) if C is an extremal (resp. sub-extremal) curve. Note that, in particular, S is non-degenerate.

Observe that Remark 2.12 and semicontinuity imply for a sectionally extremal surface S that every hyperplane section of S, which is a curve, is an extremal curve of degree d and genus g.

Example 2.14. If S is cone over an extremal (resp. a sub-extremal) curve, then S is sectionally extremal (resp. sub-extremal).

Example 2.15. Let S be a non-degenerate surface and assume that there is a hyperplane H such that $\dim(S \cap H) = 2$ and $\deg(S \cap H) = d - 1$. Then we have a residual exact sequence:

$$0 \to \mathcal{I}_{S'}(-1) \to \mathcal{I}_S \to \mathcal{I}_{S\cap H,H} \to 0,$$

where S' is a plane by Lemma 2.7. Then S is sectionally extremal by Corollary 2.9.

It follows that the union of a surface $\tilde{S}$ spanning a hyperplane H and a plane $\pi \not\subseteq H$ is a sectionally extremal curve, which is not a cone unless $\tilde{S}$ is a cone with vertex in π.

We will see that every sectionally extremal surface has the above structure (see Corollary 3.7).

Example 2.16. Let $\tilde{S}$ be a surface spanning a hyperplane H and let S' be a quadric surface spanning a hyperplane $H' \neq H$. Then by direct calculations it follows that $S := \tilde{S} \cup S'$ is sectionally sub-extremal. Moreover as in the previous example in general S is not a cone.

We will show that every sectionally sub-extremal surface of degree ≥ 7 has a structure similar to the above (see Proposition 3.9 below).

The next results provides more examples of sectionally extremal and sectionally subextremal surfaces. There we use $M^\vee$ to denote the graded K-dual $\oplus_{j\in\mathbb{Z}} \operatorname{Hom}_K([M]_{-j}, K)$ of the graded R-module M.

Proposition 2.17. *For any $d \geq 5$ and any $g \leq \binom{d-2}{2}$ there is a surface $S \subset \mathbb{P}^4$ with the following properties:*

- (i) *S is supported by a plane and is sectionally extremal of degree d and sectional genus g;*
- (ii) *S is arithmetically Cohen-Macaulay if and only if $g = \binom{d-2}{2}$;*
- (iii) *the embedding dimension of S at a general point is 3;*
- (iv) *S is not the scheme-theoretical union of two smaller surfaces;*
- (v) *S is not a cone;*
- (vi) *$H^2_*(\mathcal{I}_S)^\vee$ is a 1-dimensional Cohen-Macaulay module.*

Proof. Fix homogeneous coordinates (x, y, z, u, v) in $\mathbb{P}^4$, and let π be the plane corresponding to the homogeneous ideal (x, y). Let Y be the subscheme corresponding to the homogeneous ideal (x, y^{d-1}). The restriction of $\mathcal{I}_Y$ to π is isomorphic, to $\mathcal{O}_\pi(-1) \oplus \mathcal{O}_\pi(1-d)$

Let now $t \geq -1$ be a fixed integer, and let $\phi : \mathcal{O}_\pi(-1) \oplus \mathcal{O}_\pi(1-d) \to \mathcal{O}_\pi(t)$ be the morphism defined by sections $f \in H^0(\mathcal{O}_\pi(t+1))$ and $g \in H^0(\mathcal{O}_\pi(d+t-1))$. Let V be the set-theoretic intersection of the divisors of π corresponding to f and g. We choose f and g general enough, so that $\dim(V) \leq 0$.

Let $\mathcal{L} := \operatorname{im}(\phi) \subseteq \mathcal{O}_\pi(t)$. Then $\mathcal{L}_P = (\mathcal{O}_\pi(t))_P$ if and only if $P \notin V$. It follows that $\operatorname{depth}(\mathcal{L}_P) = 2$ if $P \notin V$ and $\operatorname{depth}(\mathcal{L}_P) = 1$ if $P \in V$.

Let now S be the scheme whose ideal sheaf is the kernel of the map $\Phi : \mathcal{I}_Y \to \mathcal{O}_\pi(t)$ induced by ϕ. Then we have an exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_Y \to \mathcal{L} \to 0 \tag{1}$$

whence the exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_S \to \mathcal{O}_Y \to 0 \tag{2}$$

From (2) we get that S is supported on π and that $\operatorname{depth}(\mathcal{O}_{S,P}) = 2$ if $P \notin V$ and $\operatorname{depth}(\mathcal{O}_{S,P}) = 1$ if $P \in V$. It follows that S is a surface which is not Cohen-Macaulay exactly at the points of V, whence, in particular, S is locally Cohen-Macaulay if and only if $t = -1$.

Let now H be a general hyperplane and set $C := S \cap H$, $D := Y \cap H$, $\ell := \pi \cap H$. By restricting (2) to H we get and exact sequence

$$0 \to \mathcal{O}_\ell(t) \to \mathcal{O}_C \to \mathcal{O}_D \to 0 \tag{3}$$

From (3) we can compute the Hilbert polynomial of C, and we get $\deg(C) = d$, $p_a(C) = p_a(D) - (t+1) = \binom{d-2}{2} - (t+1)$.

It follows that $\deg(S) = d$. Moreover C is non-degenerate and since it contains the planar subcurve D it is extremal by Corollary 2.9. Thus, S is sectionally extremal. Moreover we can choose $t \geq -1$ in such a way the sectional genus of S is g.

This proves (i).

To show (ii) observe that S is arithmetically Cohen-Macaulay if and only if C is arithmetically Cohen-Macaulay, if and only if $g = \binom{d-2}{2}$.

Let now $F = F(z,u,v)$ and $G = G(z,u,v)$ be homogeneous polynomials in $K[x,y,z,u,v]$ lifting f and g respectively. Since $\Phi(x) = f$ and $\Phi(y^{d-1}) = g$ it follows that $xG - y^{d-1}F \in I_S$. An easy calculation shows that the hypersurface $xG - y^{d-1}F = 0$ is nonsingular at every point of $\pi \setminus \{G = 0\}$, whence (iii).

For proving claim (iv) we have to exclude the possibility that I_C is the intersection of two primary ideals which properly contain I_C. Now (iii) implies that S is generically a complete intersection. Hence if η is the generic point of S the local ring $\mathcal{O}_\eta$ is Gorenstein, and (iv) follows.

Now we show that for general f and g the surface S is not a cone. Assume first that $t = -1$. Since $f \neq 0$ by our previous choice, we have $\mathcal{L} = \mathcal{O}_\pi$. This implies, by a straightforward calculation, that I_S is minimally generated by x^2, xy and $xG - y^{d-1}$. Now assume that S is a cone with vertex A. Then $A \in \pi$, and we may assume, after a linear change of coordinates z,u,v, that $A = (0,0,0,0,1)$. This easily implies that G must be a polynomial in z and u only, that is the corresponding curve in π has a point of multiplicity $d-2$ in A. Then by a general choice of g the surface S is not a cone.

Let now if $t \geq 0$. If f and g are general enough, we have $\deg(V) = (t+1)(d+t-1) > 1$, whence S has more then one non-Cohen-Macaulay point, and therefore it cannot be a cone.

Finally, we show (vi). Let M denote the graded R-module $(F,G,u,v)/(u,v)$. By construction, its sheafification is the sheaf $\mathcal{L}$. Using Sequence (1) and the fact that the surface Y is arithmetically Cohen-Macaulay we obtain the following isomorphisms

$$H^2_*(\mathcal{I}_S) \cong H^1_*(\mathcal{L}) \cong H^2_{\mathfrak{m}}(M) \cong H^1_{\mathfrak{m}}(R/(F,G,u,v)).$$

The claim follows since F, G, u, v is a regular sequence. □

Corollary 2.18. *For any $d' \geq 7$ and any $g' \leq \binom{d-3}{2} + 1$ there is a surface $S' \subset \mathbb{P}^4$ with the following properties:*

(i) *S' is supported by a plane and is sectionally sub-extremal of degree d' and sectional genus g';*
(ii) *S' is arithmetically Cohen-Macaulay if and only if $g' = \binom{d-3}{2} + 1$;*
(iii) *S' is not a cone.*

Proof. Let S be a surface as in Proposition 2.17. Then, with the same notation as in the proof of Proposition 2.17 we have $x^2 \in I_S$ and $y^d \in I_S$. Let S' be the surface linked to S by the complete intersection corresponding to the ideal (x^2, y^{d+1}). It follows easily that S' is sectionally sub-extremal of degree $d' = d+2$ and sectional genus $g' = \binom{d'-3}{2} + 1 - (t+1)$.

By construction it follows that the support of S' is π. Hence if $P \in \pi$ the ideal $\mathcal{I}_{S,P}$ and $\mathcal{I}_{S',P}$ of $\mathcal{O}_{\mathbb{P}^4,P}$ are linked by a complete intersection, whence $\mathcal{O}_{S,P}$ is Cohen-Macaulay if and only if $\mathcal{O}_{S',P}$ is such. Then if $t \geq 0$ it follows that S' is not a cone.

Assume now $t = -1$. Then a direct calculation shows that $I_{S'}$ is minimally generated by x^2, xy^2 and $xyG - y^d$. Then an argument similar to the one used in the proof of Proposition 2.17 shows that if G is general then S' is not a cone. □

3. The first cohomology

In this section we study the first cohomology of a surface in $\mathbb{P}^4$ whose general hyperplane section is extremal or sub-extremal.

We use the notation of Subsection 2.6. In particular, C denotes the general hyperplane section of the surface S.

The main result of this section is the following vanishing Theorem.

Theorem 3.1. *If $S \subseteq \mathbb{P}^4$ is a sectionally extremal surface, then $H^1_*(\mathcal{I}_S) = 0$.*

The proof of this theorem needs several steps.

Remark 3.2. If $a = 0$, then C is arithmetically Cohen-Macaulay and then also S is arithmetically Cohen-Macaulay. So we may assume $a > 0$.

Lemma 3.3. *Assume that there is a hyperplane H such that $\dim(H \cap S) = 2$ and $\deg(H \cap S) = d - 1$. Then $H^1_*(\mathcal{I}_S) = 0$.*

Proof. By Lemma 2.7 we have the residual exact sequence:

$$0 \to \mathcal{I}_{S'}(-1) \to \mathcal{I}_S \to \mathcal{I}_{Z,H}(-d+1) \to 0,$$

where S' is a plane and $\dim Z < \dim(H \cap S) = 2$.

If Z is non-empty, then it is a curve because, from the above sequence, we get $H^1(\mathcal{I}_{Z,H}(j)) = 0$ for $j \ll 0$ (see Remark 2.1).

Moreover $Z \subseteq S'$ by Lemma 2.8, hence in any case $H^1_*(\mathcal{I}_Z) = 0$ and this implies, by the exact sequence above, that $H^1_*(\mathcal{I}_S) = 0$. □

Corollary 3.4. *If $S \subseteq \mathbb{P}^4$ is sectionally extremal, then $H^1_*(\mathcal{I}_S) = 0$ in each of the following cases:*

(i) $d = 2$ *(i.e. $\ell = 0$);*
(ii) *S is contained in a quadric hypersurface;*
(iii) $a + \ell \geq 3$.

Proof. (i). Since C is non-integral (see Remark 2.5) S is non-integral by Bertini's Theorem. Hence S contains a plane π and it is easy to see that Lemma 3.3 can be applied to any hyperplane H containing π.

(ii). By (i) we may assume $\ell > 0$. Hence by Remark 2.5 the quadric surfaces containing C have a common component which is a plane π such that $\pi \cap C$ is a 1-dimensional scheme of degree $d - 1$. Then, if Q is a quadric containing S, its general hyperplane section is non-integral by Bertini's Theorem. It follows easily that there is a hyperplane H contained in Q which satisfies the assumptions of Lemma 3.3.

(iii) Let L be a general hyperplane and let $\varphi : H^1_*(\mathcal{I}_S)(-1) \to H^1_*(\mathcal{I}_S)$ be the map induced by L. If φ is injective the conclusion is clear. So assume φ is not injective. Then by the Socle Lemma (see [12]) we have $a(\ker\varphi) > a(\mathrm{soc}(\mathrm{coker}\,\varphi))$. Now $\mathrm{coker}\,\varphi \neq 0$ is isomorphic to a submodule of M_C, and hence by Remark 2.6 we have $a(\mathrm{soc}(\mathrm{coker}\,\varphi)) = a + \ell - 1$, whence $a(\ker\varphi) \geq a + \ell$. It follows that the map $H^1(\mathcal{I}_S(j)) \to H^1(\mathcal{I}_S(j+1))$ is injective for $j \leq a + \ell - 2$, whence the restriction map $H^0(\mathcal{I}_S(j)) \to H^0(\mathcal{I}_{L\cap S}(j))$ is surjective for all $j \leq a + \ell - 1$. Since $2 \leq a + \ell - 1$ it follows easily from Remark 2.5 that S is contained in a quadric hypersurface and we can apply (ii). □

The arguments above give a proof of Theorem 3.1 for all pairs (a, ℓ) with $a \geq 0$ and $\ell \geq 0$, except for $a = \ell = 1$. Now we treat this case by a direct approach.

Lemma 3.5. *$H^1_*(\mathcal{I}_S) = 0$ if $a = \ell = 1$.*

Proof. We show that S is contained in a quadric hypersurface, whence Corollary 3.4(ii) applies. We argue by contradiction, assuming that $H^0(\mathcal{I}_S(2)) = 0$.

By Lemma 2.3 and the assumption we have:

- $d = 3$, $g = -1$;
- $\rho_C(0) = \rho_C(1) = 1$ and $\rho_C(j) = 0$ for $j \neq 0, 1$;

- $h^0(\mathcal{I}_C(2)) = 2$.

Claim 1. *We have: $h^1(\mathcal{I}_S) = 1$, $h^1(\mathcal{I}_S(1)) = 2$ and $h^1(\mathcal{I}_S(j)) = 0$ for $j \neq 0, 1$.*

Proof of Claim 1. We use the restriction sequences determined by L. It is easy to see that the map $H^1(\mathcal{I}_S(j-1)) \to H^1(\mathcal{I}_S(j))$ is surjective for $j \leq -1$ and for $j \geq 2$. It is injective for $j \leq 1$.

Then by Remark 2.1 we have $H^1(\mathcal{I}_S(j)) = 0$ for $j < 0$.

Now using $H^1(\mathcal{I}_S(-1)) = 0$ and $\rho_C(0) = 1$ we get $h^1(\mathcal{I}_S) \leq 1$. From this and the equalities $h^0(\mathcal{I}_S(2)) = 0$ and $h^0(\mathcal{I}_C(2)) = 2$ it follows:

$$2 \leq h^1(\mathcal{I}_S(1)) \leq h^1(\mathcal{I}_S) + h^1(\mathcal{I}_C(1)) = h^1(\mathcal{I}_S) + 1 \leq 2,$$

whence $h^1(\mathcal{I}_S) = 1$ and $h^1(\mathcal{I}_S(1)) = 2$.

Finally from $h^1(\mathcal{I}_S(1)) = h^0(\mathcal{I}_C(2)) = 2$ we get $h^1(\mathcal{I}_S(2)) = 0$, whence $h^1(\mathcal{I}_S(j)) = 0$ for $j \geq 2$.

Claim 2. *S is locally Cohen-Macaulay and $H^2_*(\mathcal{I}_S) = 0$.*

Proof of Claim 2. Using the restriction sequences and the values of ρ_C it follows easily that $h^2(\mathcal{I}_S(j)) = 0$ for $j \geq 1$, and that $h^2(\mathcal{I}_S(j-1)) \leq h^2(\mathcal{I}_S(j))$ for $j \leq -1$. Thus, we have an exact sequence:

$$0 \to H^1(\mathcal{I}_S) \to H^1(\mathcal{I}_S(1)) \to H^1(\mathcal{I}_C(1)) \to H^2(\mathcal{I}_S) \to 0$$

and since by Claim 1 $h^1(\mathcal{I}_S(1)) = 1 = h^1(\mathcal{I}_C(1))$ and $h^1(\mathcal{I}_S(1)) = 2$ we get $h^2(\mathcal{I}_S) = 0$. This implies, by a similar argument, that $h^2(\mathcal{I}_S(-1)) = 0$. Then $H^2_*(\mathcal{I}_S) = 0$ and S is locally Cohen-Macaulay by Remark 2.1.

Claim 3. *There is a 2-dimensional linear system Φ of hyperplanes such that for every $M \in \Phi$ the multiplication map $H^1(\mathcal{I}_S) \to H^1(\mathcal{I}_S(1))$ induced by M is injective.*

Proof of Claim 3. Let $V \subseteq (\mathbb{P}^4)^*$ be the set of points corresponding to the hyperplanes M such that $\dim(M \cap S) = 1$. Then V is the complement of finitely many linear subvarieties of dimension ≤ 1 of $(\mathbb{P}^4)^*$, and hence it contains a two-dimensional linear variety. Let Φ be the corresponding linear system of hyperplanes of $\mathbb{P}^4$. Then if $M \in \Phi$ we have that $M \cap S$ is a curve because S is locally Cohen-Macaulay by Claim 2. The conclusion follows, because $H^0(\mathcal{I}_{M \cap S}(1)) = 0$ by Remark 2.12.

Now we can conclude our proof of Lemma 3.5. Indeed by Claims 1 and 3 and by [2], Lemma 3.1, we get

$$1 = h^1(\mathcal{I}_S) \leq \max\{0, h^1(\mathcal{I}_S(1)) - 2\} = 0$$

which is the desired contradiction. □

Now we give some applications of Theorem 3.1.

Corollary 3.6. *A sectionally extremal surface in $\mathbb{P}^4$ is locally Cohen-Macaulay if and only if it is arithmetically Cohen-Macaulay.*

Proof. Let S be a sectionally extremal surface. By Theorem 3.1 it follows that the map $H^0(\mathcal{I}_S(j)) \to H^0(\mathcal{I}_C(j))$ is surjective for all j. Then by Remark 2.5 S is contained in a quadric hypersurface and in a hypersurface of degree d which form a regular sequence. Hence it is possible to link S to a surface S' by a $(2, d)$ complete intersection. We have that $C' := S' \cap L$ is linked to C by a $(2, d)$ complete intersection (see [15], Proof of Proposition 5.2.17). Then by liaison C' has degree d and genus g, and $M_{C'} \cong M_C^\vee(d-2)$. This implies that C' is an extremal curve. Then Theorem 3.1 shows that $H^1_*(\mathcal{I}_{S'}) = 0$. Now, since S is locally Cohen-Macaulay we have, again by liaison, $H^2_*(\mathcal{I}_S) = H^1_*(\mathcal{I}_{S'})^\vee(d-3) = 0$. Then S is arithmetically Cohen-Macaulay, because $H^1_*(\mathcal{I}_S) = 0$ by Theorem 3.1.

The converse is clear. □

This corollary proves Theorem 1.1 of the Introduction.

Our next result describes the structure of sectionally extremal surfaces.

Corollary 3.7. *Let $S \subseteq \mathbb{P}^4$ be a sectionally extremal surface of degree d and sectional genus g. Then there is a subsurface $T \subseteq S$ of degree $d-1$ contained in a hyperplane H. Moreover the residual sequence* (see Lemma 2.7) *is:*

$$0 \to \mathcal{I}_{S'}(-1) \to \mathcal{I}_S \to \mathcal{I}_{Z,H}(1-d) \to 0,$$

where S' is a plane and Z is a planar curve of degree $a := \binom{d-2}{2} - g$.

Conversely, if $d \geq 5$ and if S contains a degenerate subsurface of degree $d-1$, then S is sectionally extremal.

Proof. Assume that S is sectionally extremal. Then by Theorem 3.1 and Remark 2.5 S is contained in a quadric hypersurface and the conclusion follows as in the proof of Corollary 3.4.

The converse follows immediately by Corollary 2.9. □

Corollary 3.8. *Let $S \subseteq \mathbb{P}^4$ be a sectionally extremal surface. Then its arithmetic genus is $p_a(S) = a(1-d) - \frac{1}{2}(a-1)(a-2)$.*

In particular, the arithmetic genus of S depends only on its degree and its sectional genus.

Proof. From the residual sequence of Corollary 3.7 we get the exact sequence:

$$0 \to \mathcal{O}_{S'}(-1) \to \mathcal{O}_S \to \mathcal{O}_Z(1-d) \to 0$$

Hence $p_a(S) = \chi(\mathcal{O}_S) - 1 = \chi(\mathcal{O}_{S'}(-1)) + \chi(\mathcal{O}_Z(1-d)) - 1$. Since S' is a plane and Z is a planar curve of degree a the conclusion follows by an easy computation. □

The methods used so far can be applied to sectionally sub-extremal surfaces, as we shall see now. We continue to use the notation of Subsection 2.6.

Proposition 3.9. *Let $S \subseteq \mathbb{P}^4$ be a sectionally sub-extremal surface of degree $d \geq 7$ Then there is a subsurface $T \subseteq S$ of degree $d-2$ lying in a hyperplane H. Moreover the corresponding residual sequence* (see Lemma 2.7) *is:*

$$0 \to \mathcal{I}_{S'}(-1) \to \mathcal{I}_S \to \mathcal{I}_{Z,H}(-d+2) \to 0, \tag{4}$$

where S' is a surface of degree 2 spanning a hyperplane and $Z \subseteq H$ is either empty or is a curve of degree b. This curve is planar if $b \neq 2$.

Proof. As in the proof of Corollary 3.4, it follows by using the Socle Lemma and the structure of the Rao module of a sub-extremal curve (see 2.4) that a general hyperplane L induces an injective map $H^1(\mathcal{I}_S(j)) \to H^1(\mathcal{I}_S(j+1))$ for $j \leq b+d-5$. Hence the restriction map $H^0(\mathcal{I}_S(j)) \to H^0(\mathcal{I}_{S\cap L}(j))$ is surjective for $j \leq b+d-4$, in particular for $j = 2$. Therefore, S is contained in a reducible quadric hypersurface, since $S \cap L$ does.

As in the proof of Lemma 3.3 we have, using Corollary 2.10, the residual exact sequence

$$0 \to \mathcal{I}_{S'}(-1) \to \mathcal{I}_S \to \mathcal{I}_{Z,H}(-d+2) \to 0,$$

where S' is a surface of degree 2 and $Z \subseteq H$ is either empty or is a curve.

By restricting the above residual sequence to L and using Corollary 2.10 we see that the curve $S' \cap L$ spans a plane, whence S' spans a hyperplane of $\mathbb{P}^4$. Moreover, if $Z \neq \emptyset$

then $Z \cap L$ is a zero-dimensional scheme of degree b contained in a line. Hence, if $b \neq 2$ then Z is either a line or a planar curve by [9] and the conclusion follows. □

Proposition 3.10. *Let $S \subseteq \mathbb{P}^4$ be a sectionally sub-extremal surface of degree $d \geq 7$. Then we have:*

(i) *if $b \neq 2$, then $H^1_*(\mathcal{I}_S) = 0$;*
(ii) *if $b = 2$, then there exists an integer t, $0 \leq t \leq d-2$, such that*

$$h^1(\mathcal{I}_S(j)) = \begin{cases} 0 & \text{if} \quad j \leq t \\ j-t & \text{if} \quad t < j \leq d-2 \\ 2d-4-t-j & \text{if} \quad d-2 < j < 2d-4-t \\ 0 & \text{if} \quad 2d-4-t \leq j \end{cases}$$

Proof. Consider the residual sequence (4) in Proposition 3.9. Since S' is arithmetically Cohen-Macaulay we obtain

$$H^1_*(\mathcal{I}_S) \cong H^1_*(\mathcal{I}_Z)(2-d)$$

and (i) follows immediately.

Assume now $b = 2$. Then Z is a curve of degree 2, thus a pair of two skew lines or a double line. Hence its first cohomology satisfies

$$h^1(\mathcal{I}_Z(j)) = \begin{cases} 0 & \text{if} \quad j \leq p_a(Z) \\ j-p_a(Z) & \text{if} \quad p_a(Z) < j \leq 0 \\ -j-p_a(Z) & \text{if} \quad 0 < j < -p_a(Z) \\ 0 & \text{if} \quad j \geq -p_a(Z) \end{cases} \tag{5}$$

Since $b = 2$, if C is a general hyperplane section of S we have $h^1(\mathcal{I}_C(j)) = 0$ for $j \leq 0$ (see Subsection 2.4). By the restriction sequence and Remark 2.1 it follows $h^1(\mathcal{I}_S(j)) = 0$ for $j \leq 0$. Thus, taking $t := d-2+p_a(Z)$ the conclusion follows by (5) and the isomorphism above. □

Corollary 3.11. *Let S be a sectionally sub-extremal surface which is locally Cohen-Macaulay. Then either $b = 2$ or S is arithmetically Cohen-Macaulay.*

Proof. Assume that $b \neq 2$ and S is locally Cohen-Macaulay. Then Proposition 3.10 and the structure of I_C (see 2.4) imply that S is contained in a complete intersection X of type $(2, d-1)$. It is easy to see that the linked surface T is sectionally extremal, and the conclusion follows from Theorem 3.1 as in the proof of Corollary 3.6. □

Lemma 3.12. *Assume that S is sectionally sub-extremal of degree ≥ 7 and $b = 2$. Let Z be the curve occurring in the residual sequence (4). Then $2-d \leq p_a(Z) \leq 0$.*

Proof. As we have seen in Proposition 3.10 and its proof we have $t := d-2+p_a(Z)$ and $0 \leq t \leq d-2$. The conclusion follows. □

Corollary 3.13. *Let $S \subseteq \mathbb{P}^4$ be a sectionally sub-extremal surface. Then we have:*

(i) *if $b \neq 2$, then $p_a(S) = b(2-d) - \frac{1}{2}(b-1)(b-2)$.*
   In particular the arithmetic genus of S depends only on its degree and its sectional genus;
(ii) *if $b = 2$, then $p_a(S) = 2(2-d) - p_a(Z)$ where Z is the curve occurring in the residual sequence (4).*
   Moreover, $2(2-d) \leq p_a(S) \leq 2-d$.

Proof. From the residual sequence (4) we get the exact sequence:

$$0 \to \mathcal{O}_{S'}(-1) \to \mathcal{O}_S \to \mathcal{O}_Z(2-d) \to 0.$$

Hence $p_a(S) = \chi(\mathcal{O}_S) - 1 = \chi(\mathcal{O}_{S'}(-1)) + \chi(\mathcal{O}_Z(1-d)) - 1$. Now S' is a quadric surface and Z is a curve of degree b, whence $p_a(S) = b(2-d) - p_a(Z)$. Now if $b \neq 2$ we have that Z is a planar curve, whence (i). If $b = 2$ we get (ii) by Lemma 3.12. □

Example 3.14. We show that the assumption $b \neq 2$ is necessary in Proposition 3.10 (i). Indeed, consider in $\mathbb{P}^4$ homogeneous coordinates (x, y, z, t, u) and let S be the surface of degree $d \geq 7$ whose homogeneous ideal is

$$I_S = (x^2, xy, y^2, xu + yt) \cap (x^2, xz, z^2, xu + zt) \cap (x, t^{d-4}).$$

Note that S is the union of a surface Y of degree $d-4$ spanning the hyperplane $H : x = 0$ and of two double planes whose supports $\pi_1 : x = y = 0$ and $\pi_2 : x = z = 0$ lie in the same hyperplane.

A straightforward calculation shows that the residual surface with respect to H is the reduced quadric $S' := \pi_1 \cup \pi_2$, and the residual of $S \cap H$ with respect to Y is the curve Z whose homogeneous ideal is $I_Z = (x, y, t) \cap (x, z, u)$. Hence Z is the union of two skew lines. From the residual sequence it follows that $h^1(\mathcal{I}_S(d-2)) = h^1(\mathcal{I}_Z) = 1$, and $h^1(\mathcal{I}_S(j)) = 0$ for $j \neq d-2$. Moreover if L is a general hyperplane it is easy to see, from the residual sequence, that $S \cap L$ is a sub-extremal curve with $b = 2$.

Remark 3.15. If we look at the proof of Proposition 3.10 we see that in case $b = 2$ the conclusion can be false only if we have a triple $Z \subseteq S' \subseteq Y$, where Z is a non-planar curve of degree 2 (compare with the previous example). We observe that a similar situation, with dimensions one less, occurs when studying curves lying in a double plane in $\mathbb{P}^3$ (see [11]). It might be possible that further examples of surfaces (with $b = 2$) not satisfying the conclusion of Proposition 3.10 (i) could be produced by extending to higher dimension the methods used in [11] to construct curves in the double plane. In particular it might be possible that all the compatible t's in Proposition 3.10 (ii) can really occur.

Remark 3.16. In [18] the notion of h-extremal curve is studied. These curves have a Koszul Hartshorne-Rao module. They are extremal curves for $h = 1$ and sub-extremal curves for $h = 2$. It might be interesting to know to which extent our results can be extended to surfaces in $\mathbb{P}^4$ whose general hyperplane section is h-extremal, or, more generally, to surfaces whose general hyperplane section has a Koszul Hartshorne-Rao module.

We feel that in most cases the first cohomology module of these surfaces will vanish, but that a number of particular exceptions will occur.

4. The second and third cohomology

We derive optimal upper bounds for the second and third cohomology groups. It turns out that surfaces achieving these bounds are sectionally extremal.

We begin with an upper bound for the second cohomology and then we study the extremal cases.

Proposition 4.1. *Let $S \subseteq \mathbb{P}^4$ be a non-degenerate surface of degree d and sectional genus g and let C be a general hyperplane section of S. Then for every $j \in \mathbb{Z}$ we have:*

(i) $h^2(\mathcal{I}_S(j)) \leq \sum_{t \geq j} h^1(\mathcal{I}_C(t+1))$;
(ii) $h^2(\mathcal{I}_S(j)) \leq \sum_{t \geq j} \rho^E_{d,g}(t+1)$;
(iii) *if equality holds in* (ii) *for some $j \leq -\binom{d-2}{2} + g$, then S is sectionally extremal.*

Proof. Restriction to a general hyperplane gives, for all $t \in \mathbb{Z}$, an exact sequence:

$$H^1(\mathcal{I}_C(t+1)) \to H^2(\mathcal{I}_S(t)) \to H^2(\mathcal{I}_S(t+1))$$

whence

$$h^2(\mathcal{I}_S(t)) - h^2(\mathcal{I}_S(t+1)) \leq h^1(\mathcal{I}_C(t+1)).$$

Adding up both sides for $t \geq j$ we get (i). Moreover (ii) follows from (i) and Remark 2.2.

Now we prove (iii). By Remark 2.2 we have $0 \leq h^1(\mathcal{I}_C(t)) \leq \rho^E_{d,g}(t)$ for every $t \in \mathbb{Z}$, whence by (i) we get

$$h^1(\mathcal{I}_C(t)) = \rho^E_{d,g}(t) \quad \text{for all} \quad t \in \mathbb{Z}$$

and the conclusion follows because $\rho^E_{d,g}(t) = 0$ for $t \leq j$. □

Definition 4.2. We say that a non-degenerate surface $S \subseteq \mathbb{P}^4$ is $H2$-*extremal* if equality holds in (ii) of Proposition 4.1 for all $j \in \mathbb{Z}$.

From the proof of Proposition 4.1 it is clear that a surface $S \subseteq \mathbb{P}^4$ is $H2$-extremal if and only if $h^2(\mathcal{I}_S(t)) - h^2(\mathcal{I}_S(t+1)) = \rho^E_{d,g}(t+1)$ for all $t \in \mathbb{Z}$.

Corollary 4.3. *Let $S \subseteq \mathbb{P}^4$ be a surface of degree d and sectional genus g. Then the following are equivalent:*

(i) *S is $H2$-extremal;*
(ii) *S is sectionally extremal and for every general hyperplane L the induced map $H^2_*(\mathcal{I}_S)(-1) \to H^2_*(\mathcal{I}_S)$ is surjective.*

Proof. (i) ⇒ (ii). If S is $H2$-extremal, then it is sectionally extremal by Proposition 4.2 (iii). Moreover by Remark 2.2, by assumption and by the exact sequence $H^1(\mathcal{I}_C(t+1)) \to H^2(\mathcal{I}_S(t)) \to H^2(\mathcal{I}_S(t+1))$ we have

$$\rho^E_{d,g}(t+1) \geq h^1(\mathcal{I}_C(t+1)) \geq h^2(\mathcal{I}_S(t)) - h^2(\mathcal{I}_S(t+1)) = \rho^E_{d,g}(t+1),$$

whence $h^1(\mathcal{I}_C(t+1)) = h^2(\mathcal{I}_S(t)) - h^2(\mathcal{I}_S(t+1))$ and the conclusion follows.

(ii) ⇒ (i). By Theorem 3.1 we have $H^1_*(\mathcal{I}_S) = 0$ whence L induces an exact sequence $0 \to H^1(\mathcal{I}_C(t+1)) \to H^2(\mathcal{I}_S(t)) \to H^2(\mathcal{I}_S(t+1)) \to 0$. The conclusion follows since C is extremal. □

Corollary 4.4. *Let S be a $H2$-extremal surface. Then the following are equivalent:*

(i) *S is locally Cohen-Macaulay;*
(ii) *S is arithmetically Cohen-Macaulay;*
(iii) *$H^2_*(\mathcal{I}_S) = 0$;*
(iv) *$g = \binom{d-2}{2}$.*

Proof. Since S is sectionally extremal the conclusion follows easily by Theorem 3.1, Corollary 3.6 and Remark 2.2. □

Now we want to give some examples of $H2$-extremal surfaces. First of all we prove the following Lemma.

Lemma 4.5. *Let $L \subseteq \mathbb{P}^{n+1}$ be a hyperplane and let $X \subseteq L$ be a closed subscheme. Let $Y \subseteq \mathbb{P}^{n+1}$ be a cone over X with vertex not in L. Then for any general linear form $\ell \in R := k[X_0, \ldots, X_{n+1}]$ the induced map $H^i_*(\mathcal{I}_Y)(-1) \to H^i_*(\mathcal{I}_Y)$ is surjective whenever $1 \leq i \leq n$.*

Proof. Clearly it is sufficient to show that there is a particular linear form with the required property. We may choose coordinates so that $L = \operatorname{Proj} S$, where $S := k[X_0, \ldots, X_n]$ and we show that $\ell := X_{n+1}$ has the required property.

By duality (see e.g. [20], Chapter 0, 4.14, 2.3 and 1.8) there is a canonical isomorphism of graded R-modules

$$[H^i_*(\mathcal{I}_Y)]^\vee \cong \operatorname{Ext}^{n-i}_R(I_Y, R)(-n-1)$$

where $M^\vee$ denotes the K-dual of a graded R-module M.

By assumption we have $I_Y = I_X R$ and since R is S-flat and I_X is a finitely generated R-module we have canonical isomorphisms

$$\operatorname{Ext}^{n-i}_R(I_Y, R) \cong \operatorname{Ext}^{n-i}_R(I_X \otimes_S R, R) \cong \operatorname{Ext}^{n-i}_S(I_X, S) \otimes_S R.$$

Then it is easy to see that X_{n+1} is a regular element for the R-module $\operatorname{Ext}^{n-i}_R(I_Y, R)$, whence it induces an injective map

$$\operatorname{Ext}^{n-i}_R(I_Y, R)(-1) \to \operatorname{Ext}^{n-i}_R(I_Y, R).$$

Hence, multiplication by X_{n+1} induces a surjective map $H^i_*(\mathcal{I}_Y)(-1) \to H^i_*(\mathcal{I}_Y)$. □

From the lemma above and Corollary 4.4 we have immediately:

Corollary 4.6. *If $S \subseteq \mathbb{P}^4$ is a cone over an extremal curve C, then S is $H2$-extremal.*

Example 4.7. We show now that the converse of the previous Corollary is false, i.e., there are more $H2$-extremal surfaces than just cones over extremal curves.

(i) First we construct explicit examples. Let $d \geq 3$ and let $Y \subseteq \mathbb{P}^4$ be a surface of degree $d-1$ spanning a hyperplane H. Let $\pi \subseteq \mathbb{P}^4$ be a plane such that $\dim(\pi \cap Y) = 0$ and set $S := \pi \cup Y$. Clearly Y and π can be chosen in such a way that S is not a cone.

Moreover from the Mayer-Vietoris sequence we have, for all $j \in \mathbb{Z}$,

$$0 = H^1(\mathcal{I}_\pi(j)) \oplus H^1(\mathcal{I}_Y(j)) \to H^1(\mathcal{I}_{\pi\cap Y}(j)) \to H^2(\mathcal{I}_S(j)) \to H^2(\mathcal{I}_\pi(j)) \oplus H^2(\mathcal{I}_Y(j)) = 0$$

whence $h^1(\mathcal{I}_{\pi\cap Y}(j)) = h^2(\mathcal{I}_S(j))$ for every $j \in \mathbb{Z}$.

Since the zero-dimensional scheme $\pi \cap Y$ has degree $d-1$ and spans a line we have

$$h^2(\mathcal{I}_S(j)) = \begin{cases} d-1 & \text{if } \ j \leq -1 \\ d-2-j & \text{if } \ 0 \leq j \leq d-2 \\ 0 & \text{if } \ d-1 \leq j \end{cases} \tag{6}$$

Now the general hyperplane section C of S is the disjoint union of a line and a planar curve of degree $d-1$, and hence the sectional genus of S is $g = \binom{d-2}{2} - 1$. It follows (see Subsection 2.3) that S is sectionally extremal and

$$\rho^E_{d,g} := \begin{cases} 0 & \text{if } \ j \leq -1 \\ 1 & \text{if } \ 0 \leq j \leq d-2 \\ 0 & \text{if } \ d-1 \leq j \end{cases} \tag{7}$$

By combining (6) and (7) we see immediately that S is $H2$-extremal.

(ii) All the surfaces constructed in Proposition 2.17 are $H2$-extremal, but not cones. This follows by Proposition 2.17 in conjunction with Corollary 4.3.

In order to get a bound for the third cohomology of a surface in $\mathbb{P}^4$ we need the following preparation.

Lemma 4.8. *Let $C \subset \mathbb{P}^3$ be a non-degenerate curve of degree d and arithmetic genus g. Then we have $h^2(\mathcal{I}_C(j)) \leq \mu(j)$ for all $j \in \mathbb{Z}$, where $\mu : \mathbb{Z} \to \mathbb{Z}$ is the function defined by*

$$\mu(j) = \begin{cases} 0 & \text{if } \ j \geq d-3 \\ \binom{d-2-j}{2} & \text{if } \ 0 \leq j \leq d-2 \\ \binom{d-2}{2} - (d-1)j - 1 & \text{if } \ g - \binom{d-2}{2} \leq j \leq -1 \\ g - 1 - dj & \text{if } \ j \leq g - \binom{d-2}{2} - 1 \end{cases}$$

Moreover, if C is an extremal curve, then $h^2(\mathcal{I}_C(j)) = \mu(j)$ for all $j \in \mathbb{Z}$. The converse is true, provided $d \geq 5$.

Proof. If $d \geq 3$, the bound for $j \geq 0$ is shown in [5], Step 2 of the proof of Theorem 2.1. It is easy to see that the bound is also true for $d = 2$.

For $j < 0$ the Riemann-Roch theorem implies

$$h^2(\mathcal{I}_C(j)) = h^1(\mathcal{I}_C(j)) - (dj - g + 1).$$

Thus, the bound follows by Remark 2.2.

Now let C be an extremal curve. Consider the restriction sequence:

$$\begin{tikzcd} H^1_*(\mathcal{I}_C)(-1) \arrow[r] & H^1_*(\mathcal{I}_C) \arrow[rr] \arrow[dr] & & H^1_*(\mathcal{I}_\Gamma) \arrow[r] & H^2_*(\mathcal{I}_C)(-1) \arrow[r] & H^2_*(\mathcal{I}_C) \arrow[r] & 0 \\ & & \text{coker} \arrow[ur] \arrow[dr] & & & & \\ & 0 \arrow[ur] & & 0 & & & \end{tikzcd}$$

where Γ denotes the general hyperplane section of C. Since $H^1_*(\mathcal{I}_C)$ is a Koszul module we obtain easily for the Hilbert function of coker:

$$h_{\text{coker}}(j) = \begin{cases} 1 & \text{if } \ g - \binom{d-2}{2} \leq j \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we have:

$$h^1(\mathcal{I}_\Gamma(j)) = \begin{cases} \max\{0, d-2-j\} & \text{if } \ j \geq 1 \\ d-1 & \text{if } \ j = 0 \\ d & \text{if } \ j < 0 \end{cases}$$

Now, an easy computation proves that $h^2(\mathcal{I}_C(j)) = \mu(j)$ for all $j \in \mathbb{Z}$.

For showing the converse assume $d \geq 5$. By assumption, we have $h^2(\mathcal{I}_C(d-4)) > 0$ and $h^2(\mathcal{I}_C(d-3)) = 0$. Hence, it follows from the restriction sequence that $h^1(\mathcal{I}_\Gamma(d-3)) > 0$. This easily implies that Γ contains a collinear closed subscheme of degree $d-1$, whence C contains a planar subcurve of degree $d-1$, being $d \geq 5$ (see [6], Corollary 4.4). Then C is extremal by [7]. □

Remark 4.9. The referee pointed out that the proof of Lemma 4.8 could be shortened by using the notion of spectrum of a curve (cf. [19]) and the computation of the spectrum of an extremal curve in [17]. We prefer to keep the present proof because it is more self-contained.

The announced bound for the third cohomology of a surface follows easily.

Proposition 4.10. *Let $S \subseteq \mathbb{P}^4$ be a non-degenerate surface of degree d and sectional genus g. Then we have for all $j \in \mathbb{Z}$:*

$$h^3(\mathcal{I}_S(j)) \leq \sum_{t>j} \mu(t).$$

Proof. Let C denote the general hyperplane section of S. Then the restriction sequence implies for all $j \in \mathbb{Z}$ (as in the proof of Proposition 4.1)

$$h^3(\mathcal{I}_S(j)) \leq \sum_{t>j} h^2(\mathcal{I}_C(t)).$$

Hence, the conclusion follows by Lemma 4.8. □

Observe that we have in particular $h^3(\mathcal{I}_S(j)) = 0$ if $j \geq d-3$ and $h^3(\mathcal{I}_S(j)) \leq \binom{d-1-j}{2}$ if $j \geq 0$.

Definition 4.11. A non-degenerate surface $S \subseteq \mathbb{P}^4$ is called $H3$-extremal if the bounds of Proposition 4.10 are achieved for every $j \in \mathbb{Z}$.

The bounds in Proposition 4.10 are optimal, i.e., $H3$-extremal surfaces do exist. Indeed, we may take the cone over an extremal curve. But more is true.

Proposition 4.12. *A $H2$-extremal surface is $H3$-extremal. The converse is also true, provided the degree of the surface is at least* 5.

In particular, a $H3$-extremal surface of degree $d \geq 5$ is sectionally extremal.

Proof. It follows from the proof of Proposition 4.10 that the surface $S \subseteq \mathbb{P}^4$ is $H3$-extremal if and only if

$$h^3(\mathcal{I}_S(j)) - h^3(\mathcal{I}_S(j-1)) = h^2(\mathcal{I}_C(j)) = \mu(j) \quad \text{for all} \quad j \in \mathbb{Z}.$$

Thus, the claims follow by Corollary 4.3 and Lemma 4.8. □

Remark 4.13. Suppose $S \subseteq \mathbb{P}^4$ is a non-degenerate surface of degree $d \geq 5$. Then we have seen that all its second cohomology groups are maximal if and only if all its third cohomology groups are maximal. But in this case, all its first cohomology groups must vanish because the general hyperplane section is an extremal curve. Moreover, S is not locally Cohen-Macaulay unless it is arithmetically Cohen-Macaulay. Thus, it follows from the proofs of our bounds that they can be improved if we restrict ourselves to locally Cohen-Macaulay surfaces. But the results at the end of Section 3 suggest that, in general, it remains open to establish optimal bounds for locally Cohen-Macaulay surfaces.

The general methods in [3], [16] provide in particular bounds for the first cohomology groups of a surface. But it seems unlikely that the resulting bounds are really best possible.

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