ON THE EVEN GORENSTEIN LIAISON CLASSES OF ROPES ON A LINE

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ABSTRACT. We describe the even Gorenstein liaison classes of ropes supported on a line which are not arithmetically Buchsbaum.

CONTENTS

Dedicated to Silvio Greco on his 60th birthday

1. INTRODUCTION

Liaison theory is well understood for subschemes of codimension two in $\mathbb{P}^n$. One cornerstone of the theory is the so-called Rao's correspondence. It says that the even liaison classes are in $1-1$ correspondence to stable equivalence classes of certain reflexive sheaves (cf., e.g., [18], [13]). For having a satisfactory liaison theory in higher codimension it has been proposed in [13] that one should allow arithmetically Gorenstein schemes for linking two subschemes directly. Results in [7] show that the resulting Gorenstein liaison theory has many advantages with respect to the classically used complete intersection liaison. Thus, in this paper liaison will always mean Gorenstein liaison.

The analogue of Rao's correspondence for curves in $\mathbb{P}^n$ would say that two curves in $\mathbb{P}^n$ belong to the same even liaison class if and only if their Hartshorne-Rao modules are isomorphic, up to a degree shift. It is one of the main open problems of liaison theory to decide if this is true. Based on the theory developed in [7], which allows to view Gorenstein liaison as a theory about divisors on arithmetically Cohen-Macaulay schemes being locally Gorenstein in codimension one, positive results in this direction have been obtained mainly for some classes of arithmetically Cohen-Macaulay curves (cf., e.g., [7], [3], [12], [6]) and for certain arithmetically Buchsbaum curves (cf., e.g., [8], [6], [4]). Some scepticism is expressed in [6].

The first more general positive result allowing curves with more complicated Hartshorne-Rao module has been obtained in [15], Theorem 5.6.

Theorem 1.1. *Let $C, E \subset \mathbb{P}^n$ be two curves of degree two. Then C and E belong to the same even liaison class if and only if their Hartshorne-Rao modules are isomorphic.*

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Since curves of degree two are typically not generically Gorenstein this result is not based on the abstract results in [7]. Instead, in [15] it is explicitly described how C and E can be linked in an even number of steps. For this it is necessary to consider not only double lines but more generally ropes supported on a line. Such a rope is a multiplicity α structure on a line L satisfying $(I_L)^2 \subset I_C \subset I_L$. If the rope has degree 2 then it is a double line. But such a rope $C \subset \mathbb{P}^n$ can have any degree $\leq n$. Our main result concerns the even liaison classes of these ropes.

Theorem 1.2. *Let $C, E \subset \mathbb{P}^n = \operatorname{Proj} R$ be two ropes, each supported on a line, which are not arithmetically Buchsbaum. Then, C and E are in the same even Gorenstein liaison class if and only if their Hartshorne-Rao modules $M(C)$ and $M(E)$ are isomorphic as R–modules.*

This result is an improvement of [15], Theorem 5.4 where we assumed that the ropes C and E have the same degree. Here we show that this assumption can be removed. The proof is based on an extension of the methods developed in [15].

For ropes in $\mathbb{P}^3$ it has been shown by Migliore (cf. [9]) that the assertion of Theorem 1.2 remains true if one drops the asumption on the ropes not being arithmetically Buchsbaum. Our Theorem 4.9 says that the same is true for ropes in $\mathbb{P}^4$.

The paper is organized as follows. In Section 2 we recall some results on ropes. In particular, we present the description of the homogeneous ideal and the Hartshorne-Rao module of a rope supported on a line. In Section 3 we describe a family of non-degenerate arithmetically Gorenstein curves in $\mathbb{P}^n$ of degree n. This family is larger than a similar family considered in [15]. It is used for linking the ropes. This is carried out in Section 4 where we prove our main results. In the final section we present examples of self-linked ropes over fields of arbitrary characteristic.

Our notation is mainly standard and follows [5]. For definitions, basic results and extensive background on liaison theory we refer to [10].

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2. The ideal and Hartshorne-Rao module of a rope on a line

In this section we recall some equivalent definitions of an α–rope supported on a smooth curve and the main results about ropes supported on a line obtained in [15]. They are used in Section 4.

Throughout this paper we will use the following notation. We put $r := n-2$ and denote by R the polynomial ring $K[x_0, \ldots, x_r, t, u]$ over an arbitrary field. It will be convenient to assume that the supporting line $L \subset \mathbb{P}^n := \operatorname{Proj}(R)$ is $L = \operatorname{Proj}(K[t, u])$.

We begin by recalling the definition of an α–rope following [11]. Let Y be a smooth and irreducible curve in $\mathbb{P}^n$ with saturated homogeneous ideal I_Y (the support of the rope), and let C be a subscheme of $\mathbb{P}^n$ with saturated homogeneous ideal I_C.

Definition 2.1. A curve $C \subset \mathbb{P}^n$ is said to be an *α–rope* supported on Y if its homogeneous ideal I_C satisfies $I_Y^2 \subseteq I_C \subseteq I_Y$ and if C is a locally Cohen–Macaulay multiplicity α structure on Y.

It has been shown in [15], Proposition 2.3 that this definition is equivalent to a generalization of the concept of *doubling* introduced by M. Boratynski and S. Greco in [1].

Proposition 2.2. *With the notation above, the 1-dimensional subscheme $C \subset \mathbb{P}^n$ is an α–rope on Y if and only if the following conditions are satisfied:*
1. $I_Y^2 \subseteq I_C \subseteq I_Y$;
2. I_C *is unmixed;*
3. $l(R_{I_Y}/I_C R_{I_Y}) = \alpha$.

Remark 2.3. It is not true that every multiple structure supported on a curve Y is an α–rope for some α. For example, the ideal $J = (x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2 + x_1t, x_0t^a + x_1u^a)$ defines a multiple structure of degree three supported on the line $L = V(x_0, x_1, x_2)$ in $\mathbb{P}^4$, but $J \not\supseteq I_L^2$.

In this paper we are interested in ropes supported on a line L. Without loss of generality, we can fix the line $L = V(x_0, \dots, x_r) = \mathrm{Proj}(S)$ where $S = K[t, u]$. If $\varphi : F \to G$ is a graded homomorphism of free modules, we denote by $I_k(\varphi)$ the ideal generated by the k-minors of a matrix representing φ. The same notation is also used if φ is a morphism of locally free sheaves.

If a rope $C \subset \mathbb{P}^n$ supported on the line L has degree n then it is defined by the ideal $(I_L)^2$. Thus, the following result characterizes the homogeneous ideal of a rope supported on a line (cf. [15], Theorem 2.6).

Theorem 2.4. *Let $C \subset \mathbb{P}^n$ be a curve of degree at most $n-1$. Then the following conditions are equivalent:*
1. *C is an $(n-k)$–rope supported on the line L;*
2. *$I_C = ((I_L)^2, [x_0, \dots, x_r]B)$ where the matrix B gives a map $\varphi_B : \oplus_{j=1}^k S(-\beta_j - 1) \to S^{r+1}(-1)$ with* $\mathrm{codim}(I_k(B)) = 2$;
3. *$I_C = ((I_L)^2, F_1, \dots, F_k)$ where $V(F_1, \dots, F_k) \subset \mathbb{P}^n$ is a scheme of codimension k which is smooth at the points of L.*

For a geometric description of ropes on a line we refer to [15], Remark 2.12.

It is easy to see that an $(n-k)$–rope $C \subset \mathbb{P}^n$ supported on L corresponds naturally to a rank $r+1-k$ subbundle E of the normal bundle $\mathcal{N}_L = \mathcal{O}_{\mathbb{P}^1}^{r+1}(1)$ via the exact sequence

$$0 \to E^* \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_L \to 0.$$

Let $E = \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(1 - \alpha_i)$ where $\alpha_i \geq 0$. If we apply $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^1})$ to the embedding of E into $\mathcal{N}_L$, we get the exact sequence

$$0 \to \oplus_{j=1}^k \mathcal{O}_{\mathbb{P}^1}(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \xrightarrow{\varphi_A} \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1) \to 0 \tag{1}$$

where the ideals of maximal minors $I_k(\varphi_B)$ and $I_{r+1-k}(\varphi_A)$ have codimension 2. Note that A^t is just the syzyzgy matrix of B^t.

The (arithmetic) genus g of C is given by $-g = \sum_{i=0}^{r-k} \alpha_i = \sum_{j=1}^k \beta_j$ (cf. [15], Proposition 2.9 and Lemma 2.8). This is a consequence of Theorem 2.4 and Sequence (1).

Throughout the paper we will use Sequence (1) frequently. In particular, the matrices A and B play a role in the description of the properties of the rope C. Sometimes we will write A_C and B_C in order to stress the dependency of these matrices on the rope C.

Theorem 2.4 shows how the matrix B is visible in the defining equations of C. The matrix A determines its Hartshorne-Rao module as proved in [15], Proposition 3.1.

Proposition 2.5. *Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope supported on the line L with homogeneous ideal $I_C = ((I_L)^2, [x_0, \dots, x_r]B)$. Then its Hartshorne-Rao module $H_*^1(\mathcal{I}_C)$ is isomorphic as R–module to* $\mathrm{coker}(S^{r+1}(-1) \xrightarrow{\varphi_A} \oplus_{j=0}^{r-k} S(\alpha_j - 1))$ *where S is identified with R/I_L.*

Moreover, the Rao function $\rho_C(i) := h^1(\mathcal{I}_C(i))$ of an $(n-k)$–rope $C \subset \mathbb{P}^n$ supported on L is

$$\rho_C(i) = \sum_{j=0}^{r-k} \binom{i+\alpha_j}{1} + \sum_{j=1}^{k} \binom{i-\beta_j}{1} - (r+1)\binom{i}{1}.$$

Taking into account the intimate relation between the matrices A and B, it is not too surprising that the knowledge of the Hartshorne-Rao module allows conclusions on the genus and the degree of a the rope. This is made precise in the next result (cf. [15], Corollary 3.4).

Proposition 2.6. *Let M be a graded R–module of finite length. Then the arithmetic genus of a rope $C \subset \mathbb{P}^n$, whose Hartshorne-Rao module is isomorphic to M as R–module, depends only on M.*

Furthermore, if in addition C is non-degenerate, then also the degree of C is uniquely determined by M.

Proposition 2.5 shows that the Hartshorne-Rao module of a rope in $\mathbb{P}^n$ is annihilated by at least $n-1$ linearly independent linear forms. The next result (cf. [15], Proposition 3.6) describes when there are more such linear forms.

Proposition 2.7. *Let $C \subset \mathbb{P}^n$ be a rope which is not arithmetically Cohen-Macaulay. Then the following conditions are equivalent:*

1. *There are at least n linearly independent linear forms annihilating the Hartshorne-Rao module $M(C)$ of C;*
2. *C is arithmetically Buchsbaum and $M(C) \cong K^m$ for some positive integer $m \leq \frac{n-1}{2}$.*

Moreover, in this case C has genus $-m$ and is cut out by quadrics and possibly linear forms. If C is non-degenerate then C has degree $n-m$.

Thus, the Hartshorne-Rao module of a rope on a line in $\mathbb{P}^n$ is annihilated by either $n-1$ or $n+1$ linearly independent linear forms.

3. Families of arithmetically Gorenstein curves

In this section we describe a family $\mathcal{F}'_n$ of arithmetically Gorenstein curves in $\mathbb{P}^n$ of degree $n+1$ supported on a fixed line for each $n \geq 3$. The family $\mathcal{F}'_n$ is larger than the family $\mathcal{F}_n$ of such curves constructed in ([15], Section 4). Contrary to $\mathcal{F}_n$ the construction of $\mathcal{F}'_n$ does not depend on the choice of generic coordinates. As in the case of $\mathcal{F}_n$, the elements of $\mathcal{F}'_n$ are given by extensions of homogeneous ideals in $T := K[x_0, \dots, x_r]$ such that the quotient ring is an Artinian Gorenstein ring.

Following [2], Example 3.2.11(b), to specify an Artinian Gorenstein ring is suffices to choose a non-trivial K–linear map $g : T_m \to K$, where T_m is the $m-th$ homogeneous part of the ring T. The socle of the corresponding Gorenstein ring appears in degree m. In our case we are interested in Gorenstein rings with h–vector $(1, n-1, 1)$ and so $m = 2$.

Given the K–linear map $g : T_2 \to K$, we define $I_j = \{a \in T_j \mid g(aT_{2-j}) = 0\}$. The ideal $I = \oplus_{j \geq 0} I_j$ is a homogeneous ideal with $I_j = T_j$ for every $j > 2$.

The following result is easy to see. We omit its proof.

Lemma 3.1. *Let $g, g' : T_2 \to K$ be two K–linear maps. Then, $I = I'$ if and only if there exists $\rho \in K \setminus 0$ such that $g' = \rho g$.*

Now, we want to compute explicitly the homogeneous part $(I)_2$ of degree 2 of I.

Proposition 3.2. *With the notation above, if* $g(x_l x_m) = 1$, *then* $(I)_2$ *is generated by* $x_i x_j - g(x_i x_j) x_l x_m$ *where* $0 \le i, j \le r$.

Proof. By definition, $a \in (I)_2$ if and only if $g(a) = 0$. In particular, we see that $\dim_K (I)_2 = \dim_K (T)_2 - 1 = \binom{r+2}{2} - 1$.

Of course, we have $g(x_i x_j - g(x_i x_j) x_l x_m) = g(x_i x_j) - g(x_i x_j) = 0$ for all (i, j) such that $0 \le i, j \le r$, thus $x_i x_j - g(x_i x_j) x_l x_m \in (I)_2$. Since these polynomials generate a vector space V of dimension $\binom{r+2}{2} - 1$ we obtain $V = \ker g$. It follows $(I)_2 = V$ completing the proof. □

Remark 3.3. The hypothesis $g(x_l x_m) = 1$ is equivalent to $g(x_l x_m) \neq 0$ by Lemma 3.1.

Every Artinian Gorenstein ring corresponding to such a map g has $(1, c, 1)$ as $h-$vector where $1 \le c \le n - 1$. Now, we want to single out the $K-$linear maps g which correspond to Artinian Gorenstein rings with $h-$vector $(1, n-1, 1)$. In particular, it turns out that the ideals constructed by this method are generated in degree 2.

Theorem 3.4. *The ideal* I *constructed as above satisfies* $(I)_1 = 0$ *if and only if the (symmetric) matrix* $G = (G_{ij})$ *is non-singular, where* $G_{ij} = g(x_i x_j)$.

Furthermore, in this case all minimal generators of I *have degree* 2.

Proof. We observe that the matrix G is clearly symmetric. Moreover, the last claim is easy to see.

Whenever a polynomial a belongs to T_1 we can write it as $a = \sum_{i=0}^{r} a_i x_i$. Then we get $g(a x_j) = \sum_{i=0}^{r} a_i g(x_i x_j) = \sum_{i=0}^{r} G_{ij} a_i$. By definition, $a \in (I)_1$ if and only if $g(a x_j) = 0$ for all $j = 0 \ldots, r$. This means that $a \in (I)_1$ if and only if the corresponding vector $[a_0, \ldots, a_r]^t$ is a solution of the system $GX = \underline{0}$. Hence, the claim follows. □

We define the family $\mathcal{F}'_n$ as the set of non-degenerate curves in $\mathbb{P}^n$ whose homogeneous ideal is an extension of an ideal $I \subset T$ as constructed above. Thus, the curves in $\mathcal{F}'_n$ are cones over the ideals of T described in Theorem 3.4.

Remark 3.5. The ideals of the family $\mathcal{F}_n$ in [15], Section 4 correspond to maps g such that $g(x_r^2) = 1$.

4. Even Gorenstein liaison classes of ropes

In this section we generalize the results on the even Gorenstein liaison classes of ropes obtained in [15] in various directions. It is always assumed that the support of the ropes is a line.

First, we adapt those results which describe the residual curve of a rope if one links by a curve of the new family $\mathcal{F}'_n$. Second, we prove that ropes supported on the same line with isomorphic Hartshorne-Rao module are always in the same even Gorenstein liaison class. At the end, we present some progresses in the arithmetically Buchsbaum case, which is the only one in which two ropes with isomorphic Hartshorne-Rao module can be supported on different lines.

Proposition 4.1. *Let* $C \subset \mathbb{P}^n$ *be an* $(n-k)$*–rope supported on the line* L *and defined by the ideal* $I_C = ((I_L)^2, \underline{x} B_C)$, *let* $X_1, X_2 \in \mathcal{F}'_n$ *be arithmetically Gorenstein curves supported on* L *with homogeneous ideal* I_{X_1} *and* I_{X_2} *and associated matrix* G_1, G_2 *(cf. Theorem 3.4), and let* D *be the curve directly linked to* C *by* X_1 *and let* E *be the curve directly linked to* D *by* X_2. *Then, we have*

1. D *is a* $(k+1)-$*rope, while* E *is an* $(n-k)-$*rope and both are supported on* L;

2. $B_D = G_1^{-1}A_C^tP$, *where P provides a graded automorphism of* $\oplus_{i=0}^{r-k}S(\alpha_i - 1)$;
3. $A_E = QA_CG_1^{-1}G_2$ *where Q provides a graded automorphism of* $\oplus_{i=0}^{r-k}S(\alpha_i - 1)$.

Proof. The proof is analogous to the arguments for Lemma 5.1, Proposition 5.2 and Corollary 5.3 in [15]. The only difference is that in the proof of [15], Proposition 5.2, we have

$$(B_C)^t(\underline{x})^t\underline{x}B_D = (B_C)^t x_r^2 G_1 B_D = 0 \mod I_{X_1}$$

whereas here we have more generally

$$(B_C)^t(\underline{x})^t\underline{x}B_D = (B_C)^t x_l x_m G_1 B_D = 0 \mod I_{X_1}.$$

□

Now, we show the roles of the various matrices in an example.

Example 4.2. Let $C \subset \mathbb{P}^4$ be the 3-rope defined by the ideal

$$I_C = ((x_0, x_1, x_2)^2, x_0u^2 + x_1tu + x_2t^2).$$

C is supported on the line $L = V(x_0, x_1, x_2)$, and its scheme structure is defined by the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-3) \xrightarrow{\varphi_{B_C}} \mathcal{O}_{\mathbb{P}^1}^3(-1) \xrightarrow{\varphi_{A_C}} \mathcal{O}_{\mathbb{P}^1}^2 \to 0$$

where $A_C = \begin{pmatrix} t & -u & 0 \\ 0 & t & -u \end{pmatrix}$ and $B_C = \begin{pmatrix} u^2 & tu & t^2 \end{pmatrix}^t$.

Let $X \in \mathcal{F}_4'$ by the arithmetically Gorensteincurve defined by the ideal

$$I_X = (x_0^2 - x_2^2, x_0x_1, x_1^2 - x_2^2, x_0x_2, x_1x_2).$$

The associated symmetric matrix is

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By direct computation, the 2-rope D linked to C by X is defined by

$$I_D = I_X : I_C = ((x_0, x_1, x_2)^2, -x_0u + x_1(t+u), x_0(t-u) + x_1u - x_2u)$$

and so the matrix B_D is

$$B_D = \begin{pmatrix} -u & t-u \\ t+u & u \\ 0 & -u \end{pmatrix}.$$

If we apply the formula of the previous Theorem, we have that

$$G^{-1}A_C^t = \begin{pmatrix} -u & t \\ t+u & -t \\ 0 & -u \end{pmatrix},$$

and so we get B_D by multiplying $G^{-1}A_C^t$ with $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If we substitute the last two generators of I_D with their linear combinations we must use a different matrix P. Hence, the matrix P depends on the choice of the generators of I_D.

Now, we can prove our main result which generalizes [15], Theorem 5.4.

Theorem 4.3. *Let $C, E \subset \mathbb{P}^n$ be ropes which are not arithmetically Buchsbaum. Then, the following conditions are equivalent:*

1. *C and E are in the same even Gorenstein liaison class;*

2. *the Hartshorne-Rao modules $M(C)$ and $M(E)$ of C and E are isomorphic as $R-$ modules.*

Remark 4.4. (i) The generalization consists in the fact that we do not assume C, E to have the same degree.

(ii) It is shown in [15], Corollary 3.8, that two non-arithmetically Buchsbaum ropes with isomorphic Hartshorne-Rao modules are neccesarily supported on the same line. This is no longer true for arithmetically Buchsbaum ropes and the only reason why we have to exclude arithmetically Buchsbaum ropes in Theorem 4.3.

Proof of Theorem 4.3. Due to the proof of [15], Theorem 5.4, we have to show only the following

Claim. *If C and E are supported on the same line L, with $\deg(C) < \deg(E)$, and $M(C), M(E)$ are isomorphic as $R-$modules, then C and E are linked in an even number of steps.*

Set $M = M(C)$, and let

$$0 \to F_2 \to F_1 \to F_0 \to M \to 0$$

be a minimal free resolution of M as $S-$module. According to Proposition 2.7, this resolution can be obtained from Sequence (1) by cancelling redundant direct summands. Since M is the Hartshorne-Rao module of ropes of different degrees we get $\mathrm{rank}_S(F_1) < r+1$. By [15], Remark 3.5, the degree of each rope D having a Harshorne-Rao module which is isomorphic to M satisfies the following estimate

$$1 + \mathrm{rank}_S(F_0) \leq \deg(D) \leq n - \mathrm{rank}_S(F_2).$$

Being $\deg(C) < \deg(E)$, we have that $\deg(C) < n - \mathrm{rank}_S(F_2)$. Thus, C is degenerate and is contained in a linear space $V \subset \mathbb{P}^n$ of dimension $\dim V = \deg(C) + \mathrm{rank}_S(F_2)$, because I_C contains $n - \deg(C) - \mathrm{rank}_S(F_2)$ independent linear forms by Theorem 2.4. Let X_1 be an arithmetically Gorenstein curve supported on L which is contained in V as the image of a curve of $\mathcal{F}'_{\deg(C)+\mathrm{rank}_S(F_2)}$ via the isomorphism $V \cong \mathbb{P}^{\deg(C)+\mathrm{rank}_S(F_2)}$. Let D be the rope which is directly linked to C via X_1. Then D is supported on L, contained in V and has degree $\deg(D) = 1 + \mathrm{rank}_S(F_2)$. Now, we choose a linear space U containing V of dimension $\dim U = \deg(E) + \mathrm{rank}_S(F_2)$. Let X_2 be an arithmetically Gorenstein curve supported on L and contained in U as the image of a curve of $\mathcal{F}'_{\deg(E)+\mathrm{rank}_S(F_2)}$ via the isomorphism $U \cong \mathbb{P}^{\deg(E)+\mathrm{rank}_S(F_2)}$. The rope C', which is directly linked to D by X_2, is supported on L and has degree $\deg(C') = \deg(E)$. Moreover, $M(C')$ is isomorphic to $M(E)$ as $R-$module by general results of Liaison theory. Hence, C' and E are in the same even Gorenstein liaison class by [15], Theorem 5.4 and the claim is shown. □

Remark 4.5. Fixing a line L the previous result characterizes all the ropes supported on L which are in the same even Gorenstein liaison class. This characterization holds in the arithmetically Buchsbaumcase, too.

According to [15], Corollary 3.7, a rope $C \subset \mathbb{P}^n$ with genus $g < -\frac{n-1}{2}$ cannot be arithmetically Buchsbaum. Therefore, Theorem 4.3 implies:

Corollary 4.6. *Let $C, E \subset \mathbb{P}^n$ be two ropes having (arithmetic) genera $< -\frac{n-1}{2}$. Then, C and E are in the same even Gorenstein liaison class if and only if their Hartshorne-Rao modules $M(C)$ and $M(E)$ are isomorphic as R–modules.*

Now, we present some partial results for the arithmetically Buchsbaum case.

Proposition 4.7. *Let $C, E \subset \mathbb{P}^n$ be ropes with Hartshorne-Rao modules $M(C) \cong M(E) \cong K$ as $R-$modules. Then, C, E are in the same even Gorenstein liaison class.*

Proof. Denote by L_C and L_E the lines supporting C and E, respectively.

A minimal free resolution of K as $S-$module is

$$0 \to S(-2) \xrightarrow{\begin{pmatrix} u \\ -t \end{pmatrix}} S^2(-1) \xrightarrow{\begin{pmatrix} t & u \end{pmatrix}} S \to K \to 0.$$

Proposition 2.5 and Theorem 2.4 show that there are 2-ropes C' and E' supported on L_C and L_E, respectively.

By the proof of Theorem 4.3, C and C' as well as E and E' belong to the same even Gorenstein liaison class. But due to [15], Theorem 5.6, C' and E' are in the same even Gorenstein liaison class. The claim follows. □

Remark 4.8. According to [15], Proposition 3.6, an arithmetically Buchsbaum rope $C \subset \mathbb{P}^n$ has Hartshorne-Rao module $M(C) \cong K^m$ for some $m \leq (n-1)/2$. Hence, in $\mathbb{P}^4$ only $M(C) \cong K$ or $M(C) = 0$ can occur as Hartshorne-Rao module of an arithmetically Buchsbaum rope. Thus, we obtain a complete characterization of the even Gorenstein liaison classes of ropes in $\mathbb{P}^4$.

Theorem 4.9. *Let $C, E \subset \mathbb{P}^4$ be two ropes. Then C, E are in the same even Gorenstein liaison class if and only if their Hartshorne-Rao modules$M(C)$ and $M(E)$ are isomorphic as R–modules.*

Proof. In spite of Theorem 4.3 and Proposition 4.7 it suffices to consider arithmetically Cohen-Macaulay ropes C and E. But then we have $I_C = I_{L_C}$ or $I_C = (I_{L_C})^2$ where L_C denotes the supporting line of C and the same description for E. This shows that C and E are standard determinantal subschemes. Therefore, both belong to the same even Gorenstein liaison class by the generaliziation of Gaeta's theorem (cf. [7], Theorem 3.6). □

We close this section with a result which restricts the cases to be checked in the arithmetically Buchsbaum case.

Proposition 4.10. *If the assertion of Theorem 4.3 holds for arithmetically Buchsbaum ropes in $\mathbb{P}^n$ with Hartshorne-Rao module K^m for a fixed m then it holds for ropes in $\mathbb{P}^{n+1}$ with Hartshorne-Rao module isomorphic to K^m, too.*

Proof. It is enough to prove that we can link in an even number of steps two ropes $C, C' \subset \mathbb{P}^{n+1}$ supported on different lines L, L', and having K^m as Hartshorne-Rao module. The proof of Theorem 4.3shows that we can link C and C' in two steps to a degenerate rope D and D' supported on $L$and L', respectively. Moreover, we can choose the linking arithmetically Gorenstein curves in such a way that D and D' are contained in a hyperplane H containing the two lines L, L'. Thus, $D, D' \subset H \cong \mathbb{P}^n$ belong to the same even Gorenstein liaison class by our hypothesis. The claim follows. □

Hence, in order to remove the assumption of being non-arithmetically Buchsbaum in Theorem 4.3, it suffices to prove that two arithmetically Buchsbaum ropes $C, C' \subset \mathbb{P}^{2m+1}$ with $M(C) \cong M(C') \cong K^m$, which are supported on different lines, are in the same even Gorenstein liaison class. The first open case is $m = 2$, i.e. 3-ropes in $\mathbb{P}^5$ with Hartshorne-Rao module K^2.

5. Self-linked ropes

In this section, we give some examples of self-linked ropes, i.e. we show that there exist ropes C and arithmetically Gorensteincurves X such that $I_X : I_C = I_C$.

At first, we compute some constraints on the self-linked ropes using Sequence (1).

Proposition 5.1. *Let $C \subset \mathbb{P}^n$ be a non-degenerate, self-linked rope. Then n is odd, $\deg(C) = \frac{n+1}{2}$ and $\alpha_i = \beta_{i+1}$ for all $i = 0, \ldots, r-k+\frac{n-3}{2}$. Furthermore, there exists a symmetric invertible matrix G with entries in S such that $B^tGB = 0$.*

Proof. If C is non-degenerate and self-linked, then $2\deg(C) = n+1$, and so n is odd, and $\deg(C) = (n+1)/2$. Moreover, $k = n - \deg(C) = (n-1)/2$.

Thus, Sequence (1) provides the following free resolution of $(M(C)$

$$0 \to \oplus_{j=1}^{k} S(-\beta_j - 1) \to S^{r+1}(-1) \to \oplus_{i=0}^{r-k} S(\alpha_i - 1) \to M(C) \to 0.$$

Then, a free resolution of $M(C)^\vee$ is

$$0 \to \oplus_{i=0}^{r-k} S(-\alpha_i - 1) \to S^{r+1}(-1) \to \oplus_{j=1}^{k} S(\beta_j - 1) \to M(C)^\vee \to 0.$$

Being C self-linked we have $M(C) \cong M(C)^\vee$. Note that the two resolutions of $M(C)$ must be minimal because C in non-degenerate. Thus, we obtain (after a suitable change of indices) $\alpha_i = \beta_{i+1}$ for all $i = 0, \ldots, r-k$.

Furthermore, by assumption there exists an arithmetically Gorenstein curve X such that $I_X : I_C = I_C$. Since C is non-degenerate X is non-degenerate as well. Thus, X is a curve of the family $\mathcal{F}'_n$ and we get $B = G^{-1}A^tP$ by Proposition 4.1. The previous equation can be written as $GB = A^tP$, but $B^tA^t = 0$, and hence $B^tGB = 0$. $\square$

J. Migliore proved in [9] that a double line in $\mathbb{P}^3_K$ is self-linked if and only if $\mathrm{char}(K) = 2$. In the proof, he explicitly constructs a complete intersection curve X such that $I_X : I_C = I_C$ whatever double line C one considers.

We generalize his construction to some $(k+1)-$ropes in $\mathbb{P}^{2k+1}_K$ when $\mathrm{char}(K) = 2$, and to some $(2k+1)-$ropes in $\mathbb{P}^{4k+1}_K$ in arbitrary characteristic, giving examples of self-linked ropes.

Definition 5.2. If $\underline{a} = (a_1, \ldots, a_l)$ is a vector, we denote by $\mathrm{rev}(\underline{a})$ the vector

$$\mathrm{rev}(\underline{a}) = (a_l, \ldots, a_1).$$

Using this notation we have.

Proposition 5.3. *Let $\mathrm{char}(K) = 2$. Let A be a $k \times 2k$ matrix which gives a graded homomorphism $S^{2k}(-1) \xrightarrow{A} \oplus_{i=1}^{k} S(\alpha_i - 1)$ with a cokernel of finite length. Denoting the rows of A by $\underline{a}_1, \ldots, \underline{a}_k$ assume that $\underline{a}_i^t$ and $rev(\underline{a}_i)^t$ are syzygies of $\underline{a}_h$ for $1 \leq i \leq h-1, 2 \leq h \leq k$. Then, the corresponding rope C is self-linked.*

Proof. Let B be the $2k \times k$ matrix having $\mathrm{rev}(\underline{a}_1)^t, \ldots, \mathrm{rev}(\underline{a}_k)^t$ as its columns. At first, we show that B gives the syzygyes of A. In fact,

$$\underline{a}_i \cdot \mathrm{rev}(\underline{a}_j)^t = \begin{cases} 0 & \text{if} \quad i \neq j \\ 2\sum_{h=1}^{k} a_{ih}a_{i,2k+1-h} & \text{if} \quad i = j \end{cases}.$$

Being $\mathrm{char}(K) = 2$, we get that $\underline{a}_i \cdot \mathrm{rev}(\underline{a}_j)^t = 0$ for every i, j. Hence, a minimal free resolution of $\mathrm{coker}(A)$ is

$$0 \to \oplus_{i=1}^{k} S(-\alpha_i - 1) \xrightarrow{B} S^{2k} \xrightarrow{A} \oplus_{i=1}^{k} S(\alpha_i - 1) \to \mathrm{coker}(A) \to 0.$$

Let L be the line $\operatorname{Proj} S$ in $\mathbb{P}^{2k+1}$ and let C be the rope supported on L associated to A and B, respectively.

Let $X \in \mathcal{F}'_{2k+1}$ be the arithmetically Gorenstein curve supported on L associated to the symmetric matrix $G = (G_{ij})$ where

$$G_{ij} = \begin{cases} 1 & \text{if } i+j = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

By construction, we have $X \in \mathcal{F}'_{2k+1} \setminus \mathcal{F}_{2k+1}$.

Notice that $G^2 = I$, and so $G^{-1} = G$. Let D be the rope which is directly linked to C by X. Then we get by Proposition 4.1 that $B_D = G^{-1}A^tP = BP$ for some invertible matrix P. This implies $D = C$, i.e. C is self-linked. $\square$

An example of a matrix which satisfies the hypotheses of Proposition 5.3 is $A = \begin{pmatrix} t & u & 0 & 0 \\ t^2u & t^3 & tu^2 & u^3 \end{pmatrix}$.

Now, we give examples of self-linked ropes in $\mathbb{P}^{4k+1}$ in arbitrary characteristic, for every $k \geq 1$.

Let F, G be two forms in S_d without common factors, and let A be the $2k \times 4k$ matrix defined as $A = \begin{pmatrix} FI_{2k} & | & GI_{2k} \end{pmatrix}$, where I_{2k} is the identity matrix of order $2k$. Let $I'_{2k} = (\delta'_{ij})$ be the square diagonal matrix of order $2k$ defined by

$$\delta'_{ij} = \begin{cases} (-1)^i & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Then, the syzygy matrix B of A is $B = \begin{pmatrix} GI'_{2k} \\ -FI'_{2k} \end{pmatrix}$. Let L be the line in $\mathbb{P}^{4k+1} = \operatorname{Proj}(K[x_1, \dots, x_{2k}, y_1, \dots, y_{2k}, t, u])$ defined as $L = V(x_1, \dots, x_{2k}, y_1, \dots, y_{2k})$.

Proposition 5.4. *The rope $C \subset \mathbb{P}^{4k+1}$ defined by $I_C = ((I_L)^2, [x_1, \dots, y_{2k}]B)$ is self-linked.*

Proof. Let $X \in \mathcal{F}'_{4k+1}$ be the arithmetically Gorenstein curve supported on L associated to the symmetric matrix $G = (G_{ij})$ where

$$G_{ij} = \begin{cases} (-1)^{i+1} & \text{if} \quad i+j = 4k+1, 1 \leq i \leq 2k \\ (-1)^i & \text{if} \quad i+j = 4k+1, 2k+1 \leq i \leq 4k \\ 0 & \text{if} \quad i+j \neq 4k+1. \end{cases}$$

By Proposition 4.1, the rope linked to C by X is associated to $G^{-1}A_C^t$ but this matrix is B_C up to reordering the columns. Then, $I_X : I_C = I_C$ and so C is self-linked. $\square$

If $\operatorname{char}(K) = 2$ then the previous result is a particular case of Proposition 5.3. However, these examples show that self-linkage of ropes is far from being well understood in $\mathbb{P}^n, n > 3$, and it is much more complicated than in $\mathbb{P}^3$.

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