THE HILBERT SCHEME OF DEGREE TWO CURVES AND CERTAIN ROPES

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Abstract. We study families of ropes of any codimension that are supported on lines. In particular, this includes all non-reduced curves of degree two. We construct suitable smooth parameter spaces and conclude that all ropes of fixed degree and genus lie in the same component of the corresponding Hilbert scheme. We show that this component is generically smooth if the genus is small enough unless the characteristic of the ground field is two and the curves under consideration have degree two. In this case the component is non-reduced.

Contents

1. Introduction

In this paper we study families of certain curves of small degree and arbitrary codimension including all curves of degree two. By a curve we always mean a closed locally Cohen-Macaulay projective subscheme of pure dimension one. If the arithmetic genus of a degree two curve is small enough then it is non-reduced, supported on a line, and contained in the second infinitesimal neighborhood of the line. Curves with these properties are called ropes. We construct smooth parameter spaces for ropes and use them to study Hilbert schemes that contain ropes.

There are only few results about general properties of the Hilbert scheme as its existence [3], its connectedness [4], or its radius [16]. More is known about Hilbert schemes parametrizing particular schemes that often have small codimension (cf., e.g., [15], [8], [2], [7], [11], [12]). But, for example, it is not even known if the locus of space curves inside its Hilbert scheme is connected (cf., e.g., [5], [14]). In this note we also consider particular schemes, certain curves of low degree, but we allow for any codimension.

Curves of degree one, i.e. lines, are parametrized by Grassmanians. Here, we study the next case(s). A curve of degree two is either an irreducible conic, a pair of two lines, or a double line. If such a curve is not planar its (arithmetic) genus is at most -1. If it is equal to -1 then the Hilbert scheme is well understood. Its general curve is a pair of skew

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lines. Somewhat surprisingly, if the genus is smaller then the characteristic of the ground field enters the picture. Let $Hilb_{d,g}(\mathbb{P}^n_K)$ denote the Hilbert scheme of 1-dimensional subschemes of $\mathbb{P}^n_K$ having degree d and arithmetic genus g where K is an algebraically closed field. Of course, $Hilb_{d,g}(\mathbb{P}^n_K)$ does not contain only curves. Part of our results on degree two curves (cf. Theorem 6.13) is summarized in the following statement.

Theorem 1.1. *Let $g \leq -2$, $n \geq 3$ be integers. Then there is a unique component, say $H_{2,g}(\mathbb{P}^n_K)$, of $Hilb_{2,g}(\mathbb{P}^n_K)$ that parametrizes all curves in $Hilb_{2,g}(\mathbb{P}^n_K)$. This component is generically smooth if and only if either $\operatorname{char} K \neq 2$ or $\operatorname{char} K = 2$ and $g = -2$. If $\operatorname{char} K = 2$ and $g \leq -3$ then $H_{2,g}(\mathbb{P}^n_K)$ is non-reduced.*

Even in the classical case of space curves, i.e. $n = 3$, this results seems new in case of characteristic two.

The general curve of $H_{2,g}(\mathbb{P}^n_K)$ is a double line if $g \leq -2$. Note that a double line is a rope of degree two. The result above is a particular case of our effort to understand families of ropes that are supported on a line. In general, it turns out that all ropes in $\mathbb{P}^n$ of fixed degree d and (arithmetic) genus g lie in the same irreducible component of $Hilb_{d,g}(\mathbb{P}^n_K)$ (cf. Lemma 6.6). We denote this component by $H_{d,g}(\mathbb{P}^n_K)$. If $d \geq 3$ then there are many more curves of degree d than just ropes. But we show, as the case of degree two curves suggests, if the genus is small enough then the general curve of $H_{d,g}(\mathbb{P}^n_K)$ is a rope. Moreover, we have.

Theorem 1.2. *Let d, g, n be integers such that $3 \leq d \leq n-1$. If $g \leq \min\{-3(d-1), d-n\}$ then the component $H_{d,g}(\mathbb{P}^n_K)$ is generically smooth of dimension $(n-1)(d+1-g)-(d-1)^2$.*

For a more precise result we refer to Theorem 6.14. It is independent of the characteristic of the ground field K.

A crucial ingredient of the results above is the computation of the global sections of the normal sheaf of a rope C. However, we were not able to compute $h^0(C, \mathcal{N}_C)$ by using general methods like vanishing theorems. Indeed, the dependence on the characteristic suggests that this might be impossible. Instead we use a very concrete approach based on our previous results in [10]. It requires information on the minimal free resolution of the rope and its structure sheaf.

Let $C \subset \mathbb{P}^n$ be a rope of degree $n-k < n$ supported on the line L. Then C is intimately related to two matrices A, B with entries in the coordinate ring S of L that fit into an exact sequence

$$0 \to \oplus_{j=1}^{k} \mathcal{O}_L(-\beta_j - 1) \xrightarrow{B} \mathcal{O}_L^{n-1}(-1) \xrightarrow{A} \oplus_{i=0}^{r-k} \mathcal{O}_L(\alpha_i - 1) \to 0.$$

Invoking a change of coordinates we may assume that the line L is defined by the ideal $(x_0, \ldots, x_{n-2})$. Then, according to [10], Theorem 2.4, the homogeneous ideal of the rope is

$$I_C = ((I_L)^2,\ [x_0, \ldots, x_{n-2}] \cdot B).$$

Using this information we compute in Section 3 the graded minimal free resolution of C including a description of the maps (Theorem 3.4). This is achieved by induction on the degree of the rope and based on a *non-minimal* resolution of the ideal $(I_L)^2$.

Section 4 is devoted to determine the global sections of the structure sheaf $\mathcal{O}_C$ of the rope C. Using the matrix A associated to C we construct explicitly global sections that form a basis of $H^0_*(C, \mathcal{O}_C) := \oplus_{j \in \mathbb{Z}} H^0(C, \mathcal{O}_C(j))$ as S-module (cf. Theorem 4.5). This generalizes Migliore's result ([9]) for double lines in $\mathbb{P}^3$. It also serves as the basis for determining the module structure of $H^0_*(C, \mathcal{O}_C)$ over the coordinate ring $R = K[x_0. \ldots, x_n]$ of $\mathbb{P}^n$. Actually, we even compute the minimal free resolution of $H^0_*(C, \mathcal{O}_C)$ in Proposition 4.7.

Let

$$0 \to G_n \xrightarrow{d'_n} \ldots \longrightarrow G_2 \xrightarrow{d'_2} G_1 \xrightarrow{d'_1} I_C \to 0$$

be the minimal free resolution of C. Then the global sections of its normal sheaf can be computed by

$$H^0(C, \mathcal{N}_C) = [\ker(\mathrm{Hom}(G_1, H^0_*(C, \mathcal{O}_C)) \xrightarrow{(d'_2)^*} \mathrm{Hom}(G_2, H^0_*(C, \mathcal{O}_C)))]_0$$

This amounts to solving a system of linear equations over the polynomial ring R and is the content of Section 5. We are able to do this if C has degree two, i.e. for a double line. In this case, the solutions of the linear system have a particular property that we formalize in Condition 5.7. For an arbitrary rope C we compute all solutions of the linear system that satisfy Condition 5.7.

In Section 6 we turn to families of ropes. Using the matrices A, B associated to a rope C we construct two types of smooth parameter spaces (cf. Propositions 6.2 and 6.3). Then a deformation argument allows to conclude that all ropes of fixed degree and genus lie in the same irreducible component $H_{d,g}(\mathbb{P}^n)$ of the corresponding Hilbert scheme. Combined with the information of Section 5 on the tangent sheaf of $Hilb_{2,g}(\mathbb{P}^n)$ at a point corresponding to a double line we obtain in particular Theorem 1.1.

For ropes of degree $d \geq 3$ the situation is more complicated because the dimension of the corresponding tangent space depends on the rope. In Proposition 6.9 we determine the ropes with smallest cohomology. As expected by semicontinuity, it turns out that these ropes are indeed general in $H_{d,g}(\mathbb{P}^n)$. Moreover, we show that the global sections of the normal sheaf of a general rope satisfy Condition 5.7. This leads to Theorem 1.2.

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2. Characterizations of ropes on a line

In this section we introduce some notation and recall results from [10] that we use later on.

Throughout the paper a curve will always be a locally Cohen-Macaulay 1-dimensional projective subscheme. A rope C is a curve that is supported on a smooth curve Y such that its homogeneous ideal I_C contains I_Y^2. In this note we will only consider ropes whose support is a line. We often will refer to these curves just as ropes, not making the assumption on the support explicit. A d-rope is just a rope of degree d. We may and will assume that the supporting line L is defined by the ideal $I_L = (x_0, \ldots, x_{n-2})$. In order to stress the particular role played by the line we denote the coordinate ring of $\mathbb{P}^n$ by $R = K[x_0, \ldots, x_r, t, u]$ where $r := n-2$ and K is an arbitrary field. Then the coordinate ring of the line L is $S := K[t, u]$.

The homogeneous ideal of a rope has been characterized in [10], Theorem 2.4.

Theorem 2.1. *Let $C \subset \mathbb{P}^n$ be a curve of degree at most $n-1$. Then the following conditions are equivalent:*

1. *C is an $(n-k)$–rope supported on the line L;*
2. *$I_C = ((I_L)^2, [x_0, \ldots, x_r]B)$ where the matrix B gives a map $\varphi_B : \oplus_{j=1}^k S(-\beta_j - 1) \to S^{r+1}(-1)$ with $\mathrm{codim}(I_k(B)) = 2$;*
3. *$I_C = ((I_L)^2, F_1, \ldots, F_k)$ where $V(F_1, \ldots, F_k) \subset \mathbb{P}^n$ is a scheme of codimension k which contains L and is smooth at the points of L.*

Actually, there is not only the matrix B that plays an important role.

Remark 2.2. According to [10], Remark 2.7, every $(n-k)$–rope supported on the line L is related to matrices A, B with entries in S such that there is an exact sequence

$$0 \to \oplus_{j=1}^{k} \mathcal{O}_{\mathbb{P}^1}(-\beta_j - 1) \xrightarrow{B} \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \xrightarrow{A} \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1) \to 0 \tag{1}$$

where the ideals of maximal minors $I_k(B)$ and $I_{r+1-k}(A)$ have codimension 2. Note that A^t is just the syzygy matrix of B^t.

For geometric characterizations of ropes we refer to [10], Remark 2.10. Now, we state a further description.

Lemma 2.3. *A curve $C \subset \mathbb{P}^n$ is a rope if and only if there is a line L such that for every hyperplane $H \subset \mathbb{P}^n$ containing L there is an exact sequence*

$$0 \to \mathcal{I}_L(-1) \to \mathcal{I}_C \to \mathcal{I}_{C\cap H, H} \to 0.$$

Proof. If C is a rope the condition is clearly satisfied. Conversely, the exact sequences provide $I_L^2 \subset I_C$. □

By the genus of a curve we always means its arithmetic genus. The genus of a rope is easily computed because in [10], Lemma 2.6, Proposition 2.7, and Corollary 2.8, it is shown.

Proposition 2.4. *Using the notation above, an $(n-k)$–rope $C \subset \mathbb{P}^n$ has genus*

$$g = -\sum_{j=1}^{k} \beta_j = -\sum_{i=0}^{r-k} \alpha_i \leq 0.$$

Moreover, if C is non-degenerate then $g \leq -k = \deg C - n$.

3. Minimal free resolutions of ropes on a line

In this section, we determine a minimal free resolution of a rope C supported on a line L. Furthermore, we compute the Hilbert function as well as the Hartshorne-Rao module $M(C)$ of C. We use the notation of the previous section. Recall in particular that $r = n - 2$.

We begin by showing an interesting property of the generators of the homogeneous ideal I_C of a rope C.

Lemma 3.1. *If $I_C = ((I_L)^2, F_1, \ldots, F_k)$ is the homogeneous ideal of an $(n-k)$–rope $C \subset \mathbb{P}^n$ supported on the line L, then we have*

$$((I_L)^2, F_1, \ldots, F_{h-1}) \cap (F_h) = (F_h) \cdot I_L$$

for each $h = 1, \ldots, k$.

Proof. Recall that $I_L = (x_0, \ldots, x_r)$. According to Theorem 2.1 it is trivial that $x_i F_h \in (I_L)^2$ for every $i = 0, \ldots, r$, and so $(F_h) \cdot I_L \subseteq ((I_L)^2, F_1, \ldots, F_{h-1})$. Vice versa, assume $GF_h \in ((I_L)^2, F_1, \ldots, F_{h-1})$. We may write G as $G = G' + G''$ where $G' \in I_L$ and $G'' \in k[t,u]$. Since $G'F_h \in (I_L)^2$ we get $G''F_h \in ((I_L)^2, F_1, \ldots, F_{h-1})$. But the fact that $G'' \in k[t,u]$ implies $G''F_h \in (F_1, \ldots, F_{h-1})$. Since $(F_1, \ldots, F_h)$ is a regular sequence (cf. Theorem 2.1), we get that $G'' = 0$. The claim follows. □

The previous proposition ensures that the sequences

$$0 \longrightarrow I_L(-\beta_h - 1) \longrightarrow I_{h-1} \oplus (F_h) \longrightarrow I_h \longrightarrow 0$$

are exact, where $I_h = ((I_L)^2, F_1, \ldots, F_h)$, $h = 1, \ldots, k$, and $I_0 = (I_L)^2$. Hence, one can use the mapping cone procedure to compute a free resolution of I_h from one of I_{h-1} if one is able to find the comparison maps. Indeed, we will carry out this program. The starting point for the inductive procedure is a free resolution of $(I_L)^2$. The minimal free resolution of $(I_L)^2$ is given by an Eagon–Northcott complex. However, we were not able to determine the comparison maps between this Eagon–Northcott complex and the Koszul complex which resolves I_L. In order to overcome this problem we use a non-minimal free resolution of $(I_L)^2$. Probably, this resolution is known to specialists, but we have no reference for it. In order to state it we need some notation. We write the Koszul complex $P_\bullet$ which resolves I_L as

$$P_\bullet \qquad\qquad 0 \longrightarrow \wedge^{n-1}P \xrightarrow{\delta_{n-1}} \cdots \wedge^2 P \xrightarrow{\delta_2} P \xrightarrow{\delta_1} I_L \longrightarrow 0$$

where $P = R^{n-1}(-1) = \oplus_{i=0}^{r} Re_i$ and $\delta_1(e_i) = x_i$ for every i.

Proposition 3.2. *The ideal $(I_L)^2$ has the following non-minimal free resolution*

$$0 \longrightarrow \begin{matrix} \wedge^{n-1}P\otimes P \\ \oplus \\ \wedge^{n-1}P \end{matrix} \xrightarrow{\partial'_{n-1}} \cdots \begin{matrix} \wedge^{2}P\otimes P \\ \oplus \\ \wedge^{2}P \end{matrix} \xrightarrow{\partial'_2} P\otimes P \xrightarrow{\partial'_1} (I_L)^2 \longrightarrow 0$$

where

$$\partial'_i = \begin{pmatrix} \delta_i \otimes \mathrm{id}_P & (-1)^i \partial_i \\ 0 & \delta_i \end{pmatrix}, \qquad i = 3, \ldots, n-1,$$

$$\partial'_2 = (\delta_2 \otimes \mathrm{id}_P \quad \partial_2), \qquad \partial'_1 = \delta_1 \otimes \delta_1$$

and $\partial_i : \wedge^i P \to \wedge^{i-1} P \otimes P$ is the canonical map defined by

$$\partial_i(u_1 \wedge \cdots \wedge u_i) = \sum_{j=1}^{i} (-1)^{j+1} u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_i \otimes u_j.$$

Proof. Consider the following surjective map

$$\alpha : I_L \otimes P \longrightarrow (I_L)^2$$

defined by $\alpha(x_i \otimes e_j) = x_i x_j$ for all i, j. The kernel of the map α is generated by the trivial relations $x_i \otimes e_j - x_j \otimes e_i$. Moreover, if we embed $I_L \otimes P$ in $R \otimes P \cong P$ we see that $\ker \alpha$ is nothing else than $\operatorname{im} \delta_2$. Therefore, we have the following exact sequence

$$0 \longrightarrow \operatorname{im} \delta_2 \longrightarrow I_L \otimes P \longrightarrow (I_L)^2 \longrightarrow 0.$$

A free resolution of $\operatorname{im} \delta_2$ is provided by the Koszul complex $P_\bullet$, while $I_L \otimes P$ is resolved by $P_\bullet \otimes P$. It is easy to check that the following diagram is commutative

$$(+) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im}\delta_2 & \longrightarrow & I_L \otimes P & \longrightarrow & (I_L)^2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^2 P & \xrightarrow{\partial_2} & P \otimes P & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \vdots & & \vdots & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^{n-1} P & \xrightarrow{\partial_{n-1}} & \wedge^{n-2} P \otimes P & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & \wedge^{n-1} P \otimes P & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Now the mapping cone procedure (cf., e.g., [19]) implies our claim. □

In order to get a minimal free resolution of $(I_L)^2$ we have to cancel redundant terms in the resolution above. For stating this formally we need some more notation.

Notation 3.3. Since the map $\partial_i : \wedge^i P \to \wedge^{i-1}P \otimes P$ is split-injective we can identify $(\wedge^{i-1}P \otimes P)/\partial_i(\wedge^i P)$ with the free R-module, say D_{i-1}, being a direct summand of $\wedge^{i-1}P \otimes P$. Furthermore, the previous diagram (+) shows that $\delta_{i-1} \otimes \mathrm{id}_P : \wedge^{i-1}P \otimes P \to \wedge^{i-2}P \otimes P$ induces a homomorphism

$$d_{i-1} : D_{i-1} \to D_{i-2}, \quad i \geq 3.$$

Similarly, putting $D_0 = R$ the map $\delta_1 \otimes \mathrm{id}_P$ induces a homomorphism $d_1 : D_1 \to (I_L)^2$. Moreover, denote by

$$\tau_B : \ Q := \oplus_{j=1}^k R(-\beta_j - 1) \to P$$

the extension of φ_B. Finally, define for $i \geq 2$

$$\mu_i : \wedge^{i-1}P \otimes Q \to D_{i-1}$$

to be the composition $\wedge^{i-1}P \otimes Q \to \wedge^{i-1}P \otimes P \to D_{i-1}$ and let μ_1 be the composition $Q \xrightarrow{\varphi_B} P \xrightarrow{\delta_1} R$.

The diagram (+) implies that

$$0 \to D_{n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow D_2 \xrightarrow{d_2} D_1 \xrightarrow{d_1} (I_L)^2 \to 0$$

is an exact sequence. Comparison with the Eagon–Northcott complex shows that it is a minimal free resolution of $(I_L)^2$. This is the starting point for the description of the minimal free resolution of an arbitrary rope on L.

Theorem 3.4. *Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope supported on the line L with homogeneous ideal $I_C = ((I_L)^2, [x_0, \ldots, x_r]B)$. Then we have the following free resolution of I_C:*

$$G_\bullet \qquad\qquad 0 \to G_n \xrightarrow{d'_n} \ldots \longrightarrow G_2 \xrightarrow{d'_2} G_1 \xrightarrow{d'_1} I_C \to 0$$

where

$$G_i := D_i \oplus (\wedge^{i-1}P \otimes Q),$$

and

$$d'_i := \begin{pmatrix} d_i & (-1)^i \mu_i \\ 0 & \delta_{i-1} \otimes \mathrm{id}_Q \end{pmatrix} \qquad \text{if } 2 \leq i \leq n-1,$$

and

$$D_n := 0, \qquad d'_1 := \begin{pmatrix} d_1 & -\mu_1 \end{pmatrix}, \quad d'_n := \begin{pmatrix} (-1)^n \mu_n \\ \delta_{n-1} \otimes \mathrm{id}_Q \end{pmatrix}.$$

This resolution is minimal if and only if the rope $C \subset \mathbb{P}^n$ is non-degenerate.

Proof. We prove the claim by induction on k. For $k = 0$ the claim is shown by the discussion above. Notice that $G_n = 0$ in this case.

Let $k \geq 1$. Write $I_C = ((I_L)^2, F_1, \ldots, F_k)$ and define $\tilde{C} \subset \mathbb{P}^n$ to be the $(n-k+1)$–rope defined by $\tilde{I} = ((I_L)^2, F_1, \ldots, F_{k-1}) = ((I_L)^2, [x_0, \ldots, x_r]\tilde{B})$, i.e. the matrix $\tilde{B}$ is obtained from B by deleting its last column. Applying the induction hypothesis to $\tilde{C}$ we denote the maps and modules in the free resolution $\tilde{G}_\bullet$ of $\tilde{I}$ depending on k by $\tilde{Q}, \tilde{\mu}_i$. Note that $Q = \tilde{Q} \oplus R(-\beta_k - 1)$. Denote by $\tau : R(-\beta_k - 1) \to P$ the map such that $\tau_B = (\tau_{\tilde{B}}, \tau)$.

Using Lemma 3.1 we have the short exact sequence

$$0 \to I_L(-\beta_k - 1) \to \tilde{I} \oplus (F_k) \to I_C \to 0.$$

Define homomorphisms $\gamma_i : \wedge^i P \to D_i \oplus (\wedge^{i-1} P \otimes \tilde{Q})$, $i = 2, \dots, n-1$ and $\gamma_1 : P \to D_1 \oplus \tilde{Q} \oplus R(-\beta_k - 1)$ as the sum of the composition

$$\wedge^i P(-\beta_k - 1) \xrightarrow{\mathrm{id}_{\wedge^i P} \otimes \tau} \wedge^i P \otimes P \twoheadrightarrow D_i, \quad i = 1, \dots, n-1,$$

and the zero maps $\wedge^i P(-\beta_k - 1) \to \wedge^{i-1} P \otimes \tilde{Q}$ and (for $i = 1$) the map $P(-\beta_k - 1) \xrightarrow{\delta_1} R(-\beta_k - 1)$. A routine computation shows that $\gamma_\bullet$ is a homomorphism between $P_\bullet$ and the direct sum of $\tilde{G}_\bullet$ and the minimal free resolution of (F_k). Hence the mapping cone procedure implies that $G_\bullet$ is a free resolution of I_C. The description of the maps shows that $G_\bullet$ is a minimal free resolution if and only if the matrix B does not contain non-zero elements of K as entries, which is equivalent to C being non-degenerate (cf. Theorem 2.1). □

Remark 3.5. Let $C \subset \mathbb{P}^n$ be a degenerate rope. Denote by $L \subset \mathbb{P}^n$ the smallest linear subspace containing C. Then Theorem 3.4 gives the minimal free resolution of I_C as module over the coordinate ring of L. From this resolution it is easy to derive the minimal free resolution of I_C as module over the coordinate ring R of $\mathbb{P}^n$. In this sense Theorem 3.4 provides the minimal free resolution for *all* ropes on a line.

From the free resolution $G_\bullet$ of I_C we easily derive some consequences. We use the convention that $\binom{a}{b} := 0$ for integers a, b such that $a < b$.

Corollary 3.6. *Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope C supported on a line. Then we have:*

(i) *The Hilbert function h_C of C is*

$$h_C(j) = \binom{j+1}{1} + (r+1)\binom{j}{1} - \sum_{h=1}^{k} \binom{j - \beta_h}{1}.$$

(ii) *The Castelnuovo-Mumford regularity of C is* $\max\{\beta_j + 1 \mid 1 \leq j \leq k\}$, *i.e. it equals the maximal degree of a minimal generator of I_C.*

Proof. (i) follows from two easy computations:

(1) the Hilbert function of the scheme associated to $(I_L)^2$ is the alternating sum of the ranks of the modules of the Eagon-Northcott complex, thus it is equal to $\binom{j+1}{1} + (r+1)\binom{j}{1}$;

(2) the alternating sum of the ranks of the modules of the twisted Koszul complex $P_\bullet(-\beta_h)$ is equal to $\binom{j-\beta_h}{1}$.

(ii) is a consequence of Theorem 3.4 because the Castelnuovo-Mumford regularity can be read off from a mimimal free resolution. □

In [10], Proposition 3.1 we have computed the Hartshorne-Rao module of a rope:

Proposition 3.7. *Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope supported on the line L with homogeneous ideal $I_C = ((I_L)^2, [x_0, \dots, x_r]B)$. Then its Hartshorne-Rao module $H^1_*(\mathcal{I}_C)$ is as R–module isomorphic to coker$(S^{r+1}(-1) \xrightarrow{\varphi_A} \oplus_{j=0}^{r-k} S(\alpha_j - 1))$.*

It is amusing to derive this result from Theorem 3.4. Indeed, since P has rank $n-1$ the last map d'_n in the resolution $\tilde{G}$ can be written as

$$d'_n = \begin{pmatrix} (-1)^n \tau_B(-n+1) \\ \delta_{n-1} \otimes \mathrm{id}_Q \end{pmatrix}.$$

It follows that

$$H^1_*(\mathcal{I}_C) \cong \mathrm{Ext}^{n-1}_R(I_C, R)^\vee(n+1) \cong (\mathrm{coker}(d'_n)^*)^\vee(n+1) \cong (\mathrm{coker}\varphi_B^*)^\vee(2) \cong \mathrm{coker}\varphi_A$$

as claimed.

For later reference we state an immediate consequence of the description of the Hartshorne-Rao module.

Corollary 3.8. *The Rao function $\rho_C : \mathbb{Z} \to \mathbb{Z}$, $\rho_C(i) = h^1_*(\mathcal{I}_C(i))$ of an $(n-k)$–rope $C \subset \mathbb{P}^n$ supported on L is*

$$\rho_C(i) = \sum_{j=0}^{r-k} \binom{i+\alpha_j}{1} + \sum_{j=1}^{k} \binom{i-\beta_j}{1} - (r+1)\binom{i}{1}.$$

4. Global sections of the structure sheaf of a rope

In this section we compute the global sections of (all twists of) the structure sheaf of a rope C supported on a line, we describe the structure of $H^0_*(C, \mathcal{O}_C)$ both as S–module and as R–module, and finally we determine the minimal free resolution of $H^0_*(C, \mathcal{O}_C)$ as R–module.

We continue using the notation of the previous sections.

Remark 4.1. Given an $(n-k)$–rope $C \subset \mathbb{P}^n$ we want to construct $r+1-k = n-1-k$ global sections of its structure sheaf $\mathcal{O}_C$ which are naturally associated to the matrix A related to C (cf. Remark 2.2).

We will construct these sections locally. To this end let $I = \{i_1, \ldots, i_k\} \subset \{0, \ldots, r\}$ be a subset of k distinct elements and put $J := \{0, \ldots, r\} \setminus I = \{j_0, \ldots, j_{r-k}\}$ where we assume $j_0 < \ldots < j_{r-k}$. We denote by A_I the submatrix of A formed by the columns of A with index in I. Similarly, let B_I be the submatrix of B consisting of the rows of B with index in I. The matrices A_J, B_J are analogously defined.

Consider the open subset

$$U_J := \{P \in \mathbb{P}^n \mid \det A_J(P) \neq 0\}.$$

Since the determinants of A_J and B_I agree up to a sign (cf. [1]) the matrix A_J is invertible over $\mathcal{O}_C(U)$ if and only if B_I has this property.

Now we define local sections $z^J_0, \ldots, z^J_{r-k} \in \mathcal{O}_C(U_J)$ by

$$(z^J_0, \ldots, z^J_{r-k}) = (x_{j_0}, \ldots, x_{j_{r-k}})A_J^{-1}.$$

Here we abuse notation by writing simply $\mathcal{O}_C(U_J)$ instead of $(\mathcal{O}_C \otimes \mathcal{O}_{\mathbb{P}^n}(d))(U_J)$ for a suitable integer d. Observe that $\deg z_i = 1 - \alpha_i$ $(0 \leq i \leq r-k)$. The next result shows that these local sections glue to global sections.

Lemma 4.2. *For all $i \in \{0, \ldots, r-k\}$ and all subsets $J, J' \subset \{0, \ldots, r\}$ of $r+1-k$ elements we have $z^J_i = z^{J'}_i$ in $\mathcal{O}_C(U_J \cap U_{J'})$.*

Proof. Since

$$0 = A \cdot B = (A_I \mid A_J) \cdot \begin{pmatrix} B_I \\ B_J \end{pmatrix}$$

we get

$$(*) \qquad A_J^{-1} A_I = -B_J B_I^{-1}.$$

Hence $[x_0, \ldots, x_r]B \subset I_C$ implies

$$x_I B_I + x_J B_J = [x_0, \ldots, x_r]B = 0$$

in $\mathcal{O}_C(U_J \cap U_{J'})$ where x_I, x_J are the suitable rows of variables. It follows

$$\begin{aligned} x_I &= -x_J B_J B_I^{-1} \\ &= x_J A_J^{-1} A_I \quad \text{(by } (*)) \\ &= (z^J_0, \ldots, z^J_{r-k}) A_I \quad \text{(by definition of } z^J_i). \end{aligned}$$

Putting $z^J = (z_0^J, \dots, z_{r-k}^J)$ we have just shown $x_I = z^J A_I$. The definition of z^J implies $x_J = z^J A_J$. Hence we obtain $[x_0, \dots, x_n] = z^J A$ and in particular $x_{J'} = z^J A_{J'}$. Plugging this into the definition of $z^{J'}$ we get

$$z^{J'} = x_{J'} A_{J'}^{-1} = z^J A_{J'} A_{J'}^{-1} = z^J$$

as claimed. □

Notation 4.3. Since the matrix A has maximal rank at all points of the line $L \cong \mathbb{P}^1$ the open subsets U_J cover the rope C if J varies in $\{0, \dots, r-k\}$. Therefore, the local sections $z_0^J, \dots, z_{r-k}^J \in \mathcal{O}_C(U_J)$ glue together to global sections of $\mathcal{O}_C$ which we denote by $z_0, \dots, z_{r-k}$. Note that $\deg z_i = 1 - \alpha_i$ (cf. the previous remark).

Remark 4.4. The end of the proof of the previous lemma shows that we have in $H^0_*(C, \mathcal{O}_C)$

$$[x_0, \dots, x_r] = [z_0, \dots, z_{r-k}]A.$$

We will use this fact soon.

Using the global sections just defined we can describe all global sections of $\mathcal{O}_C(d)$ for all $d \in \mathbb{Z}$. This generalizes Migliore's result for double lines in $\mathbb{P}^3$ (cf. [9], Proposition 3.1).

Theorem 4.5. *Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope supported on the line $L = \operatorname{Proj} S$. Then its global sections of degree d are*

$$H^0(C, \mathcal{O}_C(d)) = \{P + \sum_{i=0}^{r-k} Q_i z_i \mid P, Q_i \in S, \deg(P) = d, \deg(Q_i) = d + \alpha_i - 1\}.$$

Furthermore, $H^0_*(C, \mathcal{O}_C)$ is as S–module a free module with basis $\{1, z_0, \dots, z_{r-k}\}$.

Proof. The first step is to show that $1, z_0, \dots, z_{r-k}$ generate a free S–submodule of $H^0_*(C, \mathcal{O}_C)$. Assume that there are $f_{-1}, f_0, \dots, f_{r-k} \in S$ which are not all trivial such that

$$0 = f_{-1} + \sum_{i=0}^{r-k} f_i z_i$$

in $H^0(C, \mathcal{O}_C(d))$ for some d. We choose $J \subset \{0, \dots, r-k\}$ such that $U_J \neq \emptyset$. Then we have in $\mathcal{O}_C(U_J)$ using Remark 4.4

$$0 = f_{-1} + x_J \cdot A_J^{-1} \cdot \begin{pmatrix} f_0 \\ \vdots \\ f_{r-k} \end{pmatrix}.$$

The exact sequence

$$0 \to \mathcal{I}_C(U_J) \to \mathcal{O}_{\mathbb{P}^n}(U_J) \to \mathcal{O}_C(U_J) \to 0$$

and the description of the homogeneous ideal of C (Theorem 2.1) imply that there are $g_1, \dots, g_k \in R_{(\det A_J)}$ such that we have in $\mathcal{O}_{\mathbb{P}^n}(U_J)$

$$\begin{aligned} f_{-1} + x_J \cdot A_J^{-1} \cdot \begin{pmatrix} f_0 \\ \vdots \\ f_{r-k} \end{pmatrix} &= [x_0, \dots, x_n] \cdot B \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} \\ &= x_I \cdot B_I \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} + x_J \cdot B_J \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}. \end{aligned}$$

It follows $f_{-1} = 0$,

$$B_I \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} = 0 \quad \text{and} \quad A_J^{-1} \cdot \begin{pmatrix} f_0 \\ \vdots \\ f_{r-k} \end{pmatrix} = B_J \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}.$$

Since the matrix B_I is invertible in $\mathcal{O}_C(U_J)$ we get $f_0 = \ldots = f_{r-k} = 0$, a contradiction. Thus, we have shown the second claim.

In the second step we compare dimensions. Put $V_d = \{P + \sum Q_i z_i \mid \deg(P) = d, \deg(Q_i) = d + \alpha_i - 1\}$. Then we have

$$\dim V_d = \binom{d+1}{1} + \sum_{i=0}^{r-k} \binom{d+\alpha_i}{1}.$$

But using Corollaries 3.6 and 3.8 we get

$$h^0(C, \mathcal{O}_C(d)) = h_C(d) + \rho_C(d) = \binom{d+1}{1} + \sum_{i=0}^{r-k} \binom{d+\alpha_i}{1},$$

i.e. $\dim V_d = h^0(C, \mathcal{O}_C(d))$ for all d and our claims follow. □

Later on we will deal with the R–module structure of $H^0_*(C, \mathcal{O}_C)$. This structure is made partially explicit in the following statement.

Lemma 4.6. *Using the notation of Theorem 4.5 we have for all $j = 0, 1, \ldots, r$*

$$x_j(P + \sum_{i=0}^{r-k} Q_i z_i) = \sum_{i=0}^{r-k} (P a_{ij}) z_i.$$

Proof. By construction of the global sections z_l we have that $[x_0, \ldots, x_r] = [z_0, \ldots, z_{r-k}]A$. These relations specify the embedding $R/I_C \longrightarrow H^0_*(C, \mathcal{O}_C)$. But in R/I_C we have $x_i x_j = 0$ for every $i, j \le r$. Thus, we obtain

$$0 = x_j \cdot [x_0, \ldots, x_r] = x_j \cdot [z_0, \ldots, z_{r-k}]A.$$

But the matrix A^t defines an injective map. Hence, we get $x_j \cdot [z_0, \ldots, z_{r-k}] = 0$ i.e. $x_j z_l = 0$ for every $j = 0, \ldots, r$ and for every $l = 0, \ldots, r-k$. Our claim follows. □

We close this section with the computation of the minimal free resolution of $H^0_*(C, \mathcal{O}_C)$ as R–module. In particular, we determine the maps. This also completes the description of the R–module structure of $H^0_*(C, \mathcal{O}_C)$.

Proposition 4.7. *A graded minimal free resolution of the R–module $H^0_*(C, \mathcal{O}_C)$ is*

$$0 \longrightarrow \oplus_{i=-1}^{r-k} \wedge^{r+1} P(\alpha_i - 1) \xrightarrow{\varepsilon_{n-1}} \cdots \xrightarrow{\varepsilon_2} \oplus_{i=-1}^{r-k} P(\alpha_i - 1) \xrightarrow{\varepsilon_1}$$
$$\oplus_{i=-1}^{r-k} R(\alpha_i - 1) \xrightarrow{\varepsilon_0} H^0_*(C, \mathcal{O}_C) \longrightarrow 0$$

where $\alpha_{-1} = 1$ and, for $i \ge 1$,

$$\varepsilon_i = \begin{pmatrix} \delta_i & & & & \\ -\delta_i(A_{0,\bullet}) & \delta_i & & & \\ -\delta_i(A_{1,\bullet}) & & \delta_i & & \\ \vdots & & & \ddots & \\ -\delta_i(A_{r-k,\bullet}) & & & & \delta_i \end{pmatrix}$$

Here, the non-specified parts are zero, δ_i denotes the i-th map in the Koszul complex with respect to $\{x_0, \dots, x_r\}$ and $\delta_i(A_{j,\bullet})$ is the i-th map in the Koszul complex with respect to the elements of the j-th row of the matrix A.

Proof. Let $f_{-1}, f_0, \dots, f_{r-k}$ be the canonical basis of $\oplus_{i=-1}^{r-k} R(\alpha_i - 1)$. Then, the homomorphism

$$\varepsilon_0 : \oplus_{i=-1}^{r-k} R(\alpha_i - 1) \longrightarrow H^0_*(C, \mathcal{O}_C)$$

mapping f_{-1} onto 1 and f_i onto the global section z_i for $i \geq 0$ is clearly surjective. Using Remark 4.4 and Lemma 4.6 one checks that the sequence in the statement is a complex. The Buchsbaum-Eisenbud exactness criterion implies that (part of) it provides a minimal free resolution of $\operatorname{im} \varepsilon_1$. This allows to compute the Hilbert function of this module. Using Theorem 4.5, we see that the Hilbert functions of $\operatorname{im} \varepsilon_1$ and $\ker \varepsilon_0$ agree. It follows that $\operatorname{im} \varepsilon_1 = \ker \varepsilon_0$ completing the proof. □

5. Global sections of the normal sheaf

In this section we compute global sections of the normal sheaf of a rope. We obtain a complete description of these global sections only for double lines. But the partial results for arbitrary ropes are powerful enough for our applications later on. The proofs use heavily the description of the minimal free resolution of a rope C as obtained in Section 3. Some results require particular care when the characteristic of the ground field is two.

It is well-known (cf., e.g., [18]) that the normal sheaf of C is

$$H^0(C, \mathcal{N}_C) = [\ker(H^0(C, \operatorname{Hom}(\mathcal{G}_1, \mathcal{O}_C)) \overset{(d_2')^*}{\longrightarrow} H^0(C, \operatorname{Hom}(\mathcal{G}_2, \mathcal{O}_C)))]_0$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the sheafifications of G_1 and G_2, respectively, and the map d_2' is defined in Theorem 3.4. Equivalently, the normal sheaf can be described as

$$H^0(C, \mathcal{N}_C) = [\ker(\operatorname{Hom}(G_1, H^0_*(C, \mathcal{O}_C)) \overset{(d_2')^*}{\longrightarrow} \operatorname{Hom}(G_2, H^0_*(C, \mathcal{O}_C)))]_0 \tag{2}$$

and we use this second formulation because it is more suitable for the computation.

We now fix some notation that we will use throughout this section. Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope. Recall that $G_1 = S_2(P) \oplus R(-\beta_1 - 1) \oplus \cdots \oplus R(-\beta_k - 1)$ and that $P = R^{n-1}(-1) = \oplus_{i=0}^r Re_i$. Then we set for $\varphi \in \operatorname{Hom}(G_1, H^0_*(C, \mathcal{O}_C))$ (cf. Theorem 4.5)

- $P^{ij} + \sum_{l=0}^{r-k} Q_l^{ij} z_l := \varphi(e_i e_j)$ where $P^{ij} \in [S]_2$, $Q_l^{ij} \in [S]_{\alpha_l+1}$ $(0 \leq i, j \leq r)$;
- $P^s + \sum_{l=0}^{r-k} Q_l^s z_l := \varphi(f_s)$ where $P^s \in [S]_{\beta_s+1}$, $Q_l^s \in [S]_{\beta_s+\alpha_l}$ $(1 \leq s \leq k)$, and $f_1, \dots, f_k$ is the canonical basis of $Q = R(-\beta_1 - 1) \oplus \cdots \oplus R(-\beta_k - 1)$. (Notice that P^s is just a polynomial with index s. It is not the s-th power of P. A similar comment applies to Q_l^s.)

Furthermore, recall that $D_2 \cong \wedge^2 P \otimes P / \wedge^3 P$, $G_2 = D_2 \oplus P(-\beta_1 - 1) \oplus \cdots \oplus P(-\beta_k - 1)$ and that the map $d_2' : G_2 \to G_1$ acts as follows

- $d_2'([e_i \wedge e_j \otimes e_h]) = -x_i e_j e_h + x_j e_i e_h$ where we denote by $e_i e_j$ the class of $e_i \otimes e_j$ in D_1;
- $d_2'(e_j^s) = \sum_{i=0}^r b_{is} e_i e_j + x_j f_s$ where $e_j^s := e_j \otimes f_s$ denotes the $(j+1)$-st basis element of $P(-\beta_s - 1)$.

Then we obtain for the map $(d_2')^*(\varphi) = \varphi \circ d_2' : G_2 \to H^0_*(C, \mathcal{O}_C)$ using Lemma 4.6

- $(d_2')^*(\varphi)([e_i \wedge e_j \otimes e_h]) = \sum_{l=0}^{r-k} (-P^{jh} a_{li} + P^{ih} a_{lj}) z_l$;
- $(d_2')^*(\varphi)(e_j^s) = \sum_{i=0}^r b_{is} P^{ij} + \sum_{l=0}^{r-k} (\sum_{i=0}^r b_{is} Q_l^{ij} + P^s a_{lj}) z_l$.

Combined with Theorem 4.5 we get the following result.

Proposition 5.1. *Adopting the notation above we have $\varphi \in H^0(C, \mathcal{N}_C)$ if and only if the following conditions are satisfied*

1. $P^{ih}a_{lj} - P^{jh}a_{li} = 0$ *for every* $l = 0, \dots, r-k,$ $i, j, h = 0, \dots, r;$
2. $\sum_{i=0}^{r} b_{is}P^{ij} = 0$ *for every* $s = 1, \dots, k,$ $j = 0, \dots, r;$
3. $\sum_{i=0}^{r} b_{is}Q_l^{ij} + P^s a_{lj} = 0$ *for every* $s = 1, \dots, k,$ $l = 0, \dots, r-k,$ $j = 0, \dots, r.$

Remark 5.2. (i) The conditions above are linear systems over the ring S with unknowns P^{ij}'s (conditions 1. and 2.) and with unknowns P^s's and Q_l^{ij}'s (condition 3.).

(ii) In the conditions 1. - 3. the Q_l^s's appearing in the definition of $\varphi(f_s)$ are not involved. Thus, they are free parameters in the description of the global sections of $\mathcal{N}_C$.

Next, we will solve the linear systems given by 1. and 2. above. In case of degenerate ropes we need some further notation.

Notation 5.3. Let $C \subset \mathbb{P}^n$ be a rope spanning a $\mathbb{P}^m$. Then C is degenerate if and only if $m < n$. In this case, we may and will assume that the matrices A, B associated to C have the shape

$$A = (A' \ \ 0)$$

and

$$B = \begin{pmatrix} B' & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

where A' and B' are the matrices associated to the non-degenerate rope C considered as subscheme of $\mathbb{P}^m = \mathrm{Proj}(K[x_0, \dots, x_{m-2}, t, u])$, thus $\mathbb{P}^m = \{x_{m-1} = \dots = x_{n-2} = 0\}$.

Now, we are ready for solving systems 1. and 2.

Proposition 5.4. *If $C \subset \mathbb{P}^n$ is an $(n-k)$-rope on L, then we have for $\varphi \in H^0(C, \mathcal{N}_C)$*

(a) $(P^{00}, P^{01}, P^{11}) = c(a_{00}^2, a_{00}a_{01}, a_{01}^2)$ *for some $c \in K$ and $P^{ij} = 0$ for all other indices in case C has degree 2 and genus -1;*

(b) $P^{ij} = 0$ *for every $i, j = 0, \dots, r$, otherwise.*

Proof. First, we assume that C is non-degenerate. We distinguish two cases.

Case 1: Let $\deg(C) > 2$.

Assume on the contrary that $P^{jh} \neq 0$ for some j, h. We denote by $\mathbf{a}_0, \dots, \mathbf{a}_r$ the columns of the matrix A. Then part of the system 1. can be written as

$$P^{ih}\mathbf{a}_j = P^{jh}\mathbf{a}_i \qquad \text{for every } i, h = 0, \dots, r.$$

Since $P^{jh} \neq 0$ we get that $P^{ih} \neq 0$ for every i because every column $\mathbf{a}_i$ is not zero. The latter is true since otherwise one β_k would vanish contradicting the assumption that C is non-degenerate (cf. Theorem 2.1). It follows that the matrix A has rank 1 at a general point of L, but rank $A = r + 1 - k > 1$. This contradiction shows that $P^{ij} = 0$ for every $i, j = 0, \dots, r$.

Case 2: Let $\deg(C) = 2$.

In this case we consider the linear system 2. It can be written as

$$(P^{0j}, \dots, P^{rj}) \cdot B = 0 \qquad \text{for every } j = 0, \dots, r.$$

Since A^t is the syzygy matrix of B^t we get $(P^{0j}, \dots, P^{rj}) = c_j A$ for some $c_j \in S$. Notice that in case $\deg C = 2$ we have $A = (a_{00}, \dots, a_{0r})$. Moreover, recalling that $P^{ij} = P^{ji}$ we see that $c_i a_{0j} = c_j a_{0i}$, for each i, j. Note that every entry of A has degree $\alpha_0 \geq 1$. We distinguish three cases:

(1) $\alpha_0 \geq 3$;

(2) $\alpha_0 = 2$;
(3) $\alpha_0 = 1$.

We use the fact that every P^{ij} has degree 2. It implies in case (1) that every c_i is zero for degree reasons, thus $P^{ij} = 0$ for every i, j.

In case (2) the c_j's are in K. If one of them is not zero then all the entries of A differ by a scalar only. Thus, A has degree zero syzygies, i.e. C is degenerate. This contradiction shows the claim in this case, too.

In case (3) the rope C has genus -1, thus it is degenerate if $n \geq 4$ (cf. 2.4). But if $n = 3$, the system 1. and the system 2. read as

$$P^{00}a_{01} - P^{01}a_{00} = 0 \qquad \text{and} \qquad P^{01}a_{01} - P^{11}a_{00} = 0$$

because $A = (a_{00}, a_{01})$ and $B = (a_{01}, -a_{00})$ where a_{00} and a_{01} are two independent linear forms. Every solution of that system has the form

$$(P^{00}, P^{01}, P^{11}) = c(a_{00}^2, a_{00}a_{01}, a_{01}^2) \qquad \text{for some } c \in K$$

completing the proof for non-degenerate ropes.

Second, we assume that C is degenerate. Then using the system 2. and the block structure of its matrix B (cf. 5.3), we can reduce to the case of non-degenerate ropes. $\square$

Having resolved the systems 1. and 2. it remains to consider the linear system given by condition 3. in Proposition 5.1. We are not able to do this in full generality except in the case of double lines, i.e. 2–ropes. We use Notation 5.3.

Proposition 5.5. *Let C be a 2–rope of genus g. Then the polynomials P^s, Q_0^{ij} form a solution of the linear system 3. if and only if*

(i) *in case* $\operatorname{char} K \neq 2$*:*

$$P^s = \begin{cases} -\frac{1}{2}\sum_{j=0}^{m-2} \lambda_j b_{js} & \text{if } 1 \leq s \leq m-2 \\ -\lambda_s & \text{if } m-1 \leq s \leq n-2 = r \end{cases}$$

and

$$Q_0^{ij} = \begin{cases} \frac{1}{2}(\lambda_i a_{0j} + \lambda_j a_{0i}) & \text{if } 0 \leq i, j \leq m-2 \\ \lambda_j a_{0i} & \text{if } 0 \leq i \leq m-2 < j \leq r \\ 0 & \text{if } m-1 \leq i, j \leq r \end{cases}$$

where $\lambda_0, \ldots, \lambda_r \in S$ are any linear forms;

(ii) *in case* $\operatorname{char} K = 2$*:*

$$Q_0^{ij} = \begin{cases} 0 & \text{if } 0 \leq i = j \leq m-2 \\ \sum_{h=1}^{m-2} \operatorname{adj}((B_j')^t)_{ih} P^k & \text{if } 0 \leq i, j \leq m-2 \text{ and } i \neq j \\ a_{0i}P^j & \text{if } 0 \leq i \leq m-2 < j \leq r \\ 0 & \text{if } m-1 \leq i, j \leq r \end{cases} \qquad \text{provided } g \leq -2$$

whereas

$$Q_0^{ij} = \begin{cases} c' a_{0j}^2 & \text{if } i = j = 0, 1 \\ c' a_{0j} a_{0i} + a_{01}P^1 & \text{if } i = 0, j = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for some } c' \in K \text{ if } g = -1.$$

Here adj(_) *denotes an adjoint matrix, B'_j is the matrix obtained from B' by deleting its row with index j and M_{ih} is the entry at position (i,h) of the matrix M.*[1]

Proof. At first, we state some notation. C being a 2–rope we have $k=r$. Moreover, A is a matrix of type $1\times(r+1)$, while B is a matrix of type $(r+1)\times r$, and they are related (using the Hilbert-Burch theorem) by

$$a_{0j}=(-1)^j\det(B_j).$$

Observe that all a_{0j} are not zero because C is assumed to be non-degenerate.

As last piece of notation, we set $\mathbf{Q}^j=(Q_0^{0j},\dots,Q_0^{rj})^t$ and $\mathbf{P}=(P^1,\dots,P^r)^t$. Note that $Q_0^{ij}=Q_0^{ji}$ by their definition. Hence, the linear system 3. can be written as

$$B^t\mathbf{Q}^j+a_{0j}\mathbf{P}=\mathbf{0},\qquad j=0,\dots,r. \tag{3}$$

Since $a_{0j}=0$ if $m-1\le j\le n-2=r$ (cf. Notation 5.3) we get for $j=m-1,\dots,r$

$$\begin{pmatrix} B' & 0\\ 0 & I_{n-m}\end{pmatrix}\cdot\mathbf{Q}^j=0.$$

The last $n-m$ equations immediately imply $\begin{pmatrix} Q_0^{m-1,j}\\ \vdots\\ Q_0^{r,j}\end{pmatrix}=0$. In particular, symmetry $Q_0^{i,j}=Q_0^{j,i}$ holds if $m-1\le i,j\le r$.

The first $m-1$ equations of the last system above read as

$$(B')^t\cdot\begin{pmatrix} Q_0^{0,j}\\ \vdots\\ Q_0^{m-2,j}\end{pmatrix}=0.$$

Since $(A')^t$ is the syzygy matrix of $(B')^t$ it follows

$$\begin{pmatrix} Q_0^{0,j}\\ \vdots\\ Q_0^{m-2,j}\end{pmatrix}=\lambda_j(A')^t\qquad(m-1\le j\le r)$$

for some $\lambda_j\in[S]_1$ by comparing degrees.

For $j=0,\dots,m-2$ system (3) provides

$$(B')^t\cdot\begin{pmatrix} Q_0^{0,j}\\ \vdots\\ Q_0^{m-2,j}\end{pmatrix}+a_{0j}(B')^t\cdot\begin{pmatrix} P^1\\ \vdots\\ P^{m-2}\end{pmatrix}=0 \text{ and } \begin{pmatrix} Q_0^{m-1,j}\\ \vdots\\ Q_0^{r,j}\end{pmatrix}=-a_{0j}\begin{pmatrix} P^{m-1}\\ \vdots\\ P^{n-2}\end{pmatrix}.$$

Hence, symmetry implies if $m-1\le i\le r$ and $0\le j\le m-2$

$$\lambda_i a_{0j}=Q_0^{ji}=Q_0^{ij}=-a_{0j}P^i.$$

Since by assumption C spans $\mathbb{P}^m$ we must have $a_{0j}\neq0$ if $0\le j\le m-2$. It follows that

$$P^i=-\lambda_i\qquad\text{and}\qquad Q_0^{ij}=-a_{0i}P^j\qquad\text{if } m-1\le i\le r \text{ and } 0\le j\le m-2.$$

It remains to show the claims for the indices $i,j\le m-2$, i.e. to consider the non-degenerate rope in $\mathbb{P}^m$ associated to the matrices A',B'. In order to simplify notation we assume that C itself is non-degenerate, i.e. $n=m,\ A'=A,\ B'=B$.

[1] For convenience, the subscripts of the rows of B range from 0 to r.

Now, we will show that system (3) can be solved if we also consider $\mathbf{P}$ as given and only the $\mathbf{Q}^j$'s as unknowns. To this end we first determine a particular solution of the system (depending on $\mathbf{P}$) and then all the solutions of the associated homogeneous system.

• Particular solution of (3).

We will describe a solution satisfying $Q_0^{jj} = 0$. We denote by $\overline{\mathbf{Q}}^j$ the vector obtained from $\mathbf{Q}^j$ by deleting the entry Q_0^{jj}. Then, assuming $Q_0^{jj} = 0$, the system (3) becomes

$$B_j^t \cdot \overline{\mathbf{Q}}^j = -a_{0j} P = -(-1)^j \det(B_j)\mathbf{P}.$$

If we multiply both sides by $\operatorname{adj}(B_j^t)$ we can cancel $\det(B_j^t)$ because it is not zero (thus it is not a zero divisor). Hence, we get $\overline{\mathbf{Q}}^j = (-1)^{j+1}\operatorname{adj}(B_j^t)\mathbf{P}$. Therefore, a particular solution $\tilde{\mathbf{Q}}^j$ is given by inserting a zero entry after the j-th entry of $\overline{\mathbf{Q}}^j$.

• Solution of the associated homogeneous system $B^t\mathbf{Q}^j = 0,\ j = 0, \dots, r$.

The syzygy argument above shows that every solution is of the form $\mathbf{Q}^j = \lambda_j A^t$, where $\lambda_j \in [S]_1$.

Hence, the general solution of (3) is

$$\mathbf{Q}^j = \tilde{\mathbf{Q}}^j + \lambda_j A^t, \qquad j = 0, \dots, r. \tag{4}$$

But, as noted above, the polynomials Q_0^{ij} also satisfy $Q_0^{ij} = Q_0^{ji}$ for all $i, j \leq r$. For $i = 0$ this extra requirement reads as

$$-[\operatorname{adj}(B_0^t)]_{j-\mathrm{row}}\mathbf{P} + (-1)^j\lambda_0 \det(B_j) = (-1)^{j+1}[\operatorname{adj}(B_j^t)]_{0-\mathrm{row}}\mathbf{P} + \lambda_j \det(B_0).$$

Thus, we get

$$\{(-1)^j[\operatorname{adj}(B_j^t)]_{0-\mathrm{row}} - [\operatorname{adj}(B_0^t)]_{j-\mathrm{row}}\}\mathbf{P} = \lambda_j \det(B_0) - (-1)^j\lambda_0 \det(B_j).$$

If we denote by $B_{ij;h}$ the submatrix obtained from B by deleting rows i and j and the column h, we obtain

$$\lambda_j \det(B_0) - (-1)^j\lambda_0 \det(B_j) = (-1)^j \sum_{h=1}^{r} (-1)^h \det(B_{0j;h})(\lambda_j b_{jh} + \lambda_0 b_{0h})$$

and

$$\begin{aligned}(-1)^j[\operatorname{adj}(B_j^t)]_{0-\mathrm{row}} &- [\operatorname{adj}(B_0^t)]_{j-\mathrm{row}} = \\ &2(-1)^j[\det(B_{0j;1}), -\det(B_{0j;2}), \dots, (-1)^{r-1}\det(B_{0j;r})].\end{aligned}$$

Substituting it follows

$$\begin{aligned}&2[(-1)^{j+1}\det(B_{0j;1}), \dots, (-1)^{j+r}\det(B_{0j;r})] \cdot \mathbf{P} = \\ &\qquad \sum_{h=1}^{r} (-1)^{j+r+1}\det(B_{0j;h})(\lambda_j b_{jh} + \lambda_0 b_{0h}), \qquad j = 1, \dots, r.\end{aligned} \tag{5}$$

Now, we distinguish according to the characteristic.

Case 1: Assume char $K \neq 2$.

Using matrix notation the previous equations read as

$$2\operatorname{adj}(B_0^t)\mathbf{P} = \mathbf{n}$$

where $\mathbf{n}$ is defined in the obvious way. Multiplying both sides by B_0^t provides

$$2a_{00}\mathbf{P} = B_0^t\mathbf{n}.$$

If we compute the s-th entry of both sides we obtain

$$2a_{00}P^s = \sum_{j=1}^{r} b_{js} \sum_{h=1}^{r} (-1)^{h+j+1} \det(B_{0j;h})(\lambda_j b_{jh} + \lambda_0 b_{0h}) = -a_{00} \sum_{j=0}^{r} \lambda_j b_{js}.$$

Being $\mathrm{char}(K) \neq 2$, it follows that

$$P^s = -\frac{1}{2} \sum_{j=0}^{r} \lambda_j b_{js}, \qquad s = 1, \ldots, r.$$

If we plug these formulas into (4), we get

$$Q_0^{ij} = Q_0^{ji} = \frac{1}{2}(\lambda_j a_{0i} + \lambda_i a_{0j})$$

as claimed.

Case 2: Assume $\mathrm{char}\, K = 2$.

Then (5) shows that the symmetry conditions $Q_0^{ij} = Q_0^{ji}$ are equivalent to

$$\lambda_j a_{0i} = \lambda_i a_{0j} \qquad \text{for all } i, j = 0, \ldots, r.$$

These equations are similar to the system 1. considered in Proposition 5.4. Analogous arguments provide $\lambda_i = 0$ for all i if $g \leq -2$. If $g = -1$ we get $\lambda_i = c' a_{0i}$ for $i = 0, 1$ and some $c' \in K$ and $\lambda_i = 0$ if $2 \leq i \leq r$. Using equations (4) the asserted necessary conditions follow.

In both cases it is easy to see that the stated conditions are sufficient as well. □

Thanks to the previous computations, we obtain the following result.

Corollary 5.6. *Let $C \subset \mathbb{P}^n$ $(n \geq 3)$ be a double line of genus g. Then we have*

$$h^0(C, \mathcal{N}_C) = \begin{cases} 4(n-1) & \text{if } g = -1 \\ (n-1)(3-g) - 1 & \text{if } \mathrm{char}\, K \neq 2 \text{ and } g \leq -2 \\ n(3-g) - 6 & \text{if } \mathrm{char}\, K = 2 \text{ and } g \leq -2 \end{cases}$$

Proof. It is clear that the two last results give not only necessary, but also sufficient conditions for the global sections of $\mathcal{N}_C$. Thus, $h^0(C, \mathcal{N}_C)$ equals the number of free parameters in the description of the solutions of the systems 1. - 3. in Proposition 5.1. Counting these parameters is easy.

Case 1: Assume $\mathrm{char}\, K \neq 2$.

Then we have

- $r+1$ arbitrary polynomials $\lambda_0, \ldots, \lambda_r$ in $[S]_1$ (cf. Proposition 5.5) giving $2(r+1)$ free parameters;
- r arbitrary polynomials $Q_0^1, \ldots, Q_0^r$ in S of degrees $\alpha_0+\beta_1, \ldots, \alpha_0+\beta_r$, respectively, (cf. Remark 5.2) providing $r(\alpha_0+1)+\alpha_0$ free parameters because $\beta_1 + \ldots + \beta_r = \alpha_0 = -g$ (cf. Proposition 2.4).

Moreover, if $g \leq -2$ then $P^{ij} = 0$ for every i, j, while, if $g = -1$ we have one more free parameter (cf. Proposition 5.4). The claim follows.

Case 2: Assume $\mathrm{char}\, K = 2$.

Then we have in case $g \leq -2$

- arbitrary polynomials $P^i \in [S]_{\beta_i+1}$ (cf. Proposition 5.5) giving $2r - g$ parameters;
- as in Case 1 arbitrary polynomials $Q_0^1, \ldots, Q_0^r$ in S of degrees $\alpha_0+\beta_1, \ldots, \alpha_0+\beta_r$, respectively.

The claim follows.

If $g = -1$ we have

- an arbitrary constant $c' \in K$ and an arbitrary $P^1 \in [S]_2$ by Proposition 5.5;
- an arbitrary $c \in K$ due to Proposition 5.4;
- an arbitrary $Q_0^1 \in [S]_2$.

Summing up, we get again $h^0(C, \mathcal{N}_C) = 4(n-1)$. □

Now, we want to exhibit a family of solutions of the linear system 3. for an arbitrary rope C of degree $n-k \geq 3$ that satisfy an additional condition.

Condition 5.7. Assume that the vector $\mathbf{P} = (P^1, \ldots, P^k)$ is a combination of the rows of B, i.e.

$$\mathbf{P} = B^t \cdot \lambda$$

where $\lambda = (\lambda_0, \ldots, \lambda_r)^t$ and $\lambda_i \in S$. Note that $\deg(\lambda_i) = 1$ for each i by comparing degrees.

We are going to determine all solutions of the system 3. satisfying this condition. Observe that the condition is automatically satisfied if C is a 2–rope and char $K \neq 2$ according to Proposition 5.5.

For the remainder of this section we assume that the rope C has degree at least 3. Setting $\mathbf{Q}_l^j = (Q_l^{0,j}, \ldots, Q_l^{rj})^t$ we can write the system 3. as

$$B^t \mathbf{Q}_l^j + a_{lj} B^t \lambda = \mathbf{0}, \qquad l = 0, \ldots, r-k,\ j = 0, \ldots, r,$$

thus

$$B^t(\mathbf{Q}_l^j + a_{lj}\lambda) = \mathbf{0}.$$

Using Remark 2.2 again, we get for each l, j

$$\mathbf{Q}_l^j + a_{lj}\lambda = A^t \mathbf{c}^{lj}$$

where $\mathbf{c}^{lj} = (c_0^{lj}, \ldots, c_{r-k}^{lj})^t$ and $\deg(c_h^{lj}) = \alpha_l - \alpha_h + 1$ for $h = 0, \ldots, r-k$ and $\alpha_l - \alpha_h + 1 < 0$ means that $c_h^{lj} = 0$. Since $Q_l^{ij} = Q_l^{ji}$, the parameters c_h^{lj} satisfy the following relations

$$-\lambda_i a_{lj} + \sum_{h=0}^{r-k} a_{hi} c_h^{lj} = -\lambda_j a_{li} + \sum_{h=0}^{r-k} a_{hj} c_h^{li}, \qquad i, j = 0, \ldots, r,$$

or equivalently

$$\sum_{h=0}^{r-k} a_{hi} c_h^{lj} - \sum_{h=0}^{r-k} a_{hj} c_h^{li} = \lambda_i a_{lj} - \lambda_j a_{li} \qquad \text{if } 0 \leq j < i \leq r \text{ and } 0 \leq l \leq r-k. \tag{6}$$

We think of the system (6) as a linear system with unknowns c_h^{lj}. It remains to compute its solutions.

It is easy to see that particular solution of (6) is

$$c_h^{lj} = \begin{cases} 0 & \text{if } h \neq l \\ -\lambda_j & \text{if } h = l \end{cases} \qquad (j = 0, \ldots, r).$$

Now, we consider the homogeneous linear system associated to (6), i.e.

$$\sum_{h=0}^{r-k} a_{hi} c_h^{lj} - \sum_{h=0}^{r-k} a_{hj} c_h^{li} = 0 \qquad 0 \leq j < i \leq r. \tag{7}$$

The coefficient matrix with respect to the variables $c_h^{l0}, c_h^{l1}, \dots, c_h^{lr}$ is

$$M_h = \begin{bmatrix} a_{h1} & -a_{h0} & 0 & \dots & 0 \\ a_{h2} & 0 & -a_{h0} & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ a_{hr} & 0 & & \dots & -a_{h0} \\ 0 & a_{h2} & -a_{h1} & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & 0 & a_{hr} & -a_{h,r-1} \end{bmatrix}.$$

Notice that M_h is the matrix that describes the map $F \to \wedge^2 F$ of the Koszul complex $K(a_{h0}, \dots, a_{hr})$ associated to the row of A with index h.

The whole coefficient matrix (7) is $M = (M_0 \mid M_1 \mid \dots \mid M_{r-k})$. Thus, it is easy to verify that the following vectors are solutions of (7):

$$(\dots, 0, A_h, 0, \dots)^t, \qquad h = 0, \dots, r-k,$$

$$(\dots, 0, A_{h'}, 0, \dots, 0, A_h, 0, \dots)^t, \qquad 0 \le h < h' \le r-k$$

where $A_0, \dots, A_{r-k}$ are the rows of A. These solutions are independent over S because the rows of A have this property. The latter is true since the maximal minors of A define an ideal of codimension two. Hence, we have found $\binom{r+2-k}{2}$ independent solutions of (7).

Now, we want to estimate the rank of M. To this end we set $M_h = (m_{h0}, m_{h1}, \dots, m_{hr})$ where m_{hj} denotes the j-th column of M_h. Consider the following submatrix of M

$$\begin{aligned} M' := (&m_{00}, \dots, m_{r-k,0}; \dots; m_{0k}, \dots, m_{r-k,k}; \\ &m_{0,k+1}, \dots, m_{r-k-1,k+1}; \dots; m_{0,r-2}, m_{1,r-2}; m_{0,r-1}) \end{aligned}$$

that is obtained by picking the first r columns of M_0, the first $r-1$ columns of $M_1, \dots,$ and the first k columns of M_{r-k} and reordering them suitably. M' is an upper triangular block matrix. If we delete the first $k-1$ rows of the first block, the first $k-2$ rows of the second block, ..., the first row of the $(k-1)$–st block, we obtain a square block matrix with square blocks on the diagonal that is upper triangular, has order $(r+k)(r+1-k)/2$, and determinant

$$a_{0r} \det(A_{0,1;r-1,r}) \cdots \det(A_{0,\dots,r-1-k;k+1,\dots,r})[\det(A_{0,\dots,r-k;k,\dots,r})]^k$$

where $A_{i_1,\dots,i_p;j_1,\dots,j_p}$ denotes the submatrix of A obtained by picking the rows $i_1, \dots, i_p$ and the columns $j_1, \dots, j_p$. Analogously, there are other square submatrices of M of the same order such that their determinants are products of the k-th power of a maximal minor of A and other subminors of it and all such possibilities occur. But A has maximal rank. It follows that the rank of M is at least $\frac{(r+k)(r+1-k)}{2}$. Since M has $(r+1)(r+1-k)$ columns, the system (7) has over the quotient field of $S = K[t,u]$ at most

$$(r+1)(r+1-k) - \frac{(r+k)(r+1-k)}{2} = \binom{r+2-k}{2}$$

independent solutions. Since we have already found that many independent solutions with entries in S we have determined all solutions of the system (7).

Using this information we get a lower bound for the dimension of $H^0(C, \mathcal{N}_C)$

Proposition 5.8. *Let $C \subset \mathbb{P}^n$ be a rope with degree $n-k > 2$ and genus g that is supported on a line. Let $\alpha_0, \dots, \alpha_{r-k}$ be the degrees of the rows of the matrix A associated* *to C. Assume $\alpha_0 \geq 2$. Then we have*

$$h^0(C,\mathcal{N}_C) \geq (r+1)(2+k-g) - k^2 + \sum_{l,h=0}^{r-k}\sum_{v=h}^{r-k} \max(0, \alpha_l - \alpha_h - \alpha_v + 2).$$

Proof. We know that $P^{ij} = 0$ for every i, j by Proposition 5.3, and that $Q_l^s \in [S]_{\beta_s+\alpha_l}$ are arbitrary polynomials. This gives

$$\sum_{s=1}^{k}\sum_{l=0}^{r-k}(\beta_s + \alpha_l + 1) = (r+1)(k-g) - k^2$$

parameters.

Now, we claim that different choices of $\lambda := (\lambda_0, \dots, \lambda_r)^t$ led to different vectors $\mathbf{P} = B^t\lambda$. Indeed, if $B^t\lambda = 0$ then λ is a linear combination of the columns of A^t. Since $\alpha_0 \geq 2$, the entries of A have degree ≥ 2. This gives a contradiction because $\lambda_0, \dots, \lambda_r$ have degree one.

Furthermore, since $\mathbf{Q}_j^l + a_{lj}\lambda = A^t\mathbf{c}^{lj}$, we see that for fixed λ different choices of $\mathbf{c}^{lj}$ lead to different solutions of the system 3. Indeed, the matrix A^t provides an injective homomorphism of S–modules.

Summing up, we have shown that different choices of $(\lambda^t, \mathbf{c}^{00}, \dots, \mathbf{c}^{r,r-h})$ lead to different solutions of the system 3. Since $\lambda_0, \dots, \lambda_r \in [S]_1$ can be chosen arbitrarily we get $2(r+1)$ further parameters in K.

The parameters c_h^{lj} have to satisfy the systems (6). Thus, they can be written as

$$(\dots, c_h^{l0}, \dots, c_h^{lr}, \dots) = \sum_{i=0}^{r-k}\sum_{i'=i}^{r-k} d_{i,i'}^l(\dots, A_{i'}, \dots, A_i, \dots) + \\ (\dots, 0, \lambda_0, \dots, \lambda_r, 0, \dots)$$

where $d_{i,i'}^l$ is any polynomial in S of degree $\alpha_l - \alpha_i - \alpha_{i'} + 1$. Hence, the c_h^{lj}'s depend on

$$\sum_{l=0}^{r-k}\sum_{i=0}^{r-k}\sum_{i'=i}^{r-k} \max\{0,\ \alpha_l - \alpha_i - \alpha_{i'} + 2\}$$

parameters in K.

Adding up the various contributions provides our claim. □

The previous proof also shows.

Corollary 5.9. *Adopting the notation and assumptions in Proposition 5.8 assume in addition that, for the rope C, Condition 5.7 is satisfied for every $\varphi \in H^0(C,\mathcal{N}_C)$. Then we have*

$$h^0(C,\mathcal{N}_C) = (r+1)(2+k-g) - k^2 + \sum_{l,h=0}^{r-k}\sum_{v=h}^{r-k} \max(0, \alpha_l - \alpha_h - \alpha_v + 2).$$

Proof. Condition 5.7 guarantees that we have found all the solutions of the systems 1. - 3. as shown in the discussion preceding Proposition 5.8. □

6. Parameter spaces for ropes

The aim of this section is to describe some natural parameter spaces for families of $(n-k)$–ropes in $\mathbb{P}^n$ of fixed genus g that are supported on a line and to study their relations. This leads to results on the Hilbert scheme of ropes.

By Proposition 2.4 we know that $g = -\sum_{j=1}^k \beta_j = -\sum_{i=0}^{r-k} \alpha_i$. Moreover, a rope is uniquely determined when we fix the line that supports the rope and we know the sequence (1) that defines the scheme structure of the rope. We introduce some further notation.

Definition 6.1. Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope whose sequence (1) is

$$0 \to \oplus_{j=1}^k \mathcal{O}_{\mathbb{P}^1}(-\beta_j - 1) \to \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \to \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1) \to 0.$$

We may assume that

$$\alpha_0 \le \alpha_1 \le \ldots \le \alpha_{r-k} \qquad \text{and} \qquad \beta_1 \le \ldots \le \beta_k.$$

In this case we call $\alpha = (\alpha_0, \ldots, \alpha_{r-k})$ the *right type* and $\beta = (\beta_1, \ldots, \beta_k)$ the *left type* of the rope C.

Now, we can describe our first parameter space.

Proposition 6.2. *The $(n-k)$–ropes with fixed right type α are parameterized by an irreducible, smooth, quasi-projective scheme V_α of dimension*

$$\dim V_\alpha = (n-1)(n-k+1-g) - \sum_{i,j=0}^{r-k} \binom{\alpha_i - \alpha_j + 1}{1}. \tag{8}$$

Proof. According to [10], Proposition 2.9, two matrices A, A' providing surjective morphisms $\mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \longrightarrow \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1)$ of maximal rank lead via their syzygy matrices to the same rope if and only if $A = P \cdot A'$ where the matrix P provides an automorphism of $\oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1)$. This is equivalent to the fact that A^t and $(A')^t$ define the same subbundle of $\mathcal{O}_{\mathbb{P}^1}^{r+1}(1)$. Thus, denoting by $\mathrm{Grass}(1,n)$ the Grassmanian of lines in $\mathbb{P}^n$ we get as parameter space

$$V_\alpha = \mathrm{Grass}(1,n) \times \left\{ \text{subbundles of } \mathcal{O}_{\mathbb{P}^1}^{r+1}(1) \text{ isomorphic to } \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(1-\alpha_i) \right\}.$$

Hence, V_α is smooth, irreducible and has dimension

$$\begin{aligned} \dim V_\alpha &= 2(r+1) + (r+1)\sum_{i=0}^{r-k}(\alpha_i + 1) - \dim \mathrm{Aut}(\oplus_{i=0}^{r-k} S(1-\alpha_i)) \\ &= (n-1)(n-k+1-g) - \sum_{i,j=0}^{r-k} \binom{\alpha_i - \alpha_j + 1}{1} \end{aligned}$$

because the supporting line fixes the base of $S^{r+1}(-1)$. $\square$

There is an analogous result using the left type. We omit its proof.

Proposition 6.3. *The $(n-k)$–ropes with fixed left type β are parameterized by the irreducible, smooth, quasi-projective scheme*

$$W_\beta = \mathrm{Grass}(1,n) \times \left\{ \textit{subbundles of } \mathcal{O}_{\mathbb{P}^1}^{r+1}(1) \textit{ isomorphic to } \oplus_{j=1}^{k} \mathcal{O}_{\mathbb{P}^1}(1-\beta_j) \right\}$$

of dimension

$$\dim W_\beta = (n-1)(k+2-g) - \sum_{i,j=1}^{k} \binom{\beta_i - \beta_j + 1}{1}.$$

Remark 6.4. Let $C \subset \mathbb{P}^n$ be a non-degenerate $(n-k)$–rope with left type β. Then its generic initial ideal with respect to the degree reverse lexicographic order is

$$J = (x_0^2, x_0x_1, x_1^2, \ldots, x_r^2, x_0t^{\beta_1}, \ldots, x_{k-1}t^{\beta_k})$$

because $J \subset R$ is the only saturated, Borel-fixed, monomial ideal compatible with the homogeneous ideal and the Hilbert function of C. Hence, all the ropes parametrized by W_β correspond to points in the closure of the stratum of the Hilbert scheme $H_{n-k,g}(\mathbb{P}^n)$ that parametrizes curves with initial ideal J (cf. [13]).

Now, we begin to compare the parameter spaces.

Remark 6.5. It is not too difficult to see that the ropes parametrized by V_α can have every possible left type β subject to the condition

$$|\beta| := \sum_{j=1}^{k} \beta_j = -g = \sum_{i=0}^{r-k} \alpha_i.$$

In fact, this is a special case of [17], Theorem 2.1. Thus, every space W_β, where β satisfies the condition above, has a point that corresponds to a point in V_α. Similarly, varying a point in W_β we meet all the spaces V_α such that the condition above is satisfied. Hence, the two types of parameter spaces are transverse to each other, in some sense.

The natural incidence relation between $\cup_\alpha V_\alpha$ and $\cup_\beta W_\beta$ can be easily described. Let $(l, \mathcal{E})$ be a point of some V_α and let $(l', \mathcal{E}')$ be a point of some W_β. The the incidence relation is just

$$\mathcal{H} = \{((l,\mathcal{E}),(l',\mathcal{E}')) \mid l = l',\ \mathcal{E}' \cong \mathcal{E}^*\}.$$

The condition $\mathcal{E}' \cong \mathcal{E}^*$ guarantees that there is an exact sequence

$$0 \to \oplus_{j=1}^{k} S(-\beta_j - 1) \xrightarrow{\psi} S^{r+1}(-1) \xrightarrow{\varphi} \oplus_{i=0}^{r-k} S(\alpha_i - 1)$$

such that $\mathcal{E}' = \widetilde{\operatorname{im}\psi}$ and $\mathcal{E}^* = \widetilde{\operatorname{im}\varphi^*}$.

There is a third natural space parameterizing $(n-k)$–ropes of genus g. It is the Hilbert scheme $\operatorname{Hilb}_{n-k,g}(\mathbb{P}^n)$ that parameterizes closed subschemes in $\mathbb{P}^n$ with Hilbert polynomial $p(z) = (n-k)z + 1 - g$. Of course, this Hilbert scheme contains not only ropes but many other schemes with different geometric properties. It is natural to investigate the relation among $\cup_\alpha V_\alpha$, $\cup_\beta W_\beta$ and $\operatorname{Hilb}_{n-k,g}(\mathbb{P}^n)$. Because of the universality of the Hilbert scheme, there are morphisms

$$j_\alpha : V_\alpha \to \operatorname{Hilb}_{n-k,g}(\mathbb{P}^n) \qquad \text{and} \qquad j_\beta : W_\beta \to \operatorname{Hilb}_{n-k,g}(\mathbb{P}^n)$$

that embed V_α and W_β into $\operatorname{Hilb}_{n-k,g}(\mathbb{P}^n)$ where $-g = |\alpha| = |\beta|$.

Now, we prove a topological result about the images of these morphisms.

Lemma 6.6. *All $(n-k)$–ropes in $\mathbb{P}^n$ with genus g lie in the same irreducible component of $Hilb_{n-k,g}(\mathbb{P}^n)$. More precisely, $\bigcup_{|\alpha|=-g} j_\alpha(V_\alpha)$ is contained in one irreducible component of $Hilb_{n-k,g}(\mathbb{P}^n)$.*

Proof. V_α being irreducible, $j_\alpha(V_\alpha)$ is irreducible. Thus, it is enough to show that for every two right types α' and α'' such that $|\alpha'| = |\alpha''| = -g$, there is a flat family of ropes which connects a point in $j_{\alpha'}(V_{\alpha'})$ and a point in $j_{\alpha''}(V_{\alpha''})$. In order to construct the family, we fix a line $l \subset \mathbb{P}^n$ and a left type β such that $-g = |\beta|$. In W_β we choose two points $(l, \widetilde{\operatorname{im}\psi'})$ and $(l, \widetilde{\operatorname{im}\psi''})$ such that the morphisms φ', and φ'' corresponding to ψ', and ψ'' via the incidence relation above have type α' and α'', i.e. $(l, \widetilde{\operatorname{im}\varphi'}) \in V_{\alpha'}$ and $(l, \widetilde{\operatorname{im}\varphi''}) \in V_{\alpha''}$. This is possible by Remark 6.5.

Now, we define the family we are looking for. All the ropes of the family are supported on the line l and the scheme structure is defined via the left morphism of the sequence (1). We set

$$\mathcal{F} = \{(l, s\psi' + (1-s)\psi'')|s \in U\}$$

where U is the non-empty open subset of $\mathbb{A}^1$ defined by $U := \{s \in \mathbb{A}^1 \mid \operatorname{codim}(I_k(s\psi' + (1-s)\psi'')) = 2\}$. Here, $I_k(\psi)$ denotes the determinantal ideal generated by the k-minors of the map ψ. Theorem 2.1 shows that $\mathcal{F}$ is a flat family and that the image of $\mathcal{F}$ via $j_\beta(W_\beta)$ in $\mathrm{Hilb}_{n-k,g}(\mathbb{P}^n)$ connects $j_{\alpha'}(V_{\alpha'})$ to $j_{\alpha''}(V_{\alpha''})$. $\square$

The preceding result justifies.

Notation 6.7. Let $H_{n-k,g}(\mathbb{P}^n)$ or shorter $H_{n-k,g}$ be the irreducible component of $\mathrm{Hilb}_{n-k,g}(\mathbb{P}^n)$ that contains $j_\alpha(V_\alpha)$ for each α with $-|\alpha| = g$.

The next step is to find large families V_α and W_β such that $j_\alpha(V_\alpha) \subset \mathrm{Hilb}_{n-k,g}(\mathbb{P}^n)$ and $j_\beta(W_\beta) \subset \mathrm{Hilb}_{n-k,g}(\mathbb{P}^n)$. We will be guided by semicontinuity of $h^1(\mathbb{P}^n, \mathcal{I}_C(z))$ on $H_{n-k,g}$. Recall that by Corollary 3.8 the Rao function of a rope C is

$$\rho_C(z) = \sum_{i=0}^{r-k} \binom{z+\alpha_i}{1} + \sum_{j=1}^{k} \binom{z-\beta_j}{1} - (r+1)\binom{z}{1}.$$

It is determined by the left type and right type of C. Moreover, the left type β does not contribute if $z \leq 0$, more precisely,

$$\rho_C(z) = \sum_{i=0}^{r-k} \binom{z+\alpha_i}{1} \qquad \text{if } z \leq 0,$$

while, for $z > 0$, the right type α gives no contribution because

$$\rho_C(z) = -kz - g + \sum_{j=1}^{k} \binom{z-\beta_j}{1} \qquad \text{if } z > 0.$$

In order to specify the smallest Rao function of a rope in $H_{n-k,g}$ we introduce.

Definition 6.8. Define integers p_1, s_1 by

$$-g = p_1 \cdot (r+1-k) + s_1 \qquad \text{where} \qquad 0 \leq s_1 \leq r-k.$$

Then we set

$$\alpha_{min} = (p_1, \ldots, p_1, p_1+1, \ldots, p_1+1)$$

where p_1+1 occurs s_1 times while p_1 occurs $r+1-k-s_1$ times.

Similarly, define integers p_2, s_2 by

$$-g = p_2 \cdot k + s_2 \qquad \text{where} \qquad 0 \leq s_2 < k.$$

We set

$$\beta_{min} = (p_2, \ldots, p_2, p_2+1, \ldots, p_2+1)$$

where p_2+1 occurs s_2 times, while p_2 occurs $(k-s_2)$ times.

Obviously, α_{min} is a right type and β_{min} is a left type of a rope of genus g. They are rather particular.

Proposition 6.9. *Let ρ_{min} be the Rao function associated to the types α_{min} and β_{min} (defined via the formula in Corollary 3.8). Then the Rao function of every $(n-k)$–rope $C \subset \mathbb{P}^n$ of genus g satisfies*

$$\rho_C(z) \geq \rho_{min}(z) \qquad \textit{for all } z \in \mathbb{Z}.$$

Proof. Let $\alpha' = (\alpha_0, \dots, \alpha_{r-k})$ and $\alpha'' = (\alpha_0, \dots, \alpha_i + 1, \dots, \alpha_j - 1, \dots, \alpha_{r-k})$ be right types of some ropes such that $|\alpha'| = |\alpha''| = -g$. This means in particular that $\alpha_i + 2 \leq \alpha_j$. We want to show that the Rao function ρ_1 associated to α' is larger than the Rao function ρ_2 associated to α'' if $z \leq 0$.

In fact, if $z \leq 0$ then we have

$$\rho_1(z) = \sum_{h \neq i,j} \binom{z + \alpha_h}{1} + \binom{z + \alpha_i}{1} + \binom{z + \alpha_j}{1}$$

and

$$\rho_2(z) = \sum_{h \neq i,j} \binom{z + \alpha_h}{1} + \binom{z + \alpha_i + 1}{1} + \binom{z + \alpha_j - 1}{1},$$

thus

$$\rho_1(z) - \rho_2(z) = \binom{z + \alpha_j - 1}{0} - \binom{z + \alpha_i}{0} \geq 0$$

because $\alpha_i + 2 \leq \alpha_j$.

Analogously, let $\beta' = (\beta_1, \dots, \beta_k)$ and $\beta'' = (\beta_1, \dots, \beta_i + 1, \dots, \beta_j - 1, \dots, \beta_k)$ be left types such that $|\beta'| = |\beta''| = -g$. The Rao functions associated to these types are in case $z > 0$

$$\rho_1(z) = -kz - g + \sum_{h \neq i,j} \binom{z - \beta_h}{1} + \binom{z - \beta_i}{1} + \binom{z - \beta_j}{1}$$

and

$$\rho_2(z) = -kz - g + \sum_{h \neq i,j} \binom{z - \beta_h}{1} + \binom{z - \beta_i - 1}{1} + \binom{z - \beta_j + 1}{1}.$$

Their difference is

$$\rho_1(z) - \rho_2(z) = \binom{z - \beta_i - 1}{0} - \binom{z - \beta_j}{0} \geq 0 \qquad \text{if } z > 0.$$

Hence, the Rao function is minimal when the difference of two entries of each type is as at most one. Our claim follows. □

Due to semicontinuity of the Rao function, we expect the parameter space $V_{\alpha_{min}}$ to have the largest dimension among the parameter spaces V_α with image in $H_{n-k,g}$. Now, we compute its dimension.

Proposition 6.10. $\dim V_{\alpha_{min}} = (n-1)(k+2-g) - k^2$.

Proof. According to the proof of Proposition 6.2 it suffices to compute the dimension of the automorphism group of $\oplus S(\alpha_i - 1)$ corresponding to the right type α_{min}. By Definition 6.8 we have

$$\alpha_i = \begin{cases} p_1 & \text{if} \quad i = 0, \dots, r-k-s_1 \\ p_1 + 1 & \text{if} \quad i = r+1-k-s_1, \dots, r-k \end{cases},$$

thus $\dim \operatorname{Aut}(\oplus_{i=0}^{r-k}(\alpha_i - 1)) = (r+1-k)^2$. Hence, using formula (8) an easy computation provides our claim. □

Remark 6.11. There is an analogous result for the left type. In fact, we get

$$\dim W_{\beta_{min}} = (n-1)(k+2-g) - k^2.$$

Now, we are ready to study the component $H_{n-k,g}(\mathbb{P}^n)$ of the Hilbert scheme $\mathrm{Hilb}_{n-k,g}(\mathbb{P}^n)$ that contains ropes supported on lines (cf. Notation 6.7). For the remainder of this paper we assume that the field K is algebraically closed.

First, we discuss double lines. Their maximal genus is -1 and the case of genus -1 is particular.

Remark 6.12. The component $H_{2,-1}(\mathbb{P}^n)$ is generically smooth of dimension $4(n-1)$. Its general element is a pair of skew lines. This is well-known (cf., e.g., [6]).

In the general case, the characteristic of the ground field plays a role.

Theorem 6.13. *Let $g \leq -2$, $n \geq 3$ be integers. Then we have:*

(a) $\dim H_{2,g}(\mathbb{P}^n_K) = (n-1)(3-g) - 1$;

(b) $H_{2,g}(\mathbb{P}^n_K)$ *is generically smooth if and only if either* $\operatorname{char} K \neq 2$ *or* $\operatorname{char} K = 2$, $g = -2$.

(c) $H_{2,g}(\mathbb{P}^n_K)$ *is non-reduced if* $\operatorname{char} K = 2$ *and* $g \leq -3$.

Furthermore, a double line $C \subset \mathbb{P}^n$ is obstructed if it has genus ≤ -3 and $\operatorname{char} K = 2$; *otherwise unobstructed.*

Proof. Every curve having degree two and genus at most -2 is a double line. Hence, Lemma 6.6 shows that the dimension of $H_{2,g}(\mathbb{P}^n)$ is the maximum of $\dim V_\alpha$ where α is the right type of a double line. Proposition 6.2 and Proposition 6.10 provide the claim about the dimension.

Corollary 5.6 gives the dimension of the tangent space of $H_{2,g}(\mathbb{P}^n_K)$ at every point corresponding to a double line C. But the proof of (a) shows that the general curves of $H_{2,g}(\mathbb{P}^n)$ are the ropes in $V_{\alpha_{min}}$. Thus, $H_{2,g}(\mathbb{P}^n)$ is generically smooth if and only if every $C \in V_{\alpha_{min}}$ satisfies $(n-1)(3-g)-1 = h^0(C, \mathcal{N}_C)$. Hence, Corollary 5.6 implies claim (b) as well as (c) because the curves form an open subset of $Hilb_{2,g}(\mathbb{P}^n)$. □

It remains to consider ropes whose degree is at least three. In this case our main result about the component $H_{n-k,g}(\mathbb{P}^n)$ is.

Theorem 6.14. *Let d, g, n be integers such that $3 \leq d \leq n-1$ and $g \leq d-n$. If either $g \leq -3(d-1)$ or $-g = 2(d-1)$ then the component $H_{d,g}(\mathbb{P}^n)$ is generically smooth of dimension $(n-1)(d+1-g)-(d-1)^2$ and its general element is a non-degenerate rope.*

Proof. Put $n-k := d$. Of course, $\dim V_{\alpha_{min}} = (n-1)(k+2-g)-k^2$ is a lower bound for the dimension of $H_{n-k,g}(\mathbb{P}^n)$. Hence, it is enough to prove that the tangent space of $H_{n-k,g}(\mathbb{P}^n)$ at a suitable point has dimension $(n-1)(k+2-g)-k^2$, i.e.

$$h^0(C, \mathcal{N}_C) = (n-1)(k+2-g) - k^2$$

for some $(n-k)$–rope $C \subset \mathbb{P}^n$ of genus g.

To this end we consider ropes of varying embedding dimension. Let $C \subset \mathbb{P}^n$ be a rope having the right type $\alpha = (\alpha_1, \dots, \alpha_{s+w}) := (p, \dots, p, p+1, \dots, p+1)$ with $w \geq 1$ occurrences of $p \geq 1$ and $s \geq 0$ occurrences of $p+1$. Then (cf. Section 2) C has degree $s+w+1$ and genus $g := -[(s+w)p+s]$. Moreover, we must have $s+w < r+1$. The rope C is non-degenerate if and only if its left type $\beta = (\beta_1, \dots, \beta_k)$ satisfies $\beta_1 \geq 1$. In this case we get $-g = \beta_1 + \ldots + \beta_k \geq k = r+1-(s+w)$ which implies

$$s + w \leq r \leq (s+w) \cdot (p+1) + s - 1.$$

We now fix the right type α but let the linear span of the ropes vary. Our goal is to show.

Claim: For every r in the range above there is a non-degenerate rope $C \subset \mathbb{P}^{r+2}$ with right type α such that Condition 5.7 is satisfied for every $\varphi \in H^0(C, \mathcal{N}_C)$. (We use the notation introduced at the beginning of Section 5. Recall in particular that P^s denotes just a polynomial with index s. It is not the s-th power of P.)

We begin by specifying a suitable rope in $\mathbb{P}^{r+2}$. According to Theorem 2.1 it suffices to describe an $(s+w) \times (r+1)$ matrix A_r. We do this with the help of three types of matrices.

First, we define the $(s+w) \times (s+w+1)$ matrix $\tilde{A}_0 = (a_{ij})$ by

$$a_{ij} = \begin{cases} t^p & \text{if } i = j,\ i = 0, \dots, w-1; \\ t^{p+1} & \text{if } i = j,\ i = w, \dots, s+w-1; \\ u^p & \text{if } j = i+1,\ i = 0, \dots, w-1; \\ u^{p+1} & \text{if } j = i+1,\ i = w, \dots, s+w-1; \\ 0 & \text{otherwise} \end{cases}$$

If $0 \leq j \leq s+w$ we denote by $\tilde{A}_j$ the matrix that is obtained from $\tilde{A}_0$ by deleting its last j rows.

Second, we put

$$A_i' := (t^{p+1}, t^p u, \dots, t^{p+2-i} u^{i-1}, u^{p+1}) \qquad (1 \leq i \leq p+1)$$

and

$$A_i'' := (t^p, t^{p-1} u, \dots, t^{p+1-i} u^{i-1}, u^p) \qquad (1 \leq i \leq p).$$

Note that the format of these matrices is $1 \times (i+1)$.

Now, we define the matrices A_r. If $r = s+w$ then we set $A_r := \tilde{A}_0$.

Let $s+w < r \leq s+w+s(p+1)$. Define integers i, j by

$$r := s+w+j(p+1)+i \qquad \text{and} \qquad 1 \leq i \leq p+1.$$

Then we put

$$A_r := \begin{pmatrix} \tilde{A}_{j+1} & & & & \\ & A_i' & & & \\ & & A_{p+1}' & & \\ & & & \ddots & \\ & & & & A_{p+1}' \end{pmatrix}$$

where all non-specified entries are zero. Observe that the block A_{p+1}' occurs j times.

If $s(p+2)+w < r \leq (s+w)(p+1)+s-1$ then we define integers i, j by

$$r := s(p+2)+w+jp+i \qquad \text{and} \qquad 1 \leq i \leq p.$$

Then we set

$$A_r := \begin{pmatrix} \tilde{A}_{j+s+1} & & & & & & & \\ & A_i'' & & & & & & \\ & & A_p'' & & & & & \\ & & & \ddots & & & & \\ & & & & A_p'' & & & \\ & & & & & A_{p+1}' & & \\ & & & & & & \ddots & \\ & & & & & & & A_{p+1}' \end{pmatrix}$$

where block A_{p+1}' occurs s times and the block A_p'' appears j times.

Furthermore, we define B_r as a syzygy matrix of A_r. It has also a block structure. It is easy to see that the maximal minors of A_r and B_r, respectively, generate an ideal of codimension two. Hence, the ideal $((I_L)^2, [x_0, \dots, x_r] B_r)$ defines a non-degenerate rope

$C \subset \mathbb{P}^{r+2}$ with the prescribed right type α. It remains to check that Condition 5.7 is satisfied.

To this end we induct on $r \geq s+w$. Let $r := s+w$. Then the entries of B_r are the maximal minors of A_e, i.e.

$$B_e = \left(u^{-g}, -t^p u^{-g-p}, t^{2p}u^{-g-2p}, \ldots, (-1)^{r-1}t^{-g-p-1}u^{p+1}, (-1)^r t^{-g}\right)^t.$$

Furthermore, in this case the vector $\mathbf{P}$ has just one entry. We denote it by P to simplify notation a bit. Then for C, the equations of the linear system 3. in Proposition 5.1 with $l=0$ are

$$(9) \qquad \begin{cases} u^{-g}Q_0^{00} + \cdots + (-1)^r t^{-g} Q_0^{0r} + t^p P &= 0 \\ u^{-g}Q_0^{01} + \cdots + (-1)^r t^{-g} Q_0^{1r} + u^p P &= 0 \\ u^{-g}Q_0^{02} + \cdots + (-1)^r t^{-g} Q_0^{2r} &= 0 \\ \vdots & \\ u^{-g}Q_0^{0r} + \cdots + (-1)^r t^{-g} Q_0^{rr} &= 0 \end{cases}$$

The first equation shows that t^p divides Q_0^{00}, thus $Q_0^{00} = t^p q_0$ for some $q_0 \in S$. Analogously, the $(i+1)$–st equation provides for $i = 2, \ldots, r$ that $Q_0^{0i} = t^p q_i$ for some $q_i \in S$. If we multiply the first equation by u^p, the second one by $-t^p$ and then add the two equations, we get

$$\begin{aligned} u^{-g+p}t^p q_0 - t^p u^{-g} Q_0^{01} + t^{3p}u^{-g-p}q_2 + \cdots + (-1)^r t^{-g+p}u^p q_r \\ -t^p u^{-g}Q_0^{01} + t^{2p}u^{-g-p}Q_0^{11} + \cdots + (-1)^{r+1}t^{-g+p}Q_0^{1r} = 0. \end{aligned}$$

Assume now that $\operatorname{char} K \neq 2$. Then the last equation shows that t^p divides $u^{-g+p}q_0 - 2u^{-g}Q_0^{01}$, i.e. $u^{-g+p}q^0 - 2u^{-g}Q_0^{01} = t^p\tau'$ for some $\tau \in S$. It follows that u^{-g} divides τ'. Hence, there is a $\tau \in S$ such that $Q_0^{01} = \frac{1}{2}(u^p q_0 + t^p \tau)$. Plugging this into the first equation of (9) we obtain

$$\frac{1}{2}u^{-g}t^p q_0 - \frac{1}{2}t^{2p}u^{-g-p}\tau + t^{3p}u^{-g-2p}q_2 + \cdots + (-1)^r t^{-g+p} q_r + t^p P = 0.$$

After cancelling t^p we see that P is a linear combination of the rows of B_e, i.e. Condition 5.7 is satisfied.

If $\operatorname{char} K = 2$ a similar argument shows the claim. This completes the initial step of the induction.

Let $r > s+w$. We carry out the induction step only in case $r = s+w+1$. It shows the ideas. The general case is only notationally more complicated.

Thus, let $r := s+w+1$. Then the syzygy matrix of A_r is

$$B_r = \begin{pmatrix} \widetilde{(B_{r-1})_1} & 0 \\ 0 & B' \end{pmatrix}$$

where $\widetilde{(B_{r-1})_1}$ is the column obtained from B_{r-1} by deleting the last row and then cancelling the common factor u^{p+1} of the entries, and

$$B' = \begin{cases} (u^{p+1}, -t^{p+1})^t & \text{if } s > 0 \\ (u^p, -t^p)^t & \text{if } s = 0. \end{cases}$$

For the rope defined by $((I_L)^2, [x_0, \ldots, x_r]B_r)$ the vector $\mathbf{P}$ is (P^1, P^2) and the linear system 3. becomes

$$\begin{cases} (\widetilde{(B_{r-1})_1})^t \mathbf{Q}_l^j + a_{lj}P^1 = 0 \\ (B')^t \mathbf{Q}_l^j + a_{lj}P^2 = 0 \end{cases}$$

Using the first set of equations we see, as in the previous case $r = s + w$ above, that P^1 belongs to the ideal generated by the first r entries of the first column of B_r. To conclude for P^2 we use the part of the system where $l = 0$. It reads as

$$B_r^t \mathbf{Q}_0^j + a_{0j} \cdot \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} = 0. \tag{10}$$

By the definition of A_r, this provides for $j = 2, \dots, r$

$$B_r^t \mathbf{Q}_0^j = 0,$$

hence $\mathbf{Q}_0^j = A_r^t \cdot \mathbf{d}_j$ for some vector $\mathbf{d}_j = (d_{0j}, \dots, d_{r-2,j})$, in particular

$$Q_0^{0j} = t^p d_{0j} \qquad \text{for all } j = 2, \dots, r.$$

For $j = 0,\ l = 0$ the system (10) gives the subsystem

$$B_r^t \mathbf{Q}_0^0 + t^p \cdot \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} = 0.$$

Using $Q_0^{0j} = Q_0^{j0}$ the second equation provides

$$u^{p+1} t^p d_{0,r} - t^{p+1} t^p d_{0,r+1} + t^p P^2 = 0.$$

Cancelling t^p the claim follows for P^2. (Using the first instead of the second equation would give an alternative, more explicit proof of the claim for P^1.) This completes the proof of the claim.

Let now d, g, n be integers as in the statement. Then the claim shows that the component $H_{n-k,g}(\mathbb{P}^n)$ contains a non-degenerate rope C to which Corollary 5.9 applies because the assumption $-g \leq 2(n-1-k)$ implies $p = \alpha_0 \geq 2$. Thus, we get

$$h^0(C, \mathcal{N}_C) = (r+1)(2+k-g) - k^2 + \sum_{l,h}^{0,\dots,r-k} \sum_{v=h}^{r-k} \max(0, \alpha_l - \alpha_h - \alpha_v + 2) - \delta_{1,\alpha_0}.$$

It follows that $h^0(C, \mathcal{N}_C) = (r+1)(2+k-g) - k^2$ if and only if $\alpha_{r-k} - 2\alpha_0 \leq -2$. The latter is true if $\alpha = (2, \dots, 2)$, i.e. $-g = 2(n-1-k)$, or $\alpha_0 \geq 3$, i.e. $-g \leq 3(n-1-k)$. Comparing with Proposition 6.2 we see that the rope C is unobstructed. □

Remark 6.15. Theorem 6.14 shows that there is an open subset in $H_{d,g}(\mathbb{P}^n)$ consisting of ropes such that the global sections of the normal sheaf computed in Section 5 are all the global sections. In other words, Condition 5.7 is an open condition in $H_{d,g}(\mathbb{P}^n)$ (cf. Corollary 5.9).

Similarly, our knowledge of the normal sheaf allows to characterize singular points of $H_{d,g}(\mathbb{P}^n)$.

Corollary 6.16. *Let d, g, n be as in Theorem 6.14. Then a rope C in $H_{d,g}(\mathbb{P}^n)$ is obstructed if and only if either the corresponding system* 3. *has more solutions than the ones we computed (i.e. the inequality in Proposition 5.8 is strict), or $\alpha_{r-k} - 2\alpha_0 \geq -1$.*

Recall that we mean by a curve always a locally Cohen-Macaulay 1-dimensional scheme. Thus, if $g \leq -2$ then the only curves in $Hilb_{2,g}(\mathbb{P}^n)$ are 2–ropes. If $3 \leq d \leq n-1$ then $Hilb_{d,g}(\mathbb{P}^n)$ does not contain only 3–ropes even if $g \ll 0$. But if we restrict to the component $H_{d,g}(\mathbb{P}^n)$ we get an analogous result.

Proposition 6.17. *Let d, g, n be integers such that $3 \leq d \leq n-1$, $g \leq d-n$, and either $g \leq -3(d-1)$ or $-g = 2(d-1)$. Then all curves in $H_{d,g}(\mathbb{P}^n)$ are ropes.*

Proof. According to Theorem 6.14, the general element of $H_{d,g}(\mathbb{P}^n)$ is a rope. Hence, every curve in the closure of the open subset of ropes is the limit of a flat, 1-dimensional family $\mathcal{F}$ of ropes. Invoking projective transformations we may assume that all the ropes in $\mathcal{F}$ are supported on the *same* line L. But then the limit C of $\mathcal{F}$ satisfies $I_C \supseteq (I_L)^2$, thus C is a rope. □

There are, of course, 1-dimensional schemes in $H_{d,g}(\mathbb{P}^n)$ that are not locally Cohen-Macaulay. It suffices to consider a rope on a line L and to deform its associated matrix B to one, say B', that drops rank at some point of L. Then the ideal $((I_L)^2, [x_0, \dots, x_r]B')$ defines a non-locally Cohen-Macaulay scheme in $H_{d,g}(\mathbb{P}^n)$.

Remark 6.18. Let $\mathcal{F}$ be the family of 1-dimensional subschemes that are the union of a double line of genus g in $\mathbb{P}^n$ and a point. Note that its elements have genus $g-1$. We have just seen that $\mathcal{F}$ and $H_{2,g-1}(\mathbb{P}^n)$ meet. The dimension of $\mathcal{F}$ is

$$\dim \mathcal{F} = \dim(H_{2,g}(\mathbb{P}^n)) + n = \dim(H_{2,g-1}(\mathbb{P}^n)) + 1.$$

It follows that the general element of $\mathcal{F}$ is not contained in $H_{2,g-1}(\mathbb{P}^n)$.

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