ALGEBRAIC SHIFTING AND EXTERIOR AND SYMMETRIC ALGEBRA METHODS

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ABSTRACT. We define and study Cartan–Betti numbers of a graded ideal J in the exterior algebra over an infinite field which include the usual graded Betti numbers of J as a special case. Following ideas of Conca regarding Koszul–Betti numbers over the symmetric algebra, we show that Cartan–Betti numbers increase by passing to the generic initial ideal and the squarefree lexsegement ideal respectively. Moreover, we characterize the cases where the inequalities become equalities. As combinatorial applications of the first part of this note and some further symmetric algebra methods we establish results about algebraic shifting of simplicial complexes and use them to compare different shifting operations. In particular, we show that each shifting operation does not decrease the number of facets, and that the exterior shifting is the best among the exterior shifting operations in the sense that it increases the number of facets the least.

1. INTRODUCTION

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field K of characteristic 0, let $I \subset S$ be a graded ideal and denote by $\beta^S_{ij}(S/I) = \dim_K \operatorname{Tor}^S_i(S/I,K)_j$ the graded Betti numbers of S/I. In the last decades the graded Betti numbers $\beta^S_{ij}(S/I)$ were studied intensively. To I one associates several important monomial ideals like the generic initial ideal $\operatorname{gin}(I)$ with respect to the reverse lexicographic term order, or the lexsegment ideal $\operatorname{lex}(I)$. It is well-known by work of Bigatti [10], Hulett [24] and Pardue [32] that there are inequalities $\beta^S_{ij}(S/I) \le \beta^S_{ij}(S/\operatorname{gin}(I)) \le \beta^S_{ij}(S/\operatorname{lex}(I))$ for all i,j. The cases where these are equalities for all i,j have been characterized by Aramova–Herzog–Hibi [4] and Herzog–Hibi [20]. The first inequality is an equality for all i,j if and only if J is a componentwise linear ideal, and we have equalities everywhere for all i,j if and only if J is a Gotzmann ideal. All these results were generalized to so–called Koszul–Betti numbers by Conca [15].

Let $E = K\langle e_1,\dots,e_n\rangle$ be the exterior algebra over an infinite field K and $J \subset E$ be a graded ideal. Aramova–Herzog–Hibi [2] showed that the constructions and results mentioned above hold similarly for J. More precisely, there exists a generic initial ideal $\operatorname{gin}(J) \subset E$ with respect to the reverse lexicographic term order, and the unique squarefree lexsegment ideal $\operatorname{lex}(J) \subset E$ with the same Hilbert function as J. Let $\beta^E_{ij}(E/J) = \dim_K \operatorname{Tor}^E_i(E/J,K)_j$ be the graded Betti numbers of E/J. Then we also have the inequalities $\beta^E_{ij}(E/J) \le \beta^E_{ij}(E/\operatorname{gin}(J)) \le \beta^E_{ij}(E/\operatorname{lex}(J))$ for all integers i,j. The first inequality is again an equality for all i,j if and only if J is a componentwise linear ideal as was observed in [4].

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The first part of this paper is devoted to extend the latter results to statements about Cartan–Betti numbers, similarly as Conca did in the symmetric algebra case for Koszul–Betti numbers. Over the exterior algebra there exists the construction of the Cartan complex and Cartan homology (see Section 2 for details), which behave in many ways like the Koszul complex and Koszul homology. Following Conca's ideas we define the Cartan–Betti numbers as $\beta^E_{ijp}(E/J) = \dim_K H_i(f_1,\dots,f_p;E/J)_j$ where $H_i(f_1,\dots,f_p;E/J)$ is the i-th Cartan homology of E/J with respect to a generic sequence of linear forms $f_1,\dots,f_p$ for $1 \leq p \leq n$. Observe that this definition does not depend on the chosen generic sequence and that these modules are naturally graded. We set $H_i(f_1,\dots,f_p;E/J) = 0$ for $i > p$. Note that $\beta^E_{ijn}(E/J) = \beta^E_{ij}(E/J)$ are the graded Betti numbers of E/J. Observe that a generic initial ideal $\mathrm{gin}_\tau(J)$ can be defined with respect to any term order τ on E. Our first main result, Theorem 2.10, shows that, for all integers i,j,p:

$$\beta^E_{ijp}(E/J) \leq \beta^E_{ijp}(E/\mathrm{gin}_\tau(J)) \leq \beta^E_{ijp}(E/\mathrm{lex}(J)).$$

Analogously to Conca's result in [15] we can characterize precisely the cases where the inequalities become equalities. At first we have:

Theorem 1.1. *Let $J \subset E$ be a graded ideal. The following conditions are equivalent:*

(i) $\beta^E_{ijp}(E/J) = \beta^E_{ijp}(E/\mathrm{gin}(J))$ *for all i,j,p;*
(ii) $\beta^E_{1jn}(E/J) = \beta^E_{1jn}(E/\mathrm{gin}(J))$ *for all j;*
(iii) *J is a componentwise linear ideal;*
(iv) *A generic sequence of linear forms $f_1,\dots,f_n$ is a proper sequence of E/J.*

Moreover, we show:

Theorem 1.2. *For each graded ideal $J \subset E$, the following statements are equivalent:*

(i) $\beta^E_{ijp}(E/J) = \beta^E_{ijp}(E/\mathrm{lex}(J))$ *for all i,j,p;*
(ii) $\beta^E_{1jn}(E/J) = \beta^E_{1jn}(E/\mathrm{lex}(J))$ *for all j;*
(iii) *J is a Gotzmann ideal in the exterior algebra;*
(iv) $\beta^E_{0jp}(E/J) = \beta^E_{0jp}(E/\mathrm{lex}(J))$ *for all j,p and J is componentwise linear.*

Section 2 ends with the discussion which generic initial ideal increases the Cartan–Betti numbers the least. In Theorem 2.14 we show that for all i,j,p:

$$\beta^E_{ijp}(E/J) \leq \beta^E_{ijp}(E/\mathrm{gin}(J)) \leq \beta^E_{ijp}(E/\mathrm{gin}_\tau(J)).$$

The second part of this note presents combinatorial applications of the results mentioned above and some symmetric algebra methods. More precisely, we study properties of simplicial complexes under shifting operations. Let $\mathscr{C}([n])$ be the set of simplicial complexes on n vertices. Following Kalai [25], [26], an algebraic shifting operation is a map $\Delta\colon \mathscr{C}([n]) \to \mathscr{C}([n])$ satisfying certain conditions. We refer to Section 3 for precise definitions. Intuitively, shifting replaces each complex Γ with a combinatorially simpler complex $\Delta(\Gamma)$ that still captures some properties of Γ. Shifting has become an important technique that has been successfully applied in various contexts and deserves to be investigated in its own right (cf., for example, [5], [6], [7], [8], [28], [29], [31]).

Several shifting operations can be interpreted algebraically. Denote by I_Γ the Stanley–Reisner ideal of Γ in the polynomial ring $S = K[x_1,\dots,x_n]$. The symmetric algebraic shifting Δ^s can be realized by passing to a kind of polarization of the generic initial ideal of I_Γ with respect to the reverse lexicographic order (cf. Example 3.1). Similarly, denote by J_Γ the exterior Stanley–Reisner ideal in the exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$. The passage from J_Γ to its generic initial ideal with respect to any term order τ on E leads to the exterior algebraic shifting operation Δ^τ (Example 3.2). If τ is the reverse-lexicographic order, we call $\Delta^e := \Delta^\tau$ the exterior algebraic shifted complex.

While each shifting operation preserves the f-vector, the number of facets may change. However, we show in Theorem 4.1 that it cannot decrease. After recalling basic definitions in Section 3, we first study symmetric algebraic shifting. To this end we use degree functions. In Section 4 we show that the smallest extended degree is preserved under symmetric algebraic shifting whereas the arithmetic degree $\operatorname{adeg}(\Gamma)$ may increase because it equals the number of facets. However, the smallest extended degree and the arithmetic degree of each shifted complex agree.

In Section 5 we use these degree functions to give a short new proof of the fact that $\Delta^s(\Gamma)$ is a pure complex if and only if Γ is Cohen–Macaulay. We then derive a combinatorial interpretation of the smallest extended degree using iterated Betti numbers and show that these numbers agree with the h-triangle if and only if Γ is sequentially Cohen–Macaulay.

We conclude this note by showing in Section 6 that the exterior shift is the best exterior shifting operation in the sense that it increases the number of facets the least, i. e.

$$\operatorname{adeg}\Gamma \leq \operatorname{adeg}\Delta^e(\Gamma) \leq \operatorname{adeg}\Delta^\tau(\Gamma).$$

Moreover, Δ^e preserves the arithmetic degree of Γ if and only Γ is sequentially Cohen–Macaulay. These results rely on the full strength of the results in Section 2. The proof also uses Alexander duality and a reinterpretation of the arithmetic degree over the exterior algebra.

For notions and results related to commutative algebra we refer to Eisenbud [18], Bruns–Herzog [13] and Vasconcelos [36]. For details on combinatorics we refer to the book of Stanley [33] and Kalai [26].

2. Cartan–Betti numbers

In this section we introduce and study Cartan–Betti numbers of graded ideals in the exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$ over an infinite field K. Our results rely on techniques from Gröbner basis theory in the exterior algebra. Its basics are treated in [2]. We begin with establishing some extensions of the theory that are analogous to results over the polynomial ring.

Recall that a monomial of degree k in E is an element $e_F = e_{a_1} \wedge \ldots \wedge e_{a_k}$ where $F = \{a_1, \ldots, a_k\}$ is a subset of $[n]$ with $a_1 < \ldots < a_k$. To simplify notation, we write sometimes $fg = f \wedge g$ for any two elements $f, g \in E$. We will use only term orders τ on E that satisfy $e_1 >_\tau e_2 >_\tau \ldots >_\tau e_n$. Given a term order τ on E and a graded ideal $J \subset E$ we denote by $\operatorname{in}_\tau(J)$ and $\operatorname{gin}_\tau(J)$ respectively the *initial ideal* of J and the *generic initial ideal* of J with respect to τ. We also write $\operatorname{in}(J)$ and $\operatorname{gin}(J)$ for the initial ideal of J and the generic initial ideal of J with respect to the reverse lexicographic order on E induced by $e_1 > \cdots > e_n$.

Recall that a monomial ideal $J \subset E$ is called a *squarefree strongly stable ideal* with respect to $e_1 > \cdots > e_n$ if for all $F \subseteq [n]$ with $e_F \in J$ and all $i \in F$, $j < i$, $j \notin F$ we have that

$e_j \wedge e_{F\setminus\{i\}} \in J$. Given an ideal $J \subset E$ and a term order τ on E, note that $\operatorname{gin}_\tau(J)$ is always squarefree strongly stable and that $\operatorname{gin}_\tau(J) = J$ iff J is squarefree strongly stable which can be seen analogously to the case of a polynomial ring. As expected, the passage to initial ideals increases the graded Betti numbers. The next result generalizes [2, Proposition 1.8]. It is the exterior analogue of a well-known fact for the polynomial ring (cf. [15, Lemma 2.1]).

Proposition 2.1. *Let $J, J' \subset E$ be graded ideals and let τ be any term order on E. Then, for all $i, j \in \mathbb{Z}$:*

$$\dim_K[\operatorname{Tor}_i^E(E/J, E/J')]_j \leq \dim_K[\operatorname{Tor}_i^E(E/\operatorname{in}_\tau(J), E/\operatorname{in}_\tau(J'))]_j.$$

Proof. This follows from standard deformation arguments. The proof is verbatim the same as the one of the symmetric version in [15, Lemma 2.1] after replacing the polynomial ring S with the exterior algebra E. □

In the following we present exterior versions of results of Conca in [15]. We follow the strategy of his proofs and replace symmetric algebra methods and the Koszul complex by exterior algebra methods and the Cartan complex. We first recall the construction of the Cartan complex (see, e.g., [2] or [19]). For important results on this complex, we refer to [1, 2, 3, 4].

For a sequence $\mathbf{v} = v_1, \ldots, v_m \subseteq E_1$, the Cartan complex $C_\bullet(\mathbf{v}; E)$ is defined to be the free divided power algebra $E\langle x_1, \ldots, x_m\rangle$ together with a differential δ. The free divided power algebra $E\langle x_1, \ldots, x_m\rangle$ is generated over E by the divided powers $x_i^{(j)}$, $i = 1, \ldots, m$ and $j \geq 0$, that satisfy the relations $x_i^{(j)} x_i^{(k)} = \frac{(j+k)!}{j!k!} x_i^{(j+k)}$. We set $x_i^{(0)} = 1$ and $x_i^{(1)} = x_i$ for $i = 1, \ldots, m$. Hence $C_\bullet(\mathbf{v}; E)$ is a free E-module with basis $x^{(a)} := x_1^{(a_1)} \ldots x_m^{(a_m)}$, $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$. We set $\deg x^{(a)} = i$ if $|a| := a_1 + \ldots + a_m = i$ and $C_i(\mathbf{v}; E) = \bigoplus_{|a|=i} E x^{(a)}$. The E-linear differential ∂ on $C_\bullet(\mathbf{v}; E)$ is defined as follows. For $x^{(a)} = x_1^{(a_1)} \ldots x_m^{(a_m)}$ we set $\partial(x^{(a)}) = \sum_{a_i > 0} v_i \cdot x_1^{(a_1)} \ldots x_i^{(a_i - 1)} \ldots x_m^{(a_m)}$. One easily checks that $\partial \circ \partial = 0$, thus $C_\bullet(\mathbf{v}; E)$ is indeed a complex.

We denote by $\mathscr{M}$ the category of finitely generated graded left and right E-modules M, satisfying $ax = (-1)^{|\deg(a)||\deg(x)|} xa$ for all homogeneous $a \in E$ and $x \in M$. For example, every graded ideal $J \subseteq E$ belongs to $\mathscr{M}$. For $M \in \mathscr{M}$ and $\mathbf{v} = v_1, \ldots, v_m \subseteq E_1$, the complex $C_\bullet(\mathbf{v}; M) = C_\bullet(\mathbf{v}; E) \otimes_E M$ is called the *Cartan complex* of M with respect to $\mathbf{v}$. Its homology is denoted by $H_\bullet(\mathbf{v}; M)$; it is called *Cartan homology*.

There is a natural grading of this complex and its homology. We set $\deg x_i = 1$ and $C_j(\mathbf{v}; M)_i := \operatorname{span}_K\{m_a x^{(b)} \mid m_a \in M_a, a + |b| = i, |b| = j\}$. Cartan homology can be computed recursively. For $j = 1, \ldots, m-1$, the following sequence is exact:

$$(1)\quad 0 \to C_\bullet(v_1, \ldots, v_j; M) \xrightarrow{\iota} C_\bullet(v_1, \ldots, v_{j+1}; M) \xrightarrow{\varphi} C_{\bullet-1}(v_1, \ldots, v_{j+1}; M)(-1) \to 0.$$

Here ι is a natural inclusion map and φ is given by

$$\varphi(g_0 + g_1 x_{j+1} + \ldots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \ldots + g_k x_{j+1}^{(k-1)} \text{ for } g_i \in C_{i-k}(v_1, \ldots, v_j; M).$$

The associated long exact homology sequence is

$$\ldots \to H_i(v_1, \ldots, v_j; M) \xrightarrow{\alpha_i} H_i(v_1, \ldots, v_{j+1}; M) \xrightarrow{\beta_i} H_{i-1}(v_1, \ldots, v_{j+1}; M)(-1)$$

$$\xrightarrow{\delta_{i-1}} H_{i-1}(v_1,\dots,v_j;M) \xrightarrow{\alpha_{i-1}} H_{i-1}(v_1,\dots,v_{j+1};M) \xrightarrow{\beta_{i-1}} \dots,$$

where α_i and β_i are induced by ι and φ, respectively. For a cycle $z = g_0 + g_1 x_{j+1} + \dots + g_{i-1}x_{j+1}^{(i-1)}$ in $C_{i-1}(v_1,\dots,v_{j+1};M)$ one has $\delta_{i-1}([z]) = [g_0 v_{j+1}]$. It is now easy to see that e. g. for $e_t,\dots,e_n$, the Cartan complex $C_\bullet(e_t,\dots,e_n;E)$ is a free resolution of $E/(e_t,\dots,e_n)$. Hence, for each module $M \in \mathscr{M}$, there are isomorphisms

$$\operatorname{Tor}_i^E(E/(e_t,\dots,e_n),M) \cong H_i(e_t,\dots,e_n;M). \tag{2}$$

In particular, for $\mathbf{e} = e_1,\dots,e_n$ we have $K \cong E/(\mathbf{e})$ and there are isomorphisms of graded K-vector spaces $\operatorname{Tor}_i^E(K,M) \cong H_i(\mathbf{e};M)$.

Cartan homology is useful to study resolutions of graded ideals in the exterior algebra. For example, it has been used to show that the Castelnuovo–Mumford regularity reg does not change by passing to the generic initial ideal with respect to the reverse lexicographic order, i. e. $\operatorname{reg}(E/J) = \operatorname{reg}(E/\operatorname{gin}(J))$ (cf. [6], Theorem 5.3).

There are several other algebraic invariants which behave similarly. See for example [23] for results in this direction. Here, we are interested in the following numbers which, for $p = n$, include the graded Betti numbers of E/J:

Definition 2.2. Let $J \subset E$ be a graded ideal and $f_1,\dots,f_p$ be a generic sequence of linear forms. Let $C(f_1,\dots,f_p;E/J)$ be the Cartan complex of E/J with respect to that sequence and $H(f_1,\dots,f_p;E/J)$ the corresponding Cartan homology. We denote by

$$\beta_{ijp}^E(E/J) = \dim_K H_i(f_1,\dots,f_p;E/J)_j$$

the *Cartan–Betti numbers* of E/J for $i = 0,\dots,p$. We set $H_i(f_1,\dots,f_p;E/J) = 0$ for $i > p$.

A generic sequence means here that there exists a non-empty Zariski open set U in $K^{p\times n}$ such that if one chooses the $p\times n$ coefficients that express $f_1,\dots,f_p$ as linear combinations of $e_1,\dots,e_n$ inside U, then the ranks of the maps in each degree in the Cartan complex are as big as possible. In particular, the K-vector space dimension of the homology is the same for each generic sequences and thus the definition of $\beta_{ijp}^E(E/J)$ does not depend on the chosen generic sequence.

We are going to explain that it is easy to compute the Cartan–Betti numbers for squarefree strongly stable ideals in the exterior algebra. In fact, given a generic sequence of linear forms $f_1,\dots,f_p$ there exists an invertible upper triangular matrix g such that the linear space spanned by $f_1,\dots,f_p$ is mapped by the induced isomorphism $g\colon E \to E$ to the one generated by $e_{n-p+1},\dots,e_n$. It follows that, as graded K-vector spaces, $H_i(f_1,\dots,f_p;E/J) \cong H_i(e_{n-p+1},\dots,e_n;E/g(J))$. Since J is squarefree strongly stable we have that $g(J) = J$ and thus we get:

Lemma 2.3. *Let $J \subset E$ be a squarefree strongly stable ideal. Then we have for all i,j,p:*

$$\beta_{ijp}^E(E/J) = \dim_K H_i(e_{n-p+1},\dots,e_n;E/J)_j.$$

Aramova, Herzog and Hibi [2] computed the Cartan–Betti numbers for squarefree strongly stable ideals which gives rise to Eliahou–Kervaire type resolutions in the exterior algebra. To present this result we need some more notation. Recall that for a monomial ideal $J \subset E$ we denote by $G(J)$ the unique minimal system of monomial generators of J.

For a monomial $e_F \in E$ where $F \subseteq [n]$ we set $\max(F) = \max\{i \in F\}$. Given a set G of monomials we define:

$$\begin{aligned} m_i(G) &= |\{e_F \in G : \max(e_F) = i\}|, \\ m_{\leq i}(G) &= |\{e_F \in G : \max(e_F) \leq i\}|, \\ m_{ij}(G) &= |\{e_F \in G : \max(e_F) = i,\ |F| = j\}|. \end{aligned}$$

For a monomial ideal or a vector space generated by monomials J, we denote by $m_i(J)$, $m_{\leq i}(J)$ and $m_{ij}(J)$ the numbers $m_i(G)$, $m_{\leq i}(G)$ and $m_{ij}(G)$ where G is the set of minimal monomial generators of J. Using Lemma 2.3 we restate the result of Aramova, Herzog and Hibi as follows:

Theorem 2.4. *Let $J \subset E$ be a squarefree strongly stable ideal. Then the Cartan–Betti numbers are given by the following formulas:*

(i) *(Aramova–Herzog–Hibi* [2]*) For all $i > 0$, $p \in [n]$ and every $j \in \mathbb{Z}$ we have*

$$\beta^E_{ijp}(E/J) = \sum_{k=n-p+1}^{n} m_{k,j-i+1}(J) \binom{k+p-n+i-2}{i-1}.$$

(ii) *For all $p \in [n]$ and every $j \in \mathbb{Z}$ we have*

$$\beta^E_{0jp}(E/J) = \binom{n-p}{j} - m_{\leq n-p}(J_j)$$

where J_j is the degree j component of the ideal J.

Proof. Only (ii) needs a proof. But since $e_{n-p+1}, \dots, e_n$ is generic for E/J we can compute $\beta^E_{0jp}(E/J)$ from the Cartan homology with respect to this sequence. Then the formula follows from a direct computation. $\square$

Note that for $p = n$ we get the graded Betti numbers of E/J. We need another preparatory result, which can be proved analogously to [15, Proposition 3.6 (i)] using Theorem 2.4.

Lemma 2.5. *Let J and J' be squarefree strongly stable ideals with the same Hilbert-function such that $m_{\leq i}(J_j) \leq m_{\leq i}(J'_j)$ for all i, j. Then $m_i(J') \leq m_i(J)$ for all i with equalities everywhere if and only if $m_{\leq i}(J_j) = m_{\leq i}(J'_j)$ for all i, j.*

Using the numbers $m_{\leq i}(J_j)$, it is possible to compare the Cartan–Betti numbers of squarefree strongly stable ideals. In fact we have the following:

Proposition 2.6. *Let $J, J' \subset E$ be squarefree strongly stable ideals with the same Hilbert function such that $m_{\leq i}(J_j) \leq m_{\leq i}(J'_j)$ for all i, j. Then $\beta^E_{ijp}(E/J') \leq \beta^E_{ijp}(E/J)$ for all i, j, p with equalities everywhere if and only if $m_{\leq i}(J_j) = m_{\leq i}(J'_j)$ for all i, j.*

Proof. It follows from Theorem 2.4 (ii) and the assumption that

$$\begin{aligned} \beta^E_{0jp}(E/J) - \beta^E_{0jp}(E/J') &= \binom{n-p}{j} - m_{\leq n-p}(J_j) - \binom{n-p}{j} + m_{\leq n-p}(J'_j) \\ &= m_{\leq n-p}(J'_j) - m_{\leq n-p}(J_j) \\ &\geq 0. \end{aligned}$$

This shows the desired inequalities for $i = 0$. Next assume that $i > 0$. We compute

$$\begin{aligned}
&\beta^E_{ijp}(E/J') \\
= &\sum_{k=n-p+1}^{n} m_{k,j-i+1}(J')\binom{k+p-n+i-2}{i-1} \\
= &\sum_{k=n-p+1}^{n} \big(m_k(J'_{j-i+1}) - m_{\leq k-1}(J'_{j-i})\big)\binom{k+p-n+i-2}{i-1} \\
= &\sum_{k=n-p+1}^{n} \big(m_{\leq k}(J'_{j-i+1}) - m_{\leq k-1}(J'_{j-i+1}) - m_{\leq k-1}(J'_{j-i})\big)\binom{k+p-n+i-2}{i-1} \\
= &\ m_{\leq n}(J'_{j-i+1})\binom{p+i-2}{i-1} - m_{\leq n-p}(J'_{j-i+1}) \\
+ &\sum_{k=n-p+1}^{n-1} m_{\leq k}(J'_{j-i+1})\left(\binom{k+p-n+i-2}{i-1} - \binom{k+p-n+i-1}{i-1}\right) \\
- &\sum_{k=n-p+1}^{n} m_{\leq k-1}(J'_{j-i})\binom{k+p-n+i-2}{i-1} \\
= &\ \dim_K(J'_{j-i+1})\binom{p+i-2}{i-1} - m_{\leq n-p}(J'_{j-i+1}) \\
- &\sum_{k=n-p+1}^{n-1} m_{\leq k}(J'_{j-i+1})\binom{k+p-n+i-2}{i-2} \\
- &\sum_{k=n-p+1}^{n} m_{\leq k-1}(J'_{j-i})\binom{k+p-n+i-2}{i-1}
\end{aligned}$$

and similarly $\beta^E_{ijp}(E/J)$. We know by assumption that $\dim_K(J'_{j-i+1}) = \dim_K(J_{j-i+1})$ and $m_{\leq i}(J_j) \leq m_{\leq i}(J'_j)$ for all i, j. Hence the last equation of the computation implies $\beta^E_{ijp}(E/J) - \beta^E_{ijp}(E/J') \geq 0$ because the left-hand side is a sum of various $m_{\leq i}(J'_j) - m_{\leq i}(J_j)$ times some binomial coefficients. Moreover, we have equalities everywhere if and only if $m_{\leq i}(J_j) = m_{\leq i}(J'_j)$ for all i, j. □

Using the above results as well as [3, Lemma 3.7] instead of Proposition 3.5 in [15], we get the following rigidity statement that is proved analogously to Proposition 3.7 in [15]:

Corollary 2.7. *Let $J, J' \subset E$ be a squarefree strongly stable ideals with the same Hilbert function and $m_{\leq i}(J_j) \leq m_{\leq i}(J'_j)$ for all i, j. Then the follows statements are equivalent:*

(i) $\beta^E_{ijp}(E/J') = \beta^E_{ijp}(E/J)$ *for all i, j and p;*
(ii) $\beta^E_{ij}(E/J') = \beta^E_{ij}(E/J)$ *for all i and j;*
(iii) $\beta^E_{1j}(E/J') = \beta^E_{1j}(E/J)$ *for all j;*
(iv) $\beta^E_{1}(E/J') = \beta^E_{1}(E/J)$*;*
(v) $m_{ij}(J') = m_{ij}(J)$ *for all i, j;*

(vi) *$m_i(J') = m_i(J)$ for all i;*
(vii) *$m_i(J'_j) = m_i(J_j)$ for all i, j;*
(viii) *$m_{\leq i}(J'_j) = m_{\leq i}(J_j)$ for all i, j.*

Given a graded ideal $J \subset E$ there exists a unique *squarefree lexsegment ideal* $\operatorname{lex}(J) \subset E$ with the same Hilbert function as J (see [2] for details). Next we present a mild variation of a crucial result in [3] which was an important step to compare Betti numbers of squarefree strongly stable ideals and their corresponding squarefree lexsegment ideals.

Theorem 2.8. *(Aramova–Herzog–Hibi* [3]*) Let $J \subset E$ be a squarefree strongly stable ideal and let $L \subset E$ be its squarefree lexsegment ideal. Then*
(i) *$m_{\leq i}(L) \leq m_{\leq i}(J)$ for all i.*
(ii) *$m_{\leq i}(L_j) \leq m_{\leq i}(J_j)$ for all i, j.*

Proof. (i) was shown in [3, Theorem 3.9] under the weaker assumption that $J \subset E$ is a squarefree strongly stable ideal and $L \subset E$ is a squarefree lexsegment ideal such that $\dim_K L_t \leq \dim_K J_t$ for all t.

(ii) is also a consequence of [3, Theorem 3.9] by observing the following facts. For $j \in \mathbb{N}$ consider the ideals $J' = (J_j)$ and $L' = (L_j)$ of E. Then J' is a squarefree strongly stable ideal, L' is a squarefree lexsegment ideal, and $\dim_K L'_t \leq \dim_K J'_t$ for all t by [2, Theorem 4.2]. □

As [14, Lemma 2] one shows for Cartan–Betti numbers:

Lemma 2.9. *Let $J \subset E$ be a graded ideal. Then*

$$\beta^E_{0jp}(E/J) = \beta^E_{0jp}(E/\operatorname{gin}(J)) \text{ for all } j, p.$$

Proof. Consider the graded K-algebra homomorphism $g\colon E \to E$ that is induced by a generic matrix in $\mathrm{GL}_n(K)$. Then the linear forms $f_i := g^{-1}(e_{n-p+i})$, $i = 1, \dots, p$, are generic too and the Hilbert functions of $E/(J+L)$ and $E/(g(J)+L')$ coincide, where $L = (f_1, \dots, f_p)$ and $L' = (e_{n-p+1}, \dots, e_n)$. Note that passing to the initial ideals does not change the Hilbert function and that $\operatorname{in}(g(J)+L') = \operatorname{in}(g(J)) + L'$ because the chosen term order is revlex. Furthermore, we have that $\operatorname{in}(g(J)) = \operatorname{gin}(J)$ because g is generic. Hence, the Hilbert function of $E/(J+L)$ equals the one of $E/(\operatorname{gin}(J)+L')$. Since $\beta^E_{0jp}(E/J) = \dim_K(E/(J+L))_j$ and $\beta^E_{0jp}(E/\operatorname{gin}(J)) = \dim_K(E/(\operatorname{gin}(J)+L'))_j$, our claim follows. □

The following result generalizes [2, Prop. 1.8, Theorem 4.4]. The proof is similar to [15, Theorem 4.3].

Theorem 2.10. *Let $J \subset E$ be a graded ideal and τ be an arbitrary term order on E. Then*
(i) *$\beta^E_{ijp}(E/J) \leq \beta^E_{ijp}(E/\operatorname{gin}_\tau(J))$ for all i, j, p.*
(ii) *$\beta^E_{ijp}(E/J) \leq \beta^E_{ijp}(E/\operatorname{lex}(J))$ for all i, j, p.*

Proof. (i): Let $g \in \mathrm{GL}_n(K)$ be a generic matrix and let f_i be the preimage of e_{n-p+1} under the induced K-algebra isomorphism $g\colon E \to E$ where $i = 1, \dots, p$. Then

$$H_i(f_1, \dots, f_p; E/J) \cong H_i(e_{n-p+1}, \dots, e_n; E/g(J)) \cong \operatorname{Tor}^E_i(E/g(J), E/(e_{n-p+1}, \dots, e_n))$$

where the last isomorphism was noted above in (2). Proposition 2.1 provides:

$$\begin{aligned}\beta^E_{ijp}(E/J) &\leq \dim_K \operatorname{Tor}^E_i(E/\operatorname{in}_\tau(g(J)), E/(e_{n-p+1},\dots,e_n))_j \\ &= \dim_K H_i(e_{n-p+1},\dots,e_n; E/\operatorname{in}_\tau(g(J)))_j.\end{aligned}$$

Since g is a generic matrix, $\operatorname{in}_\tau(g(J)) = \operatorname{gin}_\tau(J)$ is squarefree strongly stable. Thus we can apply Lemma 2.3 and (i) follows.

(ii): By (i) we may replace J by $\operatorname{gin}(J)$, thus we may assume that J is squarefree strongly stable. Now (ii) follows from 2.8 (ii) and 2.6. □

The next results answer the natural question, in which cases we have equalities in Theorem 2.10. Denote by $J_{\langle t\rangle}$ the ideal that is generated by all degree t elements of the graded ideal J. Then $J \subset E$ is called *componentwise linear* if, for all $t \in \mathbb{N}$, the ideal has an t-linear resolution, i. e. $\operatorname{Tor}^E_i(J_{\langle t\rangle}, K)_{i+j} = 0$ for $j \neq t$.

Given a module $M \in \mathscr{M}$, we call, analogously to the symmetric case, the sequence $v_1,\dots,v_m \in E$ a *proper M-sequence* if $(v_1,\dots,v_m)\cdot H_i(v_1,\dots,v_t;M) = 0$ for $t = 1,\dots,m$ and all $i > 0$.

The following result generalizes [4, Theorem 2.1]. We follow the proof of [15, Theorem 4.5].

Theorem 2.11. *Let $J \subset E$ be a graded ideal. The following conditions are equivalent:*

(i) $\beta^E_{ijp}(E/J) = \beta^E_{ijp}(E/\operatorname{gin}(J))$ *for all i, j, p;*
(ii) $\beta^E_{1jn}(E/J) = \beta^E_{1jn}(E/\operatorname{gin}(J))$ *for all j;*
(iii) *J is a componentwise linear ideal;*
(iv) *A generic sequence of linear forms $f_1,\dots,f_n$ is a proper sequence of E/J.*

Proof. (i) ⇒ (ii): This is trivial.

(ii) ⇒ (iii): This follows from of [4, Theorem 1.1 and Theorem 2.1]. In fact, the proof of Aramova, Herzog an Hibi shows this stronger result.

(iii) ⇒ (iv): Assume that J is componentwise linear. Let $f_1,\dots,f_p$ be a generic sequence of linear forms. We have to show that $\mathfrak{m}H_i(f_1,\dots,f_p;E/J) = 0$ for all $i > 0$ and $p = 0,\dots,n$ where $\mathfrak{m} = (e_1,\dots,e_n)$ denotes the graded maximal ideal of E.

Assume first that J is generated in a single degree d. Since the regularity does not change by passing to the generic initial ideal with respect to revlex, it follows that also $\operatorname{gin}(I)$ is generated in degree d. By Theorem 2.4 (i) we get that for $i > 0$ the homology module $H_i(f_1,\dots,f_p;E/\operatorname{gin}(J))$ is concentrated in the single degree $i+d-1$. Theorem 2.10 (i) shows that the same is true also for $H_i(f_1,\dots,f_p;E/J)$, and this implies $\mathfrak{m}H_i(f_1,\dots,f_p;E/J) = 0$.

Now we consider the general case. For a fixed p, let φ_i be the map from $C_i(f_1,\dots,f_p;E)$ to $C_{i-1}(f_1,\dots,f_p;E)(-1)$ in the Sequence (1). Let $a \in H_i(f_1,\dots,f_p;E/J)$ be a homogeneous element of degree s and choose any homogeneous element $z \in C_i(f_1,\dots,f_p;E)$ such that $a = \bar{z}$. Then $\varphi_i(z)$ is in $JC_{i-1}(f_1,\dots,f_p;E)$. Since $\varphi_i(z)$ is homogeneous of degree $s-1$, we obtain that $\varphi_i(z)$ is in $J_kC_{i-1}(f_1,\dots,f_p;E)$ for $k = s-i$. Set $J' := J_{<k>}$. Then the homology class of z is in $H_i(f_1,\dots,f_p;E/J')$. By construction, J' is generated in a single degree and, by assumption, it has a linear resolution. Hence we know already that $\mathfrak{m}H_i(f_1,\dots,f_p;E/J') = 0$. It follows that $\mathfrak{m}z \subset \operatorname{Im}\varphi_{i+1} + J'C_i(f_1,\dots,f_p;E) \subset$

$\operatorname{Im}\varphi_{i+1} + JC_i(f_1,\dots,f_p;E)$ where the second containment is due to $J' \subset J$. This implies that $\mathfrak{m}a = 0$ in $H_i(f_1,\dots,f_p;E/J)$.

(iv) $\Rightarrow$ (i): By Lemma 2.9 we know that $\beta^E_{0jp}(E/J) = \beta^E_{0jp}(E/\operatorname{gin}(J))$ for all j,p. Observe that $\beta^E_{ij1}(E/J) = \beta^E_{ij1}(E/\operatorname{gin}(J))$ for $i > 0$ and $j \in \mathbb{Z}$ as one easily checks using properties of gin. We prove (i) by showing that the numbers $\beta^E_{ijp}(E/J)$ only depend on the numbers $\beta^E_{0jp}(E/J)$ and $\beta^E_{ij1}(E/J)$.

Since $f_1,\dots,f_n$ is a proper sequence of E/J, the long exact Cartan homology sequence splits into short exact sequences

$$\begin{aligned} 0 \to H_1(f_1,\dots,f_{p-1};E/J) \to H_1(f_1,\dots,f_p;E/J) \to H_0(f_1,\dots,f_p;E/J)(-1) \\ \to H_0(f_1,\dots,f_{p-1};E/J) \to H_0(f_1,\dots,f_p;E/J) \to 0 \end{aligned}$$

and, for $i > 1$,

$$0 \to H_i(f_1,\dots,f_{p-1};E/J) \to H_i(f_1,\dots,f_p;E/J) \to H_{i-1}(f_1,\dots,f_p;E/J)(-1) \to 0.$$

Thus

$$\beta^E_{1jp}(E/J) = \beta^E_{1jp-1}(E/J) + \beta^E_{0j-1p}(E/J) - \beta^E_{0jp-1}(E/J) + \beta^E_{0jp}(E/J)$$

and, for $i > 1$,

$$\beta^E_{ijp}(E/J) = \beta^E_{ijp-1}(E/J) + \beta^E_{i-1j-1p}(E/J).$$

Since $\beta^E_{ijp}(E/J) = 0$ for all $i > p$, it is easy to see that these equalities imply (i). □

Recall that a graded ideal $J \subset E$ is called a *Gotzmann ideal* if the growth of its Hilbert function is the least possible, i. e. $\dim_K E_1 \cdot J_j = \dim_K E_1 \cdot \operatorname{lex}(J)_j$ for all j. The next result is an exterior version of the corresponding result [15, Theorem 4.6] in the polynomial ring.

Theorem 2.12. *For each graded ideal $J \subset E$, the following statements are equivalent:*

(i) $\beta^E_{ijp}(E/J) = \beta^E_{ijp}(E/\operatorname{lex}(J))$ *for all i,j,p;*
(ii) $\beta^E_{1jn}(E/J) = \beta^E_{1jn}(E/\operatorname{lex}(J))$ *for all j;*
(iii) *J is a Gotzmann ideal in the exterior algebra;*
(iv) $\beta^E_{0jp}(E/J) = \beta^E_{0jp}(E/\operatorname{lex}(J))$ *for all j,p and J is componentwise linear.*

Proof. Set $L = \operatorname{lex}(J)$.

(i) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (iii): This proof is analogous to one of Corollary 1.4 in [20] in the symmetric case by using the exterior persistence theorem proved in [2, Theorem 4.6].

(iii) $\Rightarrow$ (iv): By Theorem 2.11, a Gotzmann ideal is componentwise linear. Thus it suffices to prove that $\beta^E_{0jp}(E/J) = \beta^E_{0jp}(E/L)$ for all j,p. By Lemma 2.9 passing to the gin does not change β^E_{0jp}. Hence, replacing J with $\operatorname{gin}(J)$, which is still Gotzmann, we may assume that J is squarefree strongly stable. Then Theorem 2.8 (ii) and Corollary 2.7 imply for all j,p:

$$\beta^E_{0jp}(E/J) = \beta^E_{0jp}(E/L).$$

(iv) $\Rightarrow$ (i): Again, Theorem 2.11 allows us to pass to $\operatorname{gin}(J)$. Thus we may assume that J is squarefree strongly stable. Then Theorem 2.4 provides $m_{\le i}(J_j) = m_{\le i}(L_j)$ for all i and j. Now (i) follows from Theorem 2.8 (ii) and Corollary 2.7. □

We conclude this section by comparing the Cartan–Betti numbers of generic initial ideals. The goal is to show that in the family

$$\mathrm{gins}(J) := \{\mathrm{gin}_\tau(J) : \tau \text{ a term order of } E\}$$

of all generic initial ideals of J, the revlex-gin has the smallest Cartan–Betti numbers.

We need a lemma and some more notation. Let $V \subset E_i$ be a d-dimensional subspace. Then $\bigwedge^d V$ is a 1-dimensional subspace of $\bigwedge^d E_i$. We identify it with any of its nonzero elements. An exterior monomial in $\bigwedge^d E_i$ is by definition an element of the form $m_1 \wedge \ldots \wedge m_d$ where $m_1, \ldots, m_d$ are distinct monomials in E_i. It is called a τ-*standard* exterior monomial if $m_1 >_\tau \ldots >_\tau m_d$. The vector space $\bigwedge^d E_i$ has a K-basis of τ-standard exterior monomials that we order lexicographically by $m_1 \wedge \ldots \wedge m_d >_\tau n_1 \wedge \ldots \wedge n_d$ if $m_i >_\tau n_i$ for the smallest index i such that $m_i \neq n_i$. Using this order one defines the initial (exterior) monomial $\mathrm{in}_\tau(f)$ of any $f \in \bigwedge^d E_i$ as the maximal τ-standard exterior monomial with respect to $>_\tau$ which appears in the unique representation of f as a sum of τ-standard exterior monomials. Similarly, the initial subspace $\mathrm{in}_\tau(V)$ of any subspace V of $\bigwedge^d E_i$ is defined as the subspace generated by all $\mathrm{in}_\tau(f)$ for $f \in V$. The τ-*support* $\mathrm{supp}_\tau(f)$ of $f \in \bigwedge^d E_i$ is the set of τ-standard monomials that appear with a nonzero coefficient in f. The following result and its proof are analogous to [14, Corollary 1.6].

Lemma 2.13. *Let τ and σ be term orders on E and let $V \subset E_i$ be any d-dimensional subspace. If $m_1 \wedge \ldots \wedge m_d$ and $n_1 \wedge \ldots \wedge n_d$ are τ-standard monomials such that* $\mathrm{gin}_\tau(V) = \langle m_1, \ldots, m_d \rangle$ *and* $\mathrm{gin}_\tau(\mathrm{in}_\sigma(V)) = \langle n_1, \ldots, n_d \rangle$*, then $m_i \geq_\tau n_i$ for all $i = 1, \ldots, d$.*

Now we get analogously to the symmetric case [15, Theorem 5.1]:

Theorem 2.14. *Let $J \subset E$ be a graded ideal and τ a term order on E. Then*

$$\beta^E_{ijp}(E/\mathrm{gin}(J)) \leq \beta^E_{ijp}(E/\mathrm{gin}_\tau(J)) \quad \textit{for all } i, j, p.$$

Proof. Set $J' = \mathrm{gin}(J)$ and $J'' = \mathrm{gin}_\tau(J)$. Note that J' and J'' are squarefree strongly stable ideals with the same Hilbert function as J. Thus by Proposition 2.6, it is enough to show that $m_{\leq i}(J''_j) \leq m_{\leq i}(J'_j)$ for all i and j.

Let $a_1, \ldots, a_k$ be the generators of J'_j and let $b_1 \ldots, b_k$ be those of J''_j. We may assume that the a_r's and the b_r's are listed in the revlex order. Then we claim:

$$a_r \geq b_r \text{ in the revlex order for all } r.$$

This implies that $\max(a_r) \leq \max(b_r)$ for all r, and hence $m_{\leq i}(J''_j) \leq m_{\leq i}(J'_j)$. Thus it remains to shows the claim.

Let $V, W \subset E_j$ be two monomial vector spaces of the same dimension where V has a monomial K-basis $v_1, \ldots, v_k$ and W is has a monomial K-basis $w_1, \ldots, w_k$. Define $V \geq_{\mathrm{revlex}} W$ if $v_i \geq_{\mathrm{revlex}} w_i$, $v_i >_{\mathrm{revlex}} v_{i+1}$ and $w_i >_{\mathrm{revlex}} w_{i+1}$ for all i. Then Lemma 2.13 provides:

$$\mathrm{gin}_{\mathrm{revlex}}(V) \geq_{\mathrm{revlex}} \mathrm{gin}_{\mathrm{revlex}}(\mathrm{in}_\tau(V)).$$

If $V \subset E_j$ is a generic subspace, then $\mathrm{in}_\tau(V) = \mathrm{gin}_\tau(V)$. Since $\mathrm{gin}_{\mathrm{revlex}}(\mathrm{gin}_\tau(V)) = \mathrm{gin}_\tau(V)$, we then get $\mathrm{gin}_{\mathrm{revlex}}(V) \geq_{\mathrm{revlex}} \mathrm{gin}_\tau(V)$. Choosing $V = J'_j$ proves the claim. □

Observe that inequalities similar to the crucial $m_{\leq i}((\mathrm{gin}_\tau(J))_j) \leq m_{\leq i}(\mathrm{gin}(J)_j)$ above were shown by Murai [27, Proposition 2.4].

3. Simplicial complexes and algebraic shifting operations

In the second part of this paper we give some combinatorial applications of the exterior algebra methods presented in Section 2 and some symmetric algebra methods which will be presented below. For this we introduce at first some notation and discuss some basic concepts. Recall that Γ is called a *simplicial complex* on the vertex set $[n]=\{1,\dots,n\}$ if Γ is a subset of the power set of $[n]$ which is closed under inclusion, i. e. if $F\subseteq G$ and $G\in\Gamma$, then $F\in\Gamma$. The elements F of Γ are called *faces*, and the maximal elements under inclusion are called *facets*. We denote the set of all facets by $\operatorname{facets}(\Gamma)$. If F contains $d+1$ elements of $[n]$, then F is called a *d-dimensional* face, and we write $\dim F=d$. The empty set is a face of dimension -1. Faces of dimension 0, 1 are called *vertices* and *edges*, respectively. Γ is called *pure* if all facets have the same dimension. If Γ is non-empty, then the *dimension* $\dim\Gamma$ is the maximum of the dimensions of the faces of Γ. Let f_i be the total number of i-dimensional faces of Γ for $i=-1,\dots,\dim\Gamma$. The vector $f(\Gamma)=(f_{-1},\dots,f_{\dim\Gamma})$ is called the *f-vector* of Γ.

Several constructions associate other simplicial complexes to a given one. For example, the *Alexander dual* of Γ is defined as $\Gamma^*=\{F\subseteq[n]:F^c\notin\Gamma\}$ where $F^c=[n]\setminus F$. This gives a duality operation on simplicial complexes in the sense that Γ^* is indeed a simplicial complex on the vertex set $[n]$ and we have that $(\Gamma^*)^*=\Gamma$.

The connection to algebra is due to the following construction. Let K be a field and $S=K[x_1,\dots,x_n]$ be the polynomial ring in n indeterminates. S is a graded ring by setting $\deg x_i=1$ for $i=1,\dots,n$. We define the *Stanley–Reisner ideal* $I_\Gamma=(\prod_{i\in F}x_i : F\subseteq[n],\ F\notin\Gamma)$ and the corresponding *Stanley–Reisner ring* $K[\Gamma]=S/I_\Gamma$. For a subset $F\subset[n]$, we write x_F for the squarefree monomial $\prod_{i\in F}x_i$. We will say that Γ has an algebraic property (like Cohen–Macaulayness) if and only if $K[\Gamma]$ has this property. Note that these definitions may depend on the ground field K.

There is an analogous exterior algebra construction due to Aramova, Herzog and Hibi (e.g. see [19]). Let $E=K\langle e_1,\dots,e_n\rangle$ be the exterior algebra on n exterior variables. E is also a graded ring by setting $\deg e_i=1$ for $i=1,\dots,n$. For $F\subseteq[n]$ we write e_F for the exterior monomial $\wedge_{i\in F}e_i$. One defines the *exterior Stanley–Reisner ideal* as $J_\Gamma=(e_F : F\notin\Gamma)$ and the *exterior face ring* $K\{\Gamma\}=E/J_\Gamma$.

We keep these notations throughout the paper. Further information on simplicial complexes can be found, for example, in the books by Bruns–Herzog [13] and by Stanley [33].

Let $\mathscr{C}([n])$ be the set of simplicial complexes on $[n]$. Following constructions of Kalai, we define axiomatically the concept of algebraic shifting. See [19] and [26] for surveys on this subject. We call a map $\Delta\colon\mathscr{C}([n])\to\mathscr{C}([n])$ a *shifting operation* if the following conditions are satisfied:

- (S1) If $\Gamma\in\mathscr{C}([n])$, then $\Delta(\Gamma)$ is a *shifted complex*, i. e. for all $F\in\Delta(\Gamma)$ and $j>i\in F$ we have that $F\setminus\{i\}\cup j\in\Delta(\Gamma)$.
- (S2) If $\Gamma\in\mathscr{C}([n])$ is shifted, then $\Delta(\Gamma)=\Gamma$.
- (S3) If $\Gamma\in\mathscr{C}([n])$, then $f(\Gamma)=f(\Delta(\Gamma))$.
- (S4) If $\Gamma',\Gamma\in\mathscr{C}([n])$ and $\Gamma'\subseteq\Gamma$ is a subcomplex, then $\Delta(\Gamma')$ is a subcomplex of $\Delta(\Gamma)$.

See [8] for variations of these axioms. Note that a simplicial complex Γ is shifted if and only if the Stanley–Reisner ideal $I_\Gamma\subset S=K[x_1,\dots,x_n]$ is a *squarefree strongly stable* *ideal* with respect to $x_1 > \cdots > x_n$, i. e. for all $F \subseteq [n]$ with $x_F \in I_\Gamma$ and all $i \in F$, $j < i$, $j \notin F$ we have that $(x_j x_F)/x_i \in I_\Gamma$. Analogously, Γ is shifted if and only if the exterior Stanley–Reisner ideal $J_\Gamma \subset E = K\langle e_1, \ldots, e_n \rangle$ is a *squarefree strongly stable ideal* with respect to $e_1 > \cdots > e_n$.

Two of the most important examples of such operations are defined as follows.

Example 3.1. (Symmetric algebraic shifting) At first we introduce the symmetric algebraic shifting operation introduced by Kalai [26]. Here, we follow the algebraic approach of Aramova, Herzog and Hibi [5].

Assume that K is a field of characteristic 0 and let $S = K[x_1, \ldots, x_n]$. We consider the following operation on monomial ideals of S. For a monomial $m = x_{i_1} \cdots x_{i_t}$ with $i_1 \leq i_2 \cdots \leq i_t$ of S we set $m^\sigma = x_{i_1} x_{i_2+1} \ldots x_{i_t+t-1}$. For a monomial ideal I with unique minimal system of generators $G(I) = \{m_1, \ldots, m_s\}$ we set $I^\sigma = (m_1^\sigma, \ldots, m_s^\sigma)$ in a suitable polynomial ring with sufficiently many variables.

Let Γ be a simplicial complex on the vertex set $[n]$ with Stanley–Reisner ideal I_Γ. The *symmetric algebraic shifted complex* of Γ is the unique simplicial complex $\Delta^s(\Gamma)$ on the vertex set $[n]$ such that

$$I_{\Delta^s(\Gamma)} = \big(\mathrm{gin}(I_\Gamma)\big)^\sigma \subset S$$

where $\mathrm{gin}(\cdot)$ is the generic initial ideal with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n$.

It is not obvious that $\Delta^s(\cdot)$ is indeed a shifting operation. The first difficulty is already to show that $I_{\Delta^s(\Gamma)}$ is an ideal of S. This and the proofs of the other properties can be found in [5] or [19].

Example 3.2. (Exterior algebraic shifting) Exterior algebraic shifting was also defined by Kalai [26]. Let $E = K\langle e_1, \ldots, e_n \rangle$ where K is any infinite field. Let Γ be a simplicial complex on the vertex set $[n]$. The *exterior algebraic shifted complex* of Γ is the unique simplicial complex $\Delta^e(\Gamma)$ on the vertex set $[n]$ such that

$$J_{\Delta^e(\Gamma)} = \mathrm{gin}(J_\Gamma)$$

where $\mathrm{gin}(\cdot)$ is the generic initial ideal with respect to the reverse lexicographic order on E induced by $e_1 > \cdots > e_n$. For an introduction to the theory of Gröbner bases in the exterior algebra see [2]. As opposed to symmetric algebraic shifting, it is much easier to see that $\Delta^e(\cdot)$ is indeed a shifting operation since generic initial ideals in the exterior algebra are already squarefree strongly stable. See again [19] for details.

There are several other shifting operations. Since the proof of [19, Proposition 8.8] works for arbitrary term orders τ, one can take a generic initial ideal $\mathrm{gin}_\tau(\cdot)$ in E with respect to any term order τ to obtain, analogously to Example 3.2, an *exterior algebraic shifting operation* $\Delta^\tau(\cdot)$.

4. Degree functions of simplicial complexes I

In commutative algebra degree functions are designed to provide measures for the size and the complexity of the structure of a given module. Given a simplicial complex Γ on the vertex set $[n]$, we recall the definition of several degree functions on the Stanley–Reisner ring $K[\Gamma]$ in terms of combinatorial data. The first important invariant is the degree (or multiplicity) $\deg\Gamma$ of Γ. By definition $\deg\Gamma = \deg K[\Gamma]$ is the degree of the Stanley–Reisner ring of Γ. It can be combinatorially described as

$$\deg\Gamma = f_{\dim\Gamma}(\Gamma), \text{ the number of faces of maximal dimension.} \tag{3}$$

Since any algebraic shifting operation $\Delta(\cdot)$ preserves the f-vector, we have that

$$\deg\Gamma = \deg\Delta(\Gamma). \tag{4}$$

In case Γ is a Cohen–Macaulay complex, this invariant gives a lot of information about Γ, because Γ is pure and thus $\deg\Gamma$ counts all facets. There are several other degree functions which take into account more information about Γ.

Next we study the *arithmetic degree*. Algebraically it is defined as

$$\operatorname{adeg}\Gamma = \operatorname{adeg} K[\Gamma] = \sum l(H^0_{\mathfrak{p}}(K[\Gamma]_{\mathfrak{p}})) \cdot \deg\big(K[\Gamma]/\mathfrak{p}\big)$$

where the sum runs over the associated prime ideals of $K[\Gamma]$ and $l(\cdot)$ denotes the length function of a module. In [35] and [7] it was noted that the results of [34] imply that

$$\operatorname{adeg}\Gamma = |\{F \in \operatorname{facets}(\Gamma)\}| \tag{5}$$

is the number of facets of Γ. Similarly, one defines for $i = 0,\dots,\dim K[\Gamma]$

$$\operatorname{adeg}_i\Gamma = \operatorname{adeg}_i K[\Gamma] = \sum l(H^0_{\mathfrak{p}}(K[\Gamma]_{\mathfrak{p}})) \cdot \deg\big(K[\Gamma]/\mathfrak{p}\big)$$

where the sum runs over the associated prime ideals of $K[\Gamma]$ such that $\dim K[\Gamma]/\mathfrak{p} = i$ and gets

$$\operatorname{adeg}_i\Gamma = |\{F \in \operatorname{facets}(\Gamma) : \dim F = i-1\}|. \tag{6}$$

By definition, shifting preserves the f-vector and the number of facets of maximal dimension of a simplicial complex. However, the total number of facets may change under shifting. But it may only increase, as we show now.

Theorem 4.1. *Let Δ be a shifting operation and Γ be a $(d-1)$-dimensional simplicial complex. Then*

$$\operatorname{adeg}_i\Gamma \le \operatorname{adeg}_i\Delta(\Gamma) \text{ for } i = 0,\dots,d.$$

In particular, $\operatorname{adeg}\Gamma \le \operatorname{adeg}\Delta(\Gamma)$.

Proof. Let $m-1$ be the smallest dimension of a facet of Γ. We prove the assertion by induction on the number $(d-1)-(m-1) = d-m \ge 0$. If $d-m = 0$, then Γ is a pure complex and only $\operatorname{adeg}_d\Gamma$ is different from zero. Using Identity (4) we get $\operatorname{adeg}_d\Gamma = \deg\Gamma = \deg\Delta(\Gamma) = \operatorname{adeg}_d\Delta(\Gamma)$ as claimed.

Let $d > m$. Observe that $\operatorname{adeg}_i\Gamma = 0$ for $i < m$. Denote by $F_1,\dots,F_s \in \Gamma$ the $m-1$-dimensional facets of Γ and let Γ' be the subcomplex of Γ whose facets are the remaining facets of Γ. Thus, we have in particular:

$$f_i(\Gamma') = f_i(\Gamma) \quad \text{for all } i > m-1.$$

Since shifting does not change the f-vector and $\Delta(\Gamma')$ is a subcomplex of $\Delta(\Gamma)$, we conclude that

$$f_i(\Delta(\Gamma')) = f_i(\Delta(\Gamma)) \quad \text{for all } i > m-1,$$

and, in particular, that each facet of $\Delta(\Gamma')$ of dimension $> m-1$ is also a facet of $\Delta(\Gamma)$. It follows from the induction hypothesis that

$$\operatorname{adeg}_i \Gamma = \operatorname{adeg}_i \Gamma' \leq \operatorname{adeg}_i \Delta(\Gamma') = \operatorname{adeg}_i \Delta(\Gamma) \text{ for } i > m.$$

Hence, it remains to show that $\Delta(\Gamma)$ has at least $s = \operatorname{adeg}_m \Gamma$ facets of dimension $m-1$.

To this end note that, by definition of Γ', $f_{m-1}(\Gamma) = f_{m-1}(\Gamma') + s$, thus we get for the shifted complexes $f_{m-1}(\Delta(\Gamma)) = f_{m-1}(\Delta(\Gamma')) + s$. Let $G \in \Delta(\Gamma)$ be any $(m-1)$-dimensional face that is not in $\Delta(\Gamma')$. Assume that G is strictly contained in a face $\widetilde{G}$ of $\Delta(\Gamma)$. Then $\dim \widetilde{G} > m-1$ and, using the fact that the faces of $\Delta(\Gamma')$ and $\Delta(\Gamma)$ of dimension $> m-1$ coincide, we conclude that $\widetilde{G} \in \Delta(\Gamma')$. Thus $G \in \Delta(\Gamma')$. This contradiction to the choice of G shows that G is a facet of $\Delta(\Gamma)$ and we are done. $\square$

Observe that we only used axioms (S3) and (S4) of a shifting operation to prove Theorem 4.1. It would be interesting to know if there are further results in this direction like:

Question 4.2. *Is there a shifting operation* $\Delta(\cdot)$ *such that* adeg *increases the least, i. e. for every simplicial complex* Γ*, we have* $\operatorname{adeg} \Delta(\Gamma) \leq \operatorname{adeg} \Delta'(\Gamma)$ *for each algebraic shifting operation* $\Delta'(\cdot)$*?*

A partial answer to Question 4.2 will be given in Section 6 after we have developed the necessary tools in Section 2.

The arithmetic degree has nice properties especially for Cohen–Macaulay K-algebras. However, there are some disadvantages for non-Cohen–Macaulay K-algebras. See Vasconcelos [36] for a discussion. Vasconcelos axiomatically defined the following concept. Recall that $S = K[x_1, \dots, x_n]$ where K is a field. A numerical function Deg that assigns to every finitely generated graded S-module a non-negative integer is said to be an *extended degree function* if it satisfies the following conditions:

(i) If $L = H^0_{\mathfrak{m}}(M)$, then $\operatorname{Deg} M = \operatorname{Deg} M/L + l(L)$.
(ii) If $y \in S_1$ is sufficiently general and M-regular, then $\operatorname{Deg} M \geq \operatorname{Deg} M/yM$.
(iii) If M is a Cohen–Macaulay module, then $\operatorname{Deg} M = \deg M$.

There exists a *smallest extended degree function* sdeg in the sense that $\operatorname{sdeg} M \leq \operatorname{Deg} M$ for every other extended degree function.

Now we recall the concept of a sequentially Cohen–Macaulay module. Let M be a finitely generated graded S-module. The module M is said to be *sequentially Cohen–Macaulay* (sequentially CM modules for short), if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M \tag{7}$$

of M by graded submodules of M such that each quotient M_i/M_{i-1} is Cohen–Macaulay and $\dim M_1/M_0 < \dim M_2/M_1 < \cdots < \dim M_r/M_{r-1}$ where dim denotes the Krull dimension of M.

Remark 4.3. We need the following properties of sdeg. (For proofs see [30].) For simplicity we state them only for a graded K-algebra S/I where $I \subset S$ is a graded ideal.

(i) S/I is Cohen–Macaulay if and only if $\operatorname{sdeg} S/I = \deg S/I$. (This is true for any extended Degree function.)

(ii) $\operatorname{adeg} S/I \leq \operatorname{sdeg} S/I$.
(iii) If S/I is sequentially Cohen–Macaulay, then $\operatorname{adeg} S/I = \operatorname{sdeg} S/I$.
(iv) $\operatorname{sdeg} S/I = \operatorname{sdeg} S/\operatorname{gin}(I)$.

Then we have:

Theorem 4.4. *Let Γ be a simplicial complex on the vertex set $[n]$. We have that:*
(i) $\operatorname{sdeg}\Delta(\Gamma) = \operatorname{adeg}\Delta(\Gamma)$ *for each algebraic shifting operation* Δ.
(ii) $\deg\Gamma \leq \operatorname{adeg}\Gamma \leq \operatorname{sdeg}\Gamma = \operatorname{sdeg}\Delta^s(\Gamma) = \operatorname{adeg}\Delta^s(\Gamma)$.

Proof. In the proof we use that $\Delta(\Gamma)$ is always sequentially Cohen–Macaulay, i. e. $K[\Delta(\Gamma)]$ is a sequentially CM-ring (cf. Proposition 5.1 below). This fact and the properties of sdeg observed in Remark 4.3 imply (i).

The only critical equality in (ii) is $\operatorname{sdeg}\Gamma = \operatorname{sdeg}\Delta^s(\Gamma)$. We compute

$$\begin{aligned}
\operatorname{sdeg}\Gamma &= \operatorname{sdeg} S/I_\Gamma \\
&= \operatorname{sdeg} S/\operatorname{gin}(I_\Gamma) \\
&= \operatorname{adeg} S/\operatorname{gin}(I_\Gamma) \\
&= \operatorname{adeg} S/I_{\Delta^s(\Gamma)} \\
&= \operatorname{sdeg} S/I_{\Delta^s(\Gamma)}
\end{aligned}$$

where the first equality is the definition of $\operatorname{sdeg}\Gamma$, the second one was noted above and the third equality follows from the fact that $S/\operatorname{gin}(I_\Gamma)$ is sequentially Cohen–Macaulay (see [22, Theorem 2.2]). The forth equality follows again from [7, Theorem 6.6] and the last equality is (i) which we proved already. □

We already mentioned that $\deg\Gamma$ is the number of facets of maximal dimension of Γ and $\operatorname{adeg}\Gamma$ is the number of facets of Γ. Theorem 4.4 provides a combinatorial interpretations of $\operatorname{sdeg}\Gamma$. It is the number of facets of $\Delta^s(\Gamma)$.

5. The Cohen–Macaulay property and iterated Betti numbers

In this section we use degree functions to relate properties of the shifted complex and iterated Betti numbers to the original complex.

Our first observation is well-known to specialists. It states that shifted complexes have a nice algebraic structure. Since we did not find a proof in the literature we include it for the sake of completeness.

Proposition 5.1. *Shifted simplicial complexes are sequentially Cohen–Macaulay.*

In particular, if Γ is a simplicial complex on the vertex set $[n]$ and $\Delta(\cdot)$ is an arbitrary algebraic shifting operation, then $\Delta(\Gamma)$ is sequentially Cohen–Macaulay.

Proof. Let Γ be a shifted simplicial complex. Recall that then $I_\Gamma \subset S$ is a squarefree strongly stable ideal with respect to $x_1 > \cdots > x_n$, i. e. for all $x_F = \prod_{l\in F} x_l \in I_\Gamma$ and i with $x_i | x_F$ we have for all $j < i$ with $x_j \nmid x_F$ that $(x_F/x_i)x_j \in I_\Gamma$.

It is easy to see that we have that I_Γ is squarefree strongly stable if and only if I_{Γ^*} is squarefree strongly stable where Γ^* is the Alexander dual of Γ. It is well-known that in this situation I_{Γ^*} is a so-called componentwise linear ideal (see Section 2 for the definition) and thus the theorem follows now from [21, Theorem 9]. □

Remark 5.2. Alternatively one can prove Proposition 5.1 using more combinatorially arguments as follows. Shifted simplicial complexes are non-pure shellable by Björner and Wachs [12] and thus sequentially Cohen–Macaulay by Stanley [33].

It is easy to decide whether a shifted simplicial complex is Cohen–Macaulay because of the following well-known fact:

Proposition 5.3. *Let Γ be a shifted simplicial complex on the vertex set $[n]$. Then the following statements are equivalent:*

(i) *Γ is Cohen–Macaulay;*
(ii) *Γ is pure.*

Proof. If Γ is Cohen–Macaulay, then it is well-known that Γ is pure. Assume now that Γ is a pure shifted simplicial complex. We compute

$$\operatorname{sdeg} S/I_\Gamma = \operatorname{adeg} S/I_{\Delta^s(\Gamma)} = \operatorname{adeg} S/I_\Gamma = \deg S/I_\Gamma.$$

The first equality was shown in Theorem 4.4, the second one holds because Γ is shifted, and the third one because Γ is pure. Hence $\operatorname{sdeg} S/I_\Gamma = \deg S/I_\Gamma$ and this implies by Remark 4.3 that Γ is Cohen–Macaulay. $\square$

Intuitively, shifting leads to a somewhat simpler complex and one would like to transfer properties from $\Delta(\Gamma)$ to Γ. In this respect, we have:

Proposition 5.4. *Let Γ be a simplicial complex on the vertex set $[n]$ and let $\Delta(\cdot)$ be any algebraic shifting operation such that* $\operatorname{sdeg}\Gamma \leq \operatorname{sdeg}\Delta(\Gamma)$. *If $\Delta(\Gamma)$ is Cohen–Macaulay, then Γ is Cohen–Macaulay.*

Proof. Suppose that $\Delta(\Gamma)$ is Cohen–Macaulay. Then $\Delta(\Gamma)$ is pure and we have that $\operatorname{adeg}\Delta(\Gamma) = \deg\Delta(\Gamma)$. Thus, we get

$$\deg\Gamma \leq \operatorname{sdeg}\Gamma \leq \operatorname{sdeg}\Delta(\Gamma) = \operatorname{adeg}\Delta(\Gamma) = \deg\Delta(\Gamma) = \deg\Gamma.$$

The only critical equality is $\operatorname{sdeg}\Delta(\Gamma) = \operatorname{adeg}\Delta(\Gamma)$ which follows from Remark 4.3 (iii) and Proposition 5.1. Hence $\deg\Gamma = \operatorname{sdeg}\Gamma$ and it follows that Γ is Cohen–Macaulay. $\square$

For exterior algebraic shifting, the following result was proved by Kalai [26]. For symmetric algebraic shifting, this result follows from some general homological arguments given in [9], as noted in [7]. Below, we provide a very short proof using only degree functions.

Theorem 5.5. *Let Γ be a simplicial complex on the vertex set $[n]$. Then the following statements are equivalent:*

(i) *Γ is Cohen–Macaulay;*
(ii) *$\Delta^s(\Gamma)$ is Cohen–Macaulay;*
(iii) *$\Delta^s(\Gamma)$ is pure.*

Proof. We already know that (ii) and (iii) are equivalent. Suppose that (i) holds. Then

$$\deg\Delta^s(\Gamma) = \deg\Gamma = \operatorname{sdeg}\Gamma = \operatorname{adeg}\Delta^s(\Gamma) \geq \operatorname{adeg}\Gamma \geq \deg\Gamma = \deg\Delta^s(\Gamma).$$

The first equality is trivial. The second one holds because Γ is Cohen–Macaulay, and the third one follows from Theorem 4.4 (ii). The first inequality is due to Theorem 4.1 and the second one follows from the definition of adeg. Hence $\operatorname{adeg}\Delta^s(\Gamma) = \deg\Delta^s(\Gamma)$, and thus $\Delta^s(\Gamma)$ is pure, as claimed in (iii).

The fact that (iii) implies (i) follows from $\operatorname{sdeg}\Gamma = \operatorname{adeg}\Delta^s(\Gamma) \le \operatorname{sdeg}\Delta^s(\Gamma)$ and Proposition 5.4. □

For the next results, we need some further notation and definitions. At first we recall a refinement of the f- and h-vector of a simplicial complex due to Björner and Wachs [11].

Definition 5.6. Let Γ be a $(d-1)$-dimensional simplicial complex on the vertex set $[n]$. We define

$$f_{i,r}(\Gamma) = |\{F \in \Gamma : \deg_\Gamma F = i \text{ and } |F| = r\}|$$

where $\deg_\Gamma F = \max\{|G| : F \subseteq G \in \Gamma\}$ is the *degree* of a face $F \in \Gamma$. The triangular integer array $f(\Gamma) = (f_{i,r}(\Gamma))_{0\le r\le i\le d}$ is called the *f-triangle* of Γ. Further we define

$$h_{i,r}(\Gamma) = \sum_{s=0}^{r} (-1)^{r-s} \binom{i-s}{r-s} f_{i,s}(\Gamma).$$

The triangular array $h(\Gamma) = (h_{i,r}(\Gamma))_{0\le r\le i\le d}$ is called the *h-triangle* of Γ.

It is easy to see that giving the h-triangle, one can also compute the f-triangle and thus these triangles determine each other. For $F \subseteq [n]$ let $\operatorname{init}(F) = \{k,\dots,n\}$ if $k,\dots,n \in F$, but $k-1 \notin F$. (Here we set $\operatorname{init}(F) = \emptyset$ if no such k exists.) Duval proved in [16, Corollary 6.2] that if Γ is sequentially Cohen–Macaulay, then $h_{i,r}(\Gamma) = |\{F \in \operatorname{facets}(\Gamma) : |\operatorname{init}(F)| = i-r,\ |F| = i\}|$. In particular, this formula holds for shifted simplicial complexes. Using this fact we define:

Definition 5.7. Let Γ be a simplicial complex on the vertex set $[n]$ and $\Delta(\cdot)$ be an arbitrary algebraic shifting operation. We call

$$b^{\Delta}_{i,r}(\Gamma) = h_{i,r}(\Delta(\Gamma)) = |\{F \in \operatorname{facets}(\Delta(\Gamma)) : |\operatorname{init}(F)| = i-r,\ |F| = i\}|$$

the *Δ-iterated Betti numbers* of Γ.

For symmetric and exterior shifting these notions were defined and studied in [7] and [17]. Since $\Delta(\Delta(\Gamma)) = \Delta(\Gamma)$ we have that

$$b^{\Delta}_{i,r}(\Gamma) = b^{\Delta}_{i,r}(\Delta(\Gamma))$$

and therefore Δ-iterated Betti numbers are invariant under each algebraic shifting operation $\Delta(\cdot)$. As noted in [7], there is the following comparison result.

Theorem 5.8. *Let Γ be a simplicial complex on the vertex set $[n]$ and $\Delta(\cdot)$ be an arbitrary shifting operation. Then the following conditions are equivalent:*

(i) *Γ is sequentially Cohen–Macaulay;*
(ii) *$b^{\Delta}_{i,r}(\Gamma) = h_{i,r}(\Gamma)$ for all i,r.*

In particular, if Γ is sequentially Cohen–Macaulay, then all iterated Betti numbers of Γ with respect to any algebraic shifting operation coincide.

Proof. Duval proved in [16, Theorem 5.1] the equivalence of (i) and (ii) for exterior algebraic shifting Δ^e. But his proof works also in this more general situation since he used only axioms (S1)–(S4) of an algebraic shifting operation. □

The next result extends Theorem 6.6 of [7] where the case of the symmetric algebraic shifting operation was studied. It follows by combining Identities (3), (5), Proposition 5.1, and Remark 4.3 (iii).

Theorem 5.9. *Let Γ be a simplicial complex on the vertex set $[n]$ of dimension $d-1$ and let $\Delta(\cdot)$ be any algebraic shifting operation. Then we have:*

(i) $\deg\Delta(\Gamma) = \sum_{r=0}^{d} b^{\Delta}_{d,r}(\Gamma) = |\{F \in \mathrm{facets}(\Delta(\Gamma)),\ \dim F = d-1\}|$.

(ii) $\mathrm{sdeg}\,\Delta(\Gamma) = \mathrm{adeg}\,\Delta(\Gamma) = \sum_{r,i} b^{\Delta}_{i,r}(\Gamma) = |\{F \in \mathrm{facets}(\Delta(\Gamma))\}|$.

The last two results suggest the following question:

Question 5.10. *What is the relationship between $b^{\Delta}_{i,r}(\Gamma)$ for different shifting operations? For example, in* [7] *it is conjectured that $b^{\Delta^s}_{i,r}(\Gamma) \leq b^{\Delta^e}_{i,r}(\Gamma)$ for all i,r.*

6. Degree functions of simplicial complexes II

We use the results in Section 2 to draw conclusions about exterior algebraic shifting operations. We begin by observing that shifting and Alexander duality commute.

Lemma 6.1. *Let Γ be a simplicial complex on the vertex set $[n]$. Let $\Delta^{\tau}(\cdot)$ be the exterior shifting operation with respect to an arbitrary term order τ on E. Then Alexander duality commutes with shifting, i. e. we have $\Delta^{\tau}(\Gamma)^* = \Delta^{\tau}(\Gamma^*)$.*

Proof. In [23] this was proved for τ being the revlex order. But the proof works for any term order τ. □

As a further preparation, we need an interpretation of the arithmetic degree over the exterior face ring using the socle. The socle $\mathrm{Soc}\,N$ of a finitely generated E-module N is the set of elements $x \in N$ such that $(e_1,\dots,e_n)x = 0$. Observe that $\mathrm{Soc}\,N$ is always a finite-dimensional K-vector space.

Lemma 6.2. *Let Γ be a $(d-1)$-dimensional simplicial complex on the vertex set $[n]$. Then*

$$\mathrm{adeg}_i\,\Gamma = \dim_K \mathrm{Soc}\,K\{\Gamma\}_i \text{ for } i = 0,\dots,d.$$

In particular, $\mathrm{adeg}\,\Gamma = \dim_K \mathrm{Soc}\,K\{\Gamma\}$.

Proof. The residue classes of the monomials x_F, $F \in \Gamma$, form a K-vector space basis of $K\{\Gamma\}$. Hence, the facets of Γ correspond to a K-basis of $\mathrm{Soc}\,K\{\Gamma\}$. □

We have seen in Theorem 4.1 that each shifting operation increases the number of facets. Below we show that among the exterior shifting operations, standard exterior shifting with respect to revlex leads to the least possible increase.

Theorem 6.3. *Let Γ be a $(d-1)$-dimensional simplicial complex on the vertex set $[n]$. Let $\Delta^{\tau}(\cdot)$ be an arbitrary exterior shifting operation, where τ is some term order on E. Then*

$$\mathrm{adeg}_i\,\Gamma \leq \mathrm{adeg}_i\,\Delta^e(\Gamma) \leq \mathrm{adeg}_i\,\Delta^{\tau}(\Gamma) \text{ for } i = 0,\dots,d.$$

In particular, $\mathrm{adeg}\,\Gamma \leq \mathrm{adeg}\,\Delta^e(\Gamma) \leq \mathrm{adeg}\,\Delta^{\tau}(\Gamma)$. *Moreover,* $\mathrm{adeg}\,\Gamma = \mathrm{adeg}\,\Delta^e(\Gamma)$ *if and only if Γ is sequentially Cohen–Macaulay.*

Proof. Using $\operatorname{adeg}_i \Gamma = \dim_K \operatorname{Soc}(E/J_\Gamma)_i$ and the definition of Alexander duality, it is easy to see that $\operatorname{adeg}_i \Gamma$ coincides with the number of minimal generators of J_{Γ^*} of degree $n-i$ which is $\beta^E_{0n-i}(J_{\Gamma^*})$, i. e. we have $\operatorname{adeg}_i \Gamma = \beta^E_{0n-i}(J_{\Gamma^*}) = \beta^E_{1n-i}(E/J_{\Gamma^*})$. Hence Lemma 6.1 provides

$$\begin{aligned}\operatorname{adeg}_i \Gamma = \beta^E_{1n-i}(E/J_{\Gamma^*}) \leq \beta^E_{1n-i}(E/\operatorname{gin}(J_{\Gamma^*})) = \beta^E_{1n-i}(E/J_{\Delta^e(\Gamma^*)})\\ = \operatorname{adeg}_i \Delta^e(\Gamma^*)^* = \operatorname{adeg}_i \Delta^e(\Gamma) \text{ for } i = 0,\dots,d.\end{aligned}$$

Since $\beta^E_{1j}(E/J_{\Gamma^*}) \leq \beta^E_{1j}(E/\operatorname{gin}(J_{\Gamma^*}))$ for all j, it follows from Theorem 2.11 that we have equality if and only if J_{Γ^*} is componentwise linear. By [20], this is the case if and only if the Stanley–Reisner ideal $I_{\Gamma^*} \subset S = K[x_1,\dots,x_n]$ is componentwise linear. But the latter is equivalent to Γ being sequentially Cohen–Macaulay according to [21].

Combining the above argument and Theorem 2.14, we obtain also

$$\operatorname{adeg}_i \Delta^e(\Gamma) = \beta^E_{1n-i}(E/\operatorname{gin}(J_{\Gamma^*})) \leq \beta^E_{1n-i}(E/\operatorname{gin}_\tau(J_{\Gamma^*})) = \operatorname{adeg}_i \Delta^\tau(\Gamma),$$

and the proof is complete. □

Remark 6.4.

(i) Observe that Theorem 2.14 is essentially equivalent to the following inequalities $m_{\leq i}((\operatorname{gin}_\tau(J))_j) \leq m_{\leq i}(\operatorname{gin}(J)_j)$. Thus one could argue in the above proof of Theorem 6.3 by using the latter numbers instead of the corresponding graded Betti-numbers. (Again see also Murai [27, Proposition 2.4] for the needed inequalities.)

(ii) The fact that $\operatorname{adeg}\Gamma = \operatorname{adeg}\Delta^e(\Gamma)$ if and only if Γ is sequentially Cohen–Macaulay can also be shown by refining the arguments in the proof of Theorem 4.1 and using the following facts: a simplicial complex Γ is sequentially Cohen–Macaulay if and only if each subcomplex $\Gamma^{[i]}$ generated by the i-faces of Γ is Cohen–Macaulay (see Duval [16]); Γ is Cohen–Macaulay if and only if $\Delta^e(\Gamma)$ is pure. (See below for a new proof of the latter fact.)

Similarly to Theorem 5.5 we can prove now the following result which is due to Kalai [26].

Corollary 6.5. *Let Γ be a simplicial complex on the vertex set $[n]$. Then the following statements are equivalent:*

(i) *Γ is Cohen–Macaulay;*

(ii) *$\Delta^e(\Gamma)$ is Cohen–Macaulay;*

(iii) *$\Delta^e(\Gamma)$ is pure.*

Proof. Clearly (ii) and (iii) are equivalent. Suppose that (i) holds, i. e. Γ is Cohen–Macaulay. Then it follows from Theorem 6.3 that

$$\deg\Delta^e(\Gamma) = \deg\Gamma = \operatorname{adeg}\Gamma = \operatorname{adeg}\Delta^e(\Gamma) = \operatorname{sdeg}\Delta^e(\Gamma).$$

Hence $\operatorname{sdeg}\Delta^e(\Gamma) = \deg\Delta^e(\Gamma)$, and thus $\Delta^e(\Gamma)$ is Cohen–Macaulay.

Assume on the other hand that $\Delta^e(\Gamma)$ is Cohen–Macaulay. Then it follows from

$$\operatorname{adeg}\Delta^e(\Gamma) = \deg\Delta^e(\Gamma) = \deg\Gamma \leq \operatorname{adeg}\Gamma \leq \operatorname{adeg}\Delta^e(\Gamma)$$

that $\operatorname{adeg}\Gamma = \operatorname{adeg}\Delta^e(\Gamma)$ and $\deg\Gamma = \operatorname{adeg}\Gamma$. Hence Γ is pure and sequentially Cohen–Macaulay and thus Cohen–Macaulay. □

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