THE EQUATIONS OF SPACE CURVES ON A QUADRIC

ROBERTA DI GENNARO[1] AND UWE NAGEL

A . The homogenous ideals of curves in a double plane have been studied by Chiarli, Greco, Nagel. Completing this work we describe the equations of any curve that is contained in some quadric. As a consequence, we classify the Hartshorne-Rao modules of such curves.

1. I

The goal of this note is to study the equations of a curve $C \subset \mathbb{P}^3 = \mathbb{P}^3_K$ that is contained in some quadric Q. By a curve, we mean a pure one-dimensional locally Cohen-Macaulay subscheme (i.e. without zero-dimensional components.) We assume that the field K is algebraically closed.

If $C \subset Q$ is arithmetically Cohen-Macaulay, then, by Dubreil's Theorem, C is defined by at most 3 equations. The converse is also true by a well-known result of Evans and Griffith ([4, Theorem 2.1]). Denoting by $\mu(I_C)$ the number of minimal generators of the homogeneous ideal of C, this gives:

Proposition 1.1. *If $C \subset \mathbb{P}^3$ is a curve lying on some quadric, then C is arithmetically Cohen-Macaulay if and only if $\mu(I_C) \leq 3$.*

In this case, C is either a complete intersection or its ideal I_C is generated by the 2-minors of a 2×3 matrix.

In order to discuss the ideal of $C \subset Q$ when C is not arithmetically Cohen-Macaulay, we take the rank of the quadric Q into account. If Q has rank one, then Q is not reduced, i.e. Q is a double plane $2H$. Building on the work of Hartshorne and Schlesinger [9], the homogeneous ideal of C has been described by Chiarli, Greco, and Nagel in [2].

In case Q has rank 2, the curves on Q have been studied by Hartshorne in [8], Section 5, from the point of view of generalized divisors. The quadric Q is a union of two distinct hyperplanes. In Section 2 we establish the following characterization of the equations of curves on such a reducible quadric:

Theorem 1.2. *Let C be a curve on a reducible reduced quadric $Q = \mathcal{H} \cup \mathcal{H}'$. Then C is not arithmetically Cohen-Macaulay if and only if $\mu(I_C) = 4$. In this case C is (up to a change of coordinates) defined by*

$$I_C = \left(xy, x^2A + xhF, y^2B + yhG, xAF + yBG + hFG\right)$$

where x, y, F, G is a regular sequence, $h \in K[z,t]$, and $y \nmid A$, $x \nmid B$ if $h = 0$.

Using this result, we find the minimal free resolution of such a curve and, finally, determine its Hartshorne-Rao module $M_C := \oplus_{j\in\mathbb{Z}} H^1(\mathcal{I}_C(j))$ in Corollary 2.4.

If Q has rank 3, then it is cone. Thus all curves on it are arithmetically Cohen-Macaulay (cf. [5, Example 5.2]). This leaves us with the case when Q has rank 4, i.e. Q is a smooth quadric.

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Curves on a smooth quadric are investigated in Section 3. We use results from liaison theory (cf. [14] for a comprehensive introduction). The Lazarsfeld-Rao property of space curves says that each curve C can be obtained from a so-called minimal curve in its even liaison class by a sequence of basic double links and, possibly, a flat deformation. The deformation can be avoided if we replace basic double links by ascending elementary biliaisons ([17]). Thus, it suffices to consider minimal curves. It is easy to see that any such curve C is linearly equivalent to $d\mathfrak{l}$ for some $d \geq 1$ where $\mathfrak{l}$ is a line on Q. This allows us to compute explicitly the Hartshorne-Rao module of C in Corollary 3.2. It improves the results in [7]. Then we determine the equations of C:

Theorem 1.3. *Let $C \subset Q$ be a curve in $|d\mathfrak{l}|$. Then there are distinct lines $L_1, \ldots, L_m$ in the same ruling of Q as $\mathfrak{l}$ such that C is defined by:*

$$I_C = I_Q + I_{L_1}^{d_1} \cdot \ldots \cdot I_{L_m}^{d_m}$$

where $d_i \geqslant 1$ and $d_1 + \ldots + d_m = d$.

In the final Section 4 we summarize the results about the Hartshorne-Rao modules of curves on a quadric. Each such module determines an even liaison class. In each class we exhibit a minimal curve.

2. C

In this section, we consider curves on a reducible quadric Q, i.e. Q is the union of two distinct planes in $\mathbb{P}^3$. We denote by $R := K[x, y, z, t]$ the coordinate ring of $\mathbb{P}^3$. Without loss of generality we may assume that Q is defined by $q = xy$. We will prove Theorem 1.2 and determine the Hartshorne-Rao module of the curves.

We first show that the equations of the curves have the shape as predicted by Theorem 1.2.

Lemma 2.1. *Let C be a curve on $\{xy = 0\}$, then*

(a) $\mu(I_C) \leq 4$.

(b) *C is not arithmetically Cohen-Macaulay if and only if $\mu(I_C) = 4$. In this case C is defined by the homogeneous ideal*

$$I_C = \left(xy, x^2A + xhF, y^2B + yhG, xAG + yBF + hFG\right)$$

where $h, F, G \in K[z, t]$, $F, G \notin K$, $A \in K[x, z, t]$, $B \in K[y, z, t]$, and $AB \neq 0$ in case $h = 0$.

Proof. As in [1] we will utilize residual sequences. We will use the following notation: $R_x = K[y, z, t], R_y = K[x, z, t]$. By Proposition 1.1, we may assume that C is not arithmetically Cohen-Macaulay.

Let $C_x \subset \mathcal{H} := \{x = 0\}$ and $C_y \subset \mathcal{H}' := \{y = 0\}$ be the planar curves defined by

$$I_{C_y} = I_C : x = (y, P_y)$$

and

$$I_{C_x} = I_C : y = (x, P_x),$$

with homogeneous polynomials $P_x \in R_x \setminus K$ and $P_y \in R_y \setminus K$. Denote by D_x the one-dimensional part of $C \cap \mathcal{H}$ and by Z_x the residual subscheme to P_x in $C \cap \mathcal{H}$. Z_x is not empty because C is not arithmetically Cohen-Macaulay. Since D_x is the largest planar subcurve of $C \cap \mathcal{H}$, we get $C_x \subset D_x$. Thus, D_x is defined by an ideal (x, fP_x) for some homogeneous polynomial $0 \neq f \in R_x$. Moreover, [1, Lemma 2.8] provides that

$$Z_x \subset \mathcal{H} \cap C_y.$$

Thus Z_x is defined by an ideal (x, y, Q_x) for some $Q_x \in K[z,t] \setminus K$. Hence, the residual sequence of C with respect to $\mathcal{H}$ reads as:

$$0 \longrightarrow (y, P_y)(-1) \xrightarrow{x} I_C \longrightarrow P_x f\,(y, Q_x) \cdot R_x \longrightarrow 0. \tag{2.1}$$

Similarly, we get for the residual sequence with respect to $\mathcal{H}'$:

$$0 \longrightarrow (x, P_x)(-1) \xrightarrow{y} I_C \longrightarrow P_y k\,(x, Q_y) \cdot R_y \longrightarrow 0$$

where $0 \neq k \in R_y$ and $Q_y \in K[z,t] \setminus K$. These sequences imply that we can write the ideal of C as:

$$\begin{aligned} I_C &= (xy,\ xP_y,\ yP_x f + xA_1,\ P_x f\, Q_x + xA_2) && \text{for some } A_1, A_2 \in R_y \\ &= (xy,\ yP_x,\ xP_y k + yB_1,\ P_y k\, Q_y + yB_2) && \text{for some } B_1, B_2 \in R_x \end{aligned}$$

This shows in particular that $\mu(I_C) \leq 4$. Furthermore, it follows that $yB_1 \in I_C$, thus B_1 is in $I_C : y = (x, P_x)$. Since B_1 and P_x are in R_x, we see that P_x must divide B_1. Analogously, we get that P_y divides A_1. Hence, we can rewrite the ideal of C as

$$\begin{aligned} I_C &= (xy,\ xP_y,\ yP_x,\ P_x f\, Q_x + xA_2) \\ &= (xy,\ xP_y,\ yP_x,\ P_y k\, Q_y + yB_2). \end{aligned}$$

Comparing with the residual sequence (2.1) we obtain:

$$\begin{aligned} I_C + xR &= xR + (yP_x, P_x f\, Q_x) \\ &= xR + P_x f\,(y, Q_x). \end{aligned}$$

Since the degree of Q_x is at least one, f must be a constant and we may assume that $f = 1$. Analogously, we get $k = 1$ without loss of generality, thus

$$I_C = (xy,\ xP_y,\ yP_x,\ P_y\, Q_y + yB_2) \tag{2.2}$$

We now distinguish two cases.

C 1. Assume that the curves C_x and C_y have a common component. This component must be the line $\mathcal{H} \cap \mathcal{H}'$. If follows that there are polynomials $0 \neq A \in R_y$ and $0 \neq B \in R_x$ such that

$$P_x = yB \quad \text{and} \quad P_y = xA.$$

Thus, the ideal of C reads as

$$I_C = (xy,\ x^2A,\ y^2B,\ x\,A\,Q_y + y\,B_2).$$

Another comparison with the residual sequence (2.1) provides

$$\begin{aligned} I_C + xR &= xR + (y^2B, yB_2) \\ &= xR + yB\,(y, Q_x). \end{aligned}$$

Since B, B_2, Q_x are in R_x, we conclude that B divides B_2, i.e. $B_2 = BF$ for some $F \in R_x \setminus K$. Setting $G := Q_y$, we get

$$I_C = (xy,\ x^2A,\ y^2B,\ xAG + yBF).$$

It follows that we may assume $F, G \in K[z,t] \setminus K$. Thus, I_C is of the required form (with $h = 0$).

C 2. Assume that C_x and C_y do not have a common component. Then we get that $C \subset C_x \cup C_y$ and $\deg C = \deg C_x + \deg C_y = \deg(C_x \cup C_y)$. Since all these curves are of pure dimension one, we conclude that $C = C_x \cup C_y$, i.e.

$$I_C = (x, P_x) \cap (y, P_y)$$

where $\dim R/(x, y, P_x, P_y) \leq 1$.

C 2.1. Assume $\dim R/(x, y, P_x, P_y) = 0$. Then C is the disjoint union of the planar curves C_x

and C_y. Write $P_x \in R_x$ as $P_x = yB + G$ for some $G \in K[z,t] \setminus K$ and some $B \in R_x$ and, similarly, $P_y = xA + F$ for some $F \in K[z,t] \setminus K$ and some $A \in R_y$. Then we get using also Identity (2.2):

$$\begin{aligned} I_C &= (x, yB+G) \cap (y, xA+F) \\ &= (xy,\ x^2A + xF,\ y^2B + yG, xAG + yBF + FG) \end{aligned}$$

showing that I_C has the required form (with $h = 1$).

C 2.2. Assume that $\dim R/(x,y,P_x,P_y) = 1$. Then the saturation of (x,y,P_x,P_y) is of the form (x,y,h) for some polynomial $h \in K[z,t] \setminus K$. It follows that

$$P_x = yB + hG \quad \text{and} \quad P_y = xA + hF$$

for some regular sequence $F, G \in K[z,t]$ and some $B \in R_x$, $A \in R_y$. Thus, we obtain:

$$(xy, x^2A + xhF, y^2B + yhG, xAG + yBF + hFG) \subset (x, yB + hG) \cap (y, xA + hF) = I_C.$$

Comparing with the identity (2.2) we conclude that the above ideals are equal. Thus, the proof is complete. □

The above lemma specifies necessary conditions on the homogeneous ideals of curves on a reducible quadric Q. In order to find sufficient conditions we determine the minimal free resolution of the ideals. This will also allow us to compute the Hartshorne-Rao module of the curves.

Lemma 2.2. *Consider the following homogeneous ideal in R:*

$$J = \left(xy, x^2A + xhF, y^2B + yhG, xAG + yBF + hFG\right)$$

where $h, F, G \in K[z,t]$, $F, G \notin K$, $A \in K[x,z,t]$, $B \in K[y,z,t]$, and $AB \neq 0$ in case $h = 0$. Denote by d_F, d_G, d_h the degree of F, G, and h, respectively. Then J defines a 1-dimensional subscheme $C \subset \mathbb{P}^3$ of degree $d = 2d_h + d_F + d_G$.

Moreover, J has the following minimal free graded resolution:

$$0 \longrightarrow F_3 \xrightarrow[\phi_3]{} F_2 \xrightarrow[\phi_2]{} F_1 \xrightarrow[\phi_1]{} J \longrightarrow 0 \tag{2.3}$$

where

$$\begin{aligned} F_1 &= R(-2) \oplus R(-d_F - d_h - 1) \oplus R(-d_G - d_h - 1) \oplus R(-d_F - d_G - d_h) \\ F_2 &= R(-d_F - d_h - 2) \oplus R(-d_G - d_h - 2) \oplus R(-d_F - d_G - d_h - 1)^2 \\ F_3 &= R(-d_F - d_G - d_h - 2) \end{aligned}$$

and, by identifying the maps with its matrices,

$$\phi_1 = \left[xy, x^2A + xhF, y^2B + yhG, xAG + yBF + hFG\right],$$

$$\phi_2 = \begin{bmatrix} xA + hF & yB + hG & AG & BF \\ -y & 0 & 0 & G \\ 0 & -x & F & 0 \\ 0 & 0 & -y & -x \end{bmatrix}, \quad \phi_3 = \begin{bmatrix} F \\ -G \\ -x \\ y \end{bmatrix}.$$

Hence, C is a curve if and only if x, y, F, G is a regular sequence.

Proof. It is immediate to verify that the Sequence (2.3) is a complex. In order to check its exactness we use the Buchsbaum-Eisenbud criterion ([3, Theorem 20.9]). Since $F, G \neq 0$ are in $K[z,t]$, the entries of ϕ_3 generate an ideal whose codimension is at least three. It remains to show that the ideal $I_3(\phi_2)$ generated by the 3-minors of ϕ_2 contains a regular sequence of length two. Clearly, we have $x^2y \in I_3(\phi_2)$. Moreover, the determinant of the matrix obtained from ϕ_2 by deleting row 3 and column 1 is $yG(yB + hG) \in I_3(\phi_2)$. It is not divisible by x. Similarly,

deleting row 2 and column 3 of ϕ_2 we get $x^2(xA + hF) \in I_3(\phi_2)$ which is not divisible by y. It follows that $(x^2y,\ yG(yB + hG),\ x^2(xA + hF)) \subset I_3(\phi_2)$ has codimension two.

Since the non-zero entries of the matrices ϕ_1, ϕ_2, ϕ_3 have positive degree, the Resolution (2.3) is minimal.

Finally, C is curve if and only if $\operatorname{Ext}^2_R(I_C, R)$ has finite length. But the resolution of I_C provides

$$\operatorname{Ext}^2_R(I_C, R) \cong (R/(x, y, F, G))\,(d_F + d_G + d_h + 2). \tag{2.4}$$

Now the last claim follows. □

Remark 2.3. Note how the polynomial h determines the geometry of the curve C. In fact, the line $\mathcal{H} \cap \mathcal{H}'$ is in the support of C if and only if $h = 0$. Furthermore, C is a union of two disjoint planar curves if $0 \neq h \in K$. If $h \notin K$, then C is the union of two planar curves that meet in a zero-dimensional scheme whose degree is less than each of the degrees of the planar curves.

Corollary 2.4. *Adopt the notation of Lemma 2.2 and let $C \subset \{xy = 0\}$ be a curve that is not arithmetically Cohen-Macaulay. Then the Hartshorne-Rao module of C is:*

$$M_C := H^1_*(\mathcal{I}_C) \cong (R/(x, y, F, G))\,(-d_h).$$

It is self-dual and $M_C^\vee \cong M_C(d - 2)$.

Proof. This follows from the Isomorphism (2.4) because $H^1_*(\mathcal{I}_C) \cong \operatorname{Ext}^2_R(I_C, R)^\vee(4)$. □

The first theorem of the introduction follows now easily.

Proof of Theorem 1.2. Lemmas 2.1 and 2.2 imply that the homogeneous ideal of C has the required form where x, y, F, G is a regular sequence and $h \in K[z, t]$, $A \in K[x, z, t]$, $B \in K[y, z, t]$, $F, G \in K[z, t] \setminus K$, and $AB \neq 0$ in case $h = 0$. Given the specific description of the minimal generators, this is equivalent to the conditions given in the statement. □

3. C

In this section we describe the equations of curves on a smooth quadric. Without loss of generality, we consider the quadric Q defined by $q = xz - yt$.

Since the hyperplane section of Q is a divisor in the class $(1, 1)$, any curve on Q is evenly linked to a curve in the class $(d, 0)$ or $(0, d)$ (cf., e.g. [10]). Hence, each minimal curve C on Q is in a linear system $|d\mathfrak{f}|$ where $d = \deg C$ and $\mathfrak{f}$ is a line on Q. We use this information to explicitly determine the Hartshorne-Rao module of any curve on Q. Finally, we determine the defining equations of the curves in $|d\mathfrak{f}|$.

We begin by computing the minimal free resolution of a particular curve in $|d\mathfrak{f}|$.

Lemma 3.1. *Let $d \geq 2$ be an integer. The minimal free resolution of the ideal $I_d = (xz - yt, (x, y)^d)$ is:*

(3.1)

$$0 \longrightarrow R^{d-1}(-d-2) \xrightarrow{\begin{bmatrix} N_d \\ -M_d \end{bmatrix}} R^{2d}(-d-1) \xrightarrow{\begin{bmatrix} 0 & -P_{d-1} \\ M_{d+1} & N_{d+1} \end{bmatrix}} R(-2) \oplus R^{d+1}(-d) \xrightarrow{\begin{bmatrix} q & P_d \end{bmatrix}} I_d \longrightarrow 0$$

where, for any $i \geq 2$, M_i and N_i are the matrices

$$M_i = \underbrace{\begin{bmatrix} y & & & 0 \\ -x & y & & \\ & -x & \ddots & \\ & & \ddots & y \\ 0 & & & -x \end{bmatrix}}_{i-1} \Bigg\} i\,, \qquad N_i = \underbrace{\begin{bmatrix} z & & & 0 \\ -t & z & & \\ & -t & \ddots & \\ & & \ddots & z \\ 0 & & & -t \end{bmatrix}}_{i-1} \Bigg\} i\,,$$

and P_i is the $1 \times (i+1)$ matrix $P_i = \begin{bmatrix} x^i & x^{i-1}y & \dots & y^i \end{bmatrix}$.

Proof. Using the relation $N_{i+1}M_i = M_{i+1}N_i$, it is easy to check that Sequence (3.1) is a complex. Its exactness follows by the Buchsbaum-Eisenbud criterion.

Alternatively, one can get the free resolution by applying the mapping cone procedure to the exact sequence that is induced by multiplication by the quadric $q = xz - yt$:

$$0 \longrightarrow R/(x,y)^{d-1}(-2) \xrightarrow{q} R/(x,y)^d \longrightarrow R/I_d \longrightarrow 0.$$

□

As a consequence, we determine the Hartshorne-Rao modules of curves in $|d\dagger|$.

Corollary 3.2. *Adopt the assumption and notation of Lemma 3.1. Then the Hartshorne-Rao module M_C of the curve C defined by the ideal $J = (xz - yt, (y,z)^d)$ is:*

$$M_C \cong \operatorname{coker}\left(R^{2d}(-2) \xrightarrow{\begin{bmatrix} N_d^t & M_d^t \end{bmatrix}} R^{d-1} \right).$$

Moreover, its minimal free resolution is of the form

$$\begin{aligned} 0 \to R^{d-1}(-d-2) \to R^{2d}(-d-1) \to \quad & R^{d+1}(-d) \\ & \oplus \\ & R^{d+1}(-2) \quad \to R^{2d}(-1) \to R^{d-1} \to M_C \to 0. \end{aligned}$$

Proof. Let D be the curve defined by the ideal $I = (xz - yt, (x,y)^d)$. Then the K-dual of its Hartshorne-Rao module is $M_D^\vee \cong \operatorname{Ext}_R^2(I,R)(-4)$. Using Lemma 3.1, we conclude that the minimal free resolution of $M_D^\vee(4)$ is of the form:

$$0 \to F_4 \to F_3 \to F_2 \oplus R^{d+1}(d) \to R^{2d}(d+1) \xrightarrow{\begin{bmatrix} N_d^t & M_d^t \end{bmatrix}} R^{d-1}(d+2) \to M_D^\vee(4) \to 0. \tag{3.2}$$

Obviously, there is an inclusion $(xz - yt, y^d) \subset I \cap J$. Since both sides have the same degree, we get equality, in other words I is geometrically linked to J by the complete intersection $(xz - yt, y^d)$. Hence it follows (cf., e.g., [16]) that $M_C^\vee \cong M_D(d-2)$. But the ideal I is transformed into J by exchanging the variables x and z. Thus, the graded Betti numbers of M_D and M_C agree. Dualizing the Resolution (3.2), our claims follow. □

Remark 3.3. (i) Since M_C has exactly $d-1$ minimal generators if $C \in |d\dagger|$, we recover the fact that each curve in $|d\dagger|$ is minimal in its even liaison class.

(ii) Using that every curve on Q is evenly linked to a curve in $|d\dagger|$ for some $d \geq 1$, we conclude that Corollary 3.2 gives a complete description of the module structure of curves on a smooth quadric. A first attempt to achieve such a classification has been made in [6], but the results there are far less explicit.

Theorem 1.3 will follow from our next result. Our original proof was based on Corollary 3.2 and complicated. Discussions with Silvio Greco lead to the much simpler proof given below.

Lemma 3.4. *Let $C \subset Q$ be a curve in $|d\dagger|$. Then, there are distinct lines $L_1, \dots, L_m$ in the same ruling of Q as $\dagger$ such that C has a primary decomposition*

$$I_C = I_{C_1} \cap I_{C_2} \cap \ldots \cap I_{C_m},$$

where the curve C_i is is defined by

$$I_{C_i} = I_Q + I_{L_i}^{d_i},$$

with $d_i \geqslant 1$ and $d_1 + \ldots + d_m = d$.

Proof. Let $C_1, \ldots, C_m$ be the components of C. Since $\dagger$ is one of the two free generators of the Picard group of Q, each curve C_i is linearly equivalent to $d_i\dagger$ for some $d_i \geq 1$. Moreover, since by assumption C_i is irreducible, it must be supported on a line L_i. If $d_i = 1$, then $C_i = L_i$. Assume $d_i \geq 2$. Then it is well-known (cf. [6]) that the ideal of C_i is minimally generated by the quadric defining Q and $d_i + 1$ polynomials of degree d_i. It follows that $I_{C_i} \subset I_Q + I_{L_i}^{d_i}$. Since both ideals have the same Hilbert fucntion, they must be equal. □

We conclude by rewriting the ideal of C such that it is generated by the maximal minors of a homogeneous matrix. We write $I_s(A)$ for the ideal generated by the s-minors of matrix A. The result covers Theorem 1.3.

Corollary 3.5. *Let $C \subset Q$ be a curve in $|d\dagger|$. Then*

$$I_C = I_Q + I_{L_1}^{d_1} \ldots I_{L_m}^{d_m} = I_Q + I_d(A),$$

where A is the block diagonal matrix

$$A = \underbrace{\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix}}_{d+m} \Bigg\} d$$

with

$$A_i = \begin{bmatrix} \ell_i & \ell_i' & & & \\ & \ell_i & \ell_i' & & \\ & & \ddots & \ddots & \\ & & & \ell_i & \ell_i' \end{bmatrix} = \begin{bmatrix} \ell_i \mathcal{I}_{d_i} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \ell_i' \mathcal{I}_{d_i} \end{bmatrix} \in R^{d_i, d_i},$$

$d_1 + \ldots + d_m = d$, and m distinct lines $L_1, \ldots, L_m$ in the same ruling as $\dagger$ defined by $I_{L_i} = (\ell_i, \ell_i')$.

Proof. It suffices to note that

$$\bigcap_{i=1}^{m} (I_Q + I_{L_i}^{d_i}) = I_Q + \prod_{i=1}^{m} I_{L_i}^{d_i}.$$

□

4. T H -R

We now describe the possible Hartshorne-Rao modules of curves and the minimal curves on any quadric. If $C \subset Q$ is arithmetically Cohen-Macaulay, then it is in the even liaison class of a line. Otherwise, we have:

Theorem 4.1. *Let $D \subset \mathbb{P}^3$ be a curve that is not arithmetically Cohen-Macaulay. Then the minimal curves in the even liaison class of D lie on a quadric Q if and only if the Hartshorne-Rao module M_D of D is (up to changes of coordinates and degree shift) presented by one of the following homogeneous $s \times (2s+2)$ matrices ($s \geq 1$):*

(i)

$$M := \begin{bmatrix} & & f_1 \\ x\mathcal{I}_s & A & \vdots \\ & & f_s \end{bmatrix},$$

where $I_s(M)$ has codimension 4, $A = (a_{i,j}) \in K[y,z,t]^{s,s+1}$, $\deg f_1 \geq \deg a_{1,1} - 1 + \sum_{j=1}^{s} \deg a_{j,j+1}$, and $I_s(A)$ has codimension 2 and is locally a complete intersection.

(ii)
$$M := \begin{bmatrix} y & -x & & & 0 & z & -t & & & 0 \\ & y & -x & & & & z & -t & & \\ & & & \ddots & & & & & \ddots & \\ & & \ddots & & & & & \ddots & & \\ 0 & & & y & -x & 0 & & & z & -t \end{bmatrix}.$$

Moreover, the quadric Q is uniquely determined by the Hartshorne-Rao module of D if it has at least three minimal generators, i.e. $s \geq 3$. In fact, if $s \geq 3$ then, in case (i)*, the quadric Q must be a double plane $2H$ where H is defined by the unique linear form in the annihilator of M_C. In case* (ii)*, the quadric Q must be smooth and is defined by the unique quadratic form in the annihilator of M_D.*

Proof. Since D is not arithmetically Cohen-Macaulay, the quadric Q cannot have rank 3. Thus, the result about the structure of the Hartshorne-Rao module M_D follows by the description of the Hartshorne-Rao modules of curves on a quadric of rank 1 ([2, Corollary 1.2]), of rank 2 (Corollary 2.4), and of rank 4 (Corollary 3.2). Note that the Hartshorne-Rao modules of curves on a reducible quadric are a subclass of those of the curves on a double plane.

Assume now that the Hartshorne-Rao module of D has at least two minimal generators. Let $C \subset Q$ be a minimal curve in the even liaison class of D. Then, in case (i), [2, Lemma 4.8] implies that the annihilator $\mathrm{Ann}_R(M_D)$ contains a unique linear form l whereas [2, Theorem 1.1] shows that l^2 is the unique quadric in I_C because (with the notation given there) $s \geq 2$ provides $\deg p \geq 2$.

In case (ii), the annihilator of M_C does not contain a linear form, thus Q must be smooth. Furthermore, $\mathrm{Ann}_R(M_C)$ contains a unique quadratic form q. Corollary 3.2 implies that $\deg C \geq s + 1 \geq 4$, thus Lemma 3.1 provides that Q must be defined by q. □

Minimal curves are particularly interesting because the Lazarsfeld-Rao property says that each even liaison class can be recovered from any of its minimal curves by applying simple operations. Using the notation of the above theorem, we conclude by describing a minimal curve in each even liaison class of curves on a quadric.

Remark 4.2. Let $D \subset \mathbb{P}^3$ be a curve as in Theorem 4.1 and let C be a minimal curve in the even liaison class of D. In general, C is not unique and we are going to specify a particular such curve in each class.

If the Hartshorne-Rao module of D is of type (ii), then we can find a minimal curve C on a smooth quadric and the homogeneous ideal of C is described in Theorem 1.3. In particular, C could be a multiple line or a union of skew lines.

Assume now that M_C is of type (i). If M_C is not cyclic, then the proof of Theorem 4.1 provides that C cannot be a reduced curve. Theorem 1.1 in [2] implies that we can choose C as the union of a double structure on a planar curve defined by the ideal (x, p) and a planar curve of degree $\deg f_1 - \left(\deg a_{1,1} - 1 + \sum_{j=1}^{s} \deg a_{j,j+1}\right)$, where p is the determinant of the matrix obtained from A by deleting its last column.

Consider now the case $s = 1$. Let us rewrite M_D as $(S/(x, p, F, G))(j)$ for some integer j. Choose a homogeneous polynomial h of degree $\deg G - \deg F - \deg p + 1$ such that also x, p, hF, G is a regular sequence. Then the curve C defined by

$$\begin{aligned} I_C &= (x^2, xp, p^2h, phF + xG) \\ &= (x^2, xp, p^2, phF + xG) \cap (x, h) \end{aligned}$$

is a minimal curve in the class of D of the form described above. If p has degree one, then C is also contained in a reducible quadric, thus its equations must be of the form described in

Theorem 1.2. Note that the curves that are contained in two quadrics are called extremal curves; they are precisely the curves with the highest-dimensional Hartshorne-Rao module among all curves with fixed degree and arithmetic genus (cf. [12] and [15]). The only reduced extremal curve is a union of two lines whose Hartshorne-Rao module is isomorphic to the field K, thus it is of type (i) and of type (ii) in Theorem 4.1.

The minimal curves on a reducible quadric are well understood.

Corollary 4.3. *Let $C \subset \mathbb{P}^3$ be a minimal curve that lies on a reducible quadric. Then C is an extremal curve.*

Proof. This follows from Corollary 2.4 and [13]. □

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[1] N. C , S. G , U. N , *Surfaces in $\mathbb{P}^4$ with extremal general hyperplane section*, J. Algebra **257**, (2002), 65–87.
[2] N. C , S. G , U. N , *Normal form for curves in the double plane*, J. Pure Appl. Algebra **190** (2004), 45–57.
[3] D. E , *Commutative algebra (With a view toward algebraic geometry)*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.
[4] E. G. E P. G , *The syzygy problem*, Ann. Math. **114** (1981), 323–333.
[5] R. F , *Weil divisors on rational normal scrolls*, in: Geometric and combinatorial aspects of commutative algebra (J. Herzog, G. Restuccia, eds.), Lecture Notes in Applied and Pure Mathematics **217** (2001), 183–198.
[6] S. G R. M , *On the resolution of a curve lying on a smooth cubic surface in $\mathbb{P}^3$*, Trans. Amer. Math. Soc. **331** (1992), 181–201.
[7] S. G R. M , *On the multiplicative structure of the Rao module of space curves lying on a smooth quadric*, Comm. Algebra **30** (2002), 2445–2467.
[8] R. H , *Generalized divisors on Gorenstein schemes*, K-Theory **8** (1994), 287–339.
[9] R. H E. S , *Curves in the double plane*, Comm. Algebra **28** (2000), 5655-5676.
[10] J. K , J. M , R.M. M ´-R , U. N , C. P , *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc. **154** (2001), No. 732.
[11] M. M -D D. P , *Sur la classification des courbes gauches*, Astérisque **184–185**, Société, Mathématique de France, 1990.
[12] M. M -D D. P , *Sur les bornes du module de Rao*, C. R. Acad. Sci. Paris Sér. **317** (1993), 1159–1162.
[13] M. Martin-Deschamps - D. Perrin, *Le schema de Hilbert de courbes localement de Cohen-Macaulay n'est (presque) jamais reduit*, Ann. Sci. Ec. Norm. Sup **29** (1996), 757–785.
[14] J. M , *Introduction to Liaison theory and deficiency modules*, Progress in Mathematics **165**, Birkhäuser, 1998.
[15] U. N , *Non-degenerate curves with maximal Hartshorne-Rao module*, Math. Z. **244** (2003), 753–773.
[16] U. N , *Even liaison classes generated by Gorenstein linkage*, J. Algebra **209** (1998), 543–584.
[17] R. S , *Biliaison classes if curves in $\mathbb{P}^3$*, Proc. Amer. Math. Soc. **132** (2004), 649–658.
[18] C. W , *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, 1994.

D M A "R. C ", U ` S N "F II", C U M S. A - 80126 N , I
E-mail address: digennar@unina.it

D M , U K , 715 P O T , L , KY 40506-0027, USA
E-mail address: uwenagel@ms.uky.edu