TETRAHEDRAL CURVES

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Abstract. A tetrahedral curve is a space curve whose defining ideal is an intersection of powers of monomial prime ideals of height two. It is supported on a tetrahedral configuration of lines. Schwartau described when certain such curves are ACM, namely he restricted to curves supported on a certain four of the six lines. We consider the general situation.

We first show that starting with an arbitrary tetrahedral curve, there is a particular reduction that produces a smaller tetrahedral curve and preserves the even liaison class. We call the curves that are minimal with respect to this reduction S-minimal curves. Given a tetrahedral curve, we describe a simple algorithm (involving only integers) that computes the S-minimal curve of the corresponding even liaison class; in the process it determines if the original curve is arithmetically Cohen-Macaulay or not. We also describe the minimal free resolution of an S-minimal curve, using the theory of cellular resolutions. This resolution is always linear. This result allows us to classify the arithmetically Buchsbaum, non-ACM tetrahedral curves. More importantly, it allows us to conclude that an S-minimal curve is minimal in its even liaison class; that is, the whole even liaison class can be built up from the S-minimal curve. Finally, we show that there is a large set of S-minimal curves such that each curve corresponds to a smooth point of a component of the Hilbert scheme and that this component has the expected dimension.

Contents

1. Introduction

Recently monomial ideals have attracted a lot of attention, both for their own sake and (for instance via generic initial ideals) as a tool for studying more general ideals,

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certain hyperplane arrangements, and problems in combinatorics and convex geometry. Many questions remain open, however. One important question is that of finding the minimal free resolution of a monomial ideal. While several results are known that give free resolutions, it is very difficult to eliminate the extraneous terms.

In this paper, as part of a broader study, we have in particular found the minimal free resolution for a natural class of monomial ideals (minimal tetrahedral curves – see Theorem 4.2), and we discuss the minimal free resolution for an even broader class (tetrahedral curves – see Remark 4.5). The minimal tetrahedral curves turn out to have a *linear* resolution, among other interesting properties. We also show that the other tetrahedral curves can be obtained from the minimal ones via a simple procedure.

The starting point of this work has been Phil Schwartau's Ph.D. thesis [20], which was never published, where he considered certain monomial ideals and the question of whether or not they were Cohen-Macaulay. Specifically, he considered ideals of the form

$$(X_0, X_1)^a \cap (X_1, X_2)^b \cap (X_2, X_3)^c \cap (X_3, X_0)^d$$

in the ring $k[X_0, X_1, X_2, X_3]$ where k is an algebraically closed field. These are unmixed ideals defining curves in $\mathbb{P}^3$. They are supported on a complete intersection (taking $(a, b, c, d) = (1, 1, 1, 1)$ gives the complete intersection of X_0X_2 and X_1X_3). Schwartau gave a complete classification of the 4-tuples of integers (a, b, c, d) that define arithmetically Cohen-Macaulay curves (see Theorem 2.4 and Theorem 5.3).

This paper arose from our desire to make the natural extension of this result to include the lines defined by (X_0, X_2) and (X_1, X_3). To simplify the notation we have changed the variables, so we are considering the ring $R = k[a, b, c, d]$ and ideals of the form

$$I = (a, b)^{a_1} \cap (a, c)^{a_2} \cap (a, d)^{a_3} \cap (b, c)^{a_4} \cap (b, d)^{a_5} \cap (c, d)^{a_6}.$$

Note that I is an unmixed monomial ideal. Since these six lines can be viewed as forming the edges of a tetrahedron, we call such a curve a *tetrahedral curve*. It is useful to consider the empty set as the *trivial curve* defined by $(0, 0, 0, 0, 0, 0)$.

More than simply considering the question of when such a curve is arithmetically Cohen-Macaulay, we are interested in getting some idea of the even liaison class of such a curve. For background on liaison, see the book [12]; for the most part we will assume the necessary definitions and basic results on liaison.

Many papers in the literature give classification results of the following kind: they consider a particular kind of curve, and ask when two such curves are linked, or ask for a description of the even liaison class of such a curve (cf. for instance [5], [13], [14], [18]). The work in this paper can be viewed, in part, as a new contribution to this kind of question. However, the main interest comes from the naturalness of these monomial ideals themselves, from the surprising effectiveness of our reduction procedure, from the simplicity of the minimal free resolution obtained, and from the interplay of strikingly different techniques and tools used to obtain these results.

The key idea that got this work started was a realization that there is a simple reduction possible for tetrahedral curves (Proposition 3.1). Using the machinery of basic double links, this reduction accomplishes two amazing things: the new curve is in the same even liaison class as the original curve, and the new curve is again a tetrahedral curve, but smaller!

Using this reduction, one of two things happens: either the process of reduction continues until the curve vanishes, i.e. we get the trivial curve, or the process stops at a curve that cannot be further reduced in this way. The curves that ultimately reduce to the trivial curve are clearly arithmetically Cohen-Macaulay, thanks to liaison theory. Curves that cannot be further reduced in this way are very important, and we call them *S-minimal curves*. We give a numerical criterion for S-minimality (Corollary 3.5 and Lemma 3.8). This leads to a simple algorithm (easily done by hand) for computing an S-minimal curve starting with any tetrahedral curve. A MAPLE implementation of it is documented in the appendix.

The first main result of this paper is Theorem 4.2, where the minimal free resolution of an S-minimal tetrahedral curve is computed. We show, in particular, that this resolution is linear. This proof starts by finding the minimal generators of the ideal, and then it translates these generators into a cell complex and uses the theory of cellular resolutions as developed in [2] to find the rest of the resolution.

An immediate consequence of this result is that an S-minimal curve is arithmetically Cohen-Macaulay if and only if it is trivial. This allows one to easily determine, for any 6-tuple of integers $(a_1, a_2, a_3, a_4, a_5, a_6)$, whether it corresponds to an arithmetically Cohen-Macaulay curve or not. Another easy consequence is that a tetrahedral curve is arithmetically Buchsbaum if and only if its Hartshorne-Rao module has diameter 1. (It was already known by Schwartau that certain arithmetically Buchsbaum tetrahedral curves existed.)

A much deeper consequence of Theorem 4.2 is the second main result of the paper (Theorem 5.1), which says that a tetrahedral curve C is S-minimal if and only if it is minimal in its even liaison class.

We give two applications of this result. The first, Theorem 5.3, is a new proof of Schwartau's theorem classifying the arithmetically Cohen-Macaulay curves supported on the complete intersection by giving the precise set of 4-tuples. The second application, Corollary 5.4, is a classification of the 6-tuples that define minimal tetrahedral curves that are arithmetically Buchsbaum. Both of these applications primarily use the earlier results described above, but need the minimality in the even liaison class to complete the proof.

Our third main line of investigation concerns the question of unobstructedness and the search for nice components of the Hilbert scheme. (A component is said to be "nice" if it has the expected dimension $4 \cdot \deg C$.) A great deal of work has been done concerning nice components of the Hilbert scheme – we refer the reader to [7] and to [9], both for important results along these lines, and also for numerous references to other work. Using the method of Dolcetti in [7], we first prove that any curve in $\mathbb{P}^3$ with linear resolution and Hartshorne-Rao module of diameter ≤ 2 is unobstructed, and its Hilbert scheme has the expected dimension $4 \cdot \deg C$ (Proposition 6.1). We then characterize the 6-tuples corresponding to minimal tetrahedral curves with Hartshorne-Rao modules of diameter ≤ 2 (Corollary 5.4 and Lemma 6.2), which then are unobstructed. Turning to a different kind of minimal tetrahedral curve, we show that a tetrahedral curve defined by $(a_1, 0, 0, 0, 0, a_6)$ is also unobstructed. Since we have shown the unobstructedness of a large class of minimal tetrahedral curves, and have shown experimentally on the computer that others are also unobstructed, we end with the question of whether in fact all minimal tetrahedral curves are unobstructed.

We end the paper with a number of questions that arise naturally from our work. In particular, while tetrahedral curves form a large class of unmixed monomial ideals in four variables, they are by no means the only such ideals. Our methods suggest that Liaison theory might be successfully used to study unmixed monomial ideals in a more general setting. The authors plan to pursue this study in the near future.

2. Background

Let $R = k[a,b,c,d]$, where k is a field. We abbreviate by ACM the term "arithmetically Cohen-Macaulay." Recall that a projective subscheme V is ACM if and only if the deficiency modules all vanish:

$$H^i(\mathbb{P}^n, \mathcal{I}_V(t)) = 0 \quad \text{for all } t \in \mathbb{Z} \text{ and all } 1 \leq i \leq \dim V.$$

We recall that the notion of *basic double linkage* was introduced by Lazarsfeld and Rao [10] as a way of adjoining a plane curve, or more generally a complete intersection, to a given curve C in such a way as to preserve the even liaison class of C. More precisely,

Definition 2.1. Let $I \subset R$ be the saturated ideal of an unmixed curve C in $\mathbb{P}^3$. Let $F \in I$ and $G \in R$ be homogeneous polynomials such that (F,G) is a regular sequence. Then the ideal $G \cdot I + (F)$ is the saturated ideal of a curve Y which is linked to C in two steps (i.e. is *bilinked* to C). Y is said to be a *basic double link* of C. As sets, Y is the union of C and the complete intersection defined by (F,G). The degree of Y is $\deg C + (\deg F)(\deg G)$.

The notion of basic double linkage has generalizations in several different directions (higher dimension, higher projective spaces, Gorenstein liaison). We refer to [12] for the details. If one has a way of recognizing an ideal as being of the above form, then one can replace that ideal by the simpler ideal I, knowing that the liaison class is preserved. For instance, if the original ideal is Cohen-Macaulay then so is the new, simpler ideal. This is the approach we take below.

Recall that for an unmixed ideal I, the *n-th symbolic power* of I, denoted $I^{(n)}$, is the saturation of the top dimensional part of the ideal I^n. Recall also that if I is a complete intersection then $I^{(n)} = I^n$.

Definition 2.2. Let T be any union of six lines $\ell_1, \dots, \ell_6$ forming the edges of a "tetrahedron." A *tetrahedral curve* is the non-reduced scheme C supported on T and defined by the saturated ideal $I_C = I_{\ell_1}^{a_1} \cap \cdots \cap I_{\ell_6}^{a_6}$, where $a_i \geq 0$ for all i. The number a_i is called the *weight* of the line l_i.

Remark 2.3. If C is a tetrahedral curve then we can perform a change of variables and obtain a monomial ideal which has a primary decomposition of the form

$$I = (a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6}.$$

For the remainder of this paper we will assume that a tetrahedral curve is a monomial ideal of this form.

In his thesis [20], Schwartau considered the curves with ideals of the form

$$(X_0, X_1)^a \cap (X_1, X_2)^b \cap (X_2, X_3)^c \cap (X_3, X_0)^d$$

(in his notation). This is equivalent to taking $a_2 = a_5 = 0$ in Definition 2.2. Schwartau was primarily interested in the question of when these curves are arithmetically Cohen-Macaulay. His main result on this problem is the following, which we now translate to our language.

Theorem 2.4 ([20]). *The ideal*

$$(a, b)^{a_1} \cap (a, c)^0 \cap (a, d)^{a_3} \cap (b, c)^{a_4} \cap (b, d)^0 \cap (c, d)^{a_6}$$

defines an arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ if and only if

Case 1.	$a_1, a_3, a_4, a_6 > 0$:	$a_1 + a_6 = a_3 + a_4 + \epsilon$, *for* $\epsilon = -1, 0, 1$.
Case 2.	$a_1, a_4, a_6 > 0,\ a_3 = 0$:	$a_1 + a_6 \leq a_4 + 1$.
Case 3A.	$a_1, a_4 > 0,\ a_3 = a_6 = 0$:	*always*
Case 3B.	$a_1, a_6 > 0,\ a_3 = a_4 = 0$:	*never*
Case 4.	$a_1 > 0,\ a_3 = a_4 = a_6 = 0$:	*always*

Remark 2.5. Clearly Schwartau intended some kind of reduction of cases, since for instance the case $a_2 > 0$, $a_1 = a_3 = a_4 = 0$ is not included in his theorem. We have given an "invariant" version in Theorem 5.3 below, which we think reflects Schwartau's intention. We also give a new proof.

3. Reduction and S-minimality

The key to our approach to this problem is the following reduction method.

Proposition 3.1. *Let* $I = (a, b)^{a_1} \cap (a, c)^{a_2} \cap (a, d)^{a_3} \cap (b, c)^{a_4} \cap (b, d)^{a_5} \cap (c, d)^{a_6}$ *where not all exponents a_i are zero. Consider the following systems of inequalities:*

$$\begin{array}{llcllll} (A): & a_1 + a_2 & \geq & a_4, & (B): & a_1 + a_4 & \geq & a_2, \\ & a_1 + a_3 & \geq & a_5, & & a_1 + a_5 & \geq & a_3, \\ & a_2 + a_3 & \geq & a_6 & & a_4 + a_5 & \geq & a_6 \\ (C): & a_2 + a_4 & \geq & a_1, & (D): & a_3 + a_5 & \geq & a_1, \\ & a_2 + a_6 & \geq & a_3, & & a_3 + a_6 & \geq & a_2, \\ & a_4 + a_6 & \geq & a_5 & & a_5 + a_6 & \geq & a_4. \end{array}$$

For $1 \leq i \leq 6$ let $a_i' = \max\{0, a_i - 1\}$. Then we have

(i) $(A) \Leftrightarrow$ *I is a basic double link of*

$$(a, b)^{a_1'} \cap (a, c)^{a_2'} \cap (a, d)^{a_3'} \cap (b, c)^{a_4} \cap (b, d)^{a_5} \cap (c, d)^{a_6}$$

using $F = b^{a_1} c^{a_2} d^{a_3}$ and $G = a$.

(ii) $(B) \Leftrightarrow$ *I is a basic double link of*

$$(a, b)^{a_1'} \cap (a, c)^{a_2} \cap (a, d)^{a_3} \cap (b, c)^{a_4'} \cap (b, d)^{a_5'} \cap (c, d)^{a_6}.$$

using $F = a^{a_1} c^{a_4} d^{a_5}$ and $G = b$.

(iii) $(C) \Leftrightarrow$ *I is a basic double link of*

$$(a,b)^{a_1} \cap (a,c)^{a'_2} \cap (a,d)^{a_3} \cap (b,c)^{a'_4} \cap (b,d)^{a_5} \cap (c,d)^{a'_6}.$$

using $F = a^{a_2} b^{a_4} d^{a_6}$ *and* $G = c$.

(iv) $(D) \Leftrightarrow$ *I is a basic double link of*

$$(a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a'_3} \cap (b,c)^{a_4} \cap (b,d)^{a'_5} \cap (c,d)^{a'_6}.$$

using $F = a^{a_3} b^{a_5} c^{a_6}$ *and* $G = d$.

Proof. We will prove (i); of course the others are proved similarly. First note that

$$a \cdot (a,b)^{n-1} + (b^n) = (a,b)^n$$

for $n \geq 1$. Now consider the monomial $F = b^{a_1} c^{a_2} d^{a_3}$. Notice that

$$F \in (a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \subset (a,b)^{a'_1} \cap (a,c)^{a'_2} \cap (a,d)^{a'_3} \tag{3.1}$$

even when one or more of the $a_i = 0$. The three inequalities are equivalent to

$$F \in (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6}. \tag{3.2}$$

Hence

$$F \in (a,b)^{a'_1} \cap (a,c)^{a'_2} \cap (a,d)^{a'_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6}.$$

Notice that (a, F) is a regular sequence. Hence we can construct a basic double link of the form

$$J = a \cdot \left[(a,b)^{a'_1} \cap (a,c)^{a'_2} \cap (a,d)^{a'_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6} \right] + (b^{a_1} c^{a_2} d^{a_3}).$$

We have to show that

$$J = I = (a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6}.$$

The ideal J is a saturated, unmixed ideal, by the theory of basic double linkage (cf. [12]), so it is enough to show that $J \subset I$ and that they define schemes of the same degree.

For the first one, the fact that

$$\begin{aligned} & a \cdot \left[(a,b)^{a'_1} \cap (a,c)^{a'_2} \cap (a,d)^{a'_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6} \right] \\ & \subseteq (a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6} \end{aligned}$$

is clear, while the fact that $b^{a_1} c^{a_2} d^{a_3} \in I$ comes from (3.1) and (3.2) above.

For the degree computation, recall that $\binom{n}{2} + n = \binom{n+1}{2}$. Then, again using the theory of basic double linkage, the degree of J is

$$\begin{aligned} \deg J &= \binom{a_1}{2} + \binom{a_2}{2} + \binom{a_3}{2} + \binom{a_4+1}{2} + \binom{a_5+1}{2} + \binom{a_6+1}{2} + (a_1 + a_2 + a_3) \\ &= \binom{a_1+1}{2} + \binom{a_2+1}{2} + \binom{a_3+1}{2} + \binom{a_4+1}{2} + \binom{a_5+1}{2} + \binom{a_6+1}{2} \\ &= \deg I \end{aligned}$$

as desired.

The converse is immediate: the fact that I is a basic double link as stated implies the inequalities of (A) by again using(3.1) and (3.2). $\square$

Notation 3.2. For the rest of this paper we will abbreviate the monomial ideal

$$(a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6}$$

(or the corresponding curve) by the 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$.

Remark 3.3. Basic double linkage is a special case of Schwartau's "liaison addition." (See [20] for the original and [8] for a generalization.) Without entering into details, we remark that many interesting tetrahedral curves arise as liaison additions. For instance, let I be the ideal of the six lines, i.e. the tetrahedral curve $(1,1,1,1,1,1)$. Let F be the polynomial $abcd$ giving the four faces of the "tetrahedron" defined by I. Note that F is double along each of the six lines. Let G be a generally chosen cubic in I. Then (F,G) self-links I (as can be see geometrically, using Bezout's theorem) and one can check that in fact the d-th symbolic power of I, $I^{(d)}$, can be expressed as

$$I^{(d)} = G^{d-1} \cdot I + F \cdot I^{(d-2)}.$$

Hence we see directly the curves (d,d,d,d,d,d) are ACM. Note however, that the d-th power I^d is not saturated, and has embedded points.

Schwartau also obtains many Buchsbaum curves using liaison addition. We will return to Buchsbaum curves below.

Definition 3.4. We say that a tetrahedral curve is *S-minimal* if there is no reduction of the type described in Proposition 3.1.

Corollary 3.5. *Consider a tetrahedral curve $C = (a_1, a_2, a_3, a_4, a_5, a_6)$ where not all a_i are 0. Assume without loss of generality that $a_6 = \max\{a_1, \dots, a_6\}$. Then C is S-minimal if and only if*

$$\begin{array}{rcl} a_1 & > & \max\{a_3 + a_5, a_2 + a_4\} \text{ and} \\ a_6 & > & \max\{a_4 + a_5, a_2 + a_3\} \end{array}$$

Proof. It is immediate to check that if the stated conditions are satisfied then each of (A), (B), (C) and (D) in Proposition 3.1 has at least one inequality that is not satisfied, hence C cannot be reduced via Proposition 3.1.

Conversely, suppose that C is S-minimal. The second and third inequalities of (C) and (D) in Proposition 3.1are forced to be true by the assumption that $a_6 = \max\{a_1, \dots, a_6\}$, so S-minimal implies that $a_1 > \max\{a_3 + a_5, a_2 + a_4\}$. In particular, $a_1 > a_2$, $a_1 > a_3$, $a_1 > a_4$ and $a_1 > a_5$, so the first two inequalities of (A) and of (B) must be true. Hence again the assumption that C is S-minimal forces $a_6 > \max\{a_4 + a_5, a_2 + a_3\}$. □

Example 3.6. Even with the assumption that $a_6 = \max\{a_1, \dots, a_6\}$, the first inequality of Corollary 3.5 alone does not imply S-minimality, as shown by the example $C = (4,2,2,1,1,4)$, which can be reduced using (A) to the curve $C' = (3,1,1,1,1,4)$. This new curve C' is S-minimal. It can be checked that C' (and hence C) is not arithmetically Cohen-Macaulay (see Corollary 4.3 as well).

Our next goal is to provide an algorithm that produces an S-minimal curve to a given tetrahedral curve C. To this end we will extend the definition of weights. We will use the tetrahedron $T = T(C)$ whose edges are part of the (potentially) supporting lines of the curve C.

Definition 3.7. The *weight of a facet* of $T = T(C)$ is the sum of the weights of the edges forming its boundary. Similarly, the weight of a pair of skew lines being a subset of the six given lines is the sum of the weights of the lines.

Using these concepts we can make Corollary 3.5 more precise. Note that each of the reductions (A) - (D) in Proposition 3.1 reduces the weights of the edges of a facet of the tetrahedron. Thus, we will say that we *reduce a facet* if we apply the corresponding reduction.

The next result says in particular that a tetrahedral curve is not minimal if and only if we can reduce a facet of maximal weight.

Lemma 3.8. *Let C be a tetrahedral curve $C = (a_1, a_2, a_3, a_4, a_5, a_6)$ where we assume without loss of generality that $a_6 = \max\{a_1, \dots, a_6\} > 0$. Let w be the maximal weight of a facet of the tetrahedron $T = T(C)$. Then the following conditions are equivalent:*

(a) *C is not S-minimal.*
(b) $a_1 + a_6 \leq w$.
(c) *One can reduce any of the facets of T having maximal weight w.*

Proof. We begin by showing the equivalence of (a) and (b). According to Corollary 3.5 we know that C is not S-minimal if and only if

$$\begin{array}{lll} a_1 & \leq & \max\{a_3 + a_5, a_2 + a_4\} \quad \text{or} \\ a_6 & \leq & \max\{a_4 + a_5, a_2 + a_3\}. \end{array}$$

But this is equivalent to the condition

$$a_1 + a_6 \leq \max\{a_1 + a_4 + a_5, a_1 + a_2 + a_3, a_3 + a_5 + a_6, a_2 + a_4 + a_6\} = w$$

as claimed.

Since (c) trivially implies (a), it remains to show that (c) is a consequence of (b). To this end we distinguish four cases.

Case 1: Let $w = a_3 + a_5 + a_6$. Then (b) provides $a_3 + a_5 \geq a_1$ showing that we can apply reduction (D).

Case 2: Let $w = a_2 + a_4 + a_6$. Then we conclude as above that we can use reduction (C).

Case 3: Let $w = a_1 + a_2 + a_3$. Then (b) implies $a_2 + a_3 \geq a_6$. Moreover, we have by assumption $w \geq a_2 + a_4 + a_6$. It implies

$$a_1 + a_3 \geq a_4 + a_6 \geq a_6 \geq a_5.$$

Similarly $w \geq a_3 + a_5 + a_6$ provides $a_1 + a_2 \geq a_4$. Thus, we have shown that we can apply reduction (A).

Case 4: Let $w = a_1 + a_4 + a_5$. Then we see as in Case 3 that we can use reduction (B). □

The last result leads to an algorithm for producing S-minimal curves that works with weights only.

Algorithm 3.9 (for computing S-minimal curves)**.**
Input: $(a_1, \dots, a_6) \in \mathbb{Z}^6$ where all $a_i \geq 0$, the weight vector of a tetrahedral curve C.
Output: $(a_1, \dots, a_6)$, the weight vector of an S-minimal curve obtained by reducing C.

1. compute i such that $a_i = \max\{a_1, \dots, a_6\}$
2. **if** $a_i = 0$ **then return** $(0, \dots, 0)$
3. determine a facet F of maximal weight w
4. **if** $a_i + a_{7-i} > w$ **then return** $(a_1, \dots, a_6)$.
5. apply the reduction corresponding to the facet F and **go** to Step 1

Remark 3.10. (i) Strictly speaking the scheme above is not an algorithm since it leaves choices in Steps 1 and 3 in case there is more than one line or facet of maximal weight. But this can be fixed easily, e.g., by using the lexicographic order. See the appendix.

(ii) Correctness of Algorithm 3.9 follows by Lemma 3.8. Note that the lines with index i and $7 - i$ do not intersect.

4. The Minimal Free Resolution of an S-minimal Curve

Notice that the 6-tuple $(0, \dots, 0)$ corresponds to the ring R. It turns out that it is useful to consider R formally as a curve as we did in Definition 2.2. We give it a name.

Definition 4.1. The *trivial* tetrahedral curve is defined by $(0, \dots, 0)$.

The following is one of the two main results of this paper.

Theorem 4.2. *Every non-trivial* S-minimal *tetrahedral curve has a linear minimal free resolution.*

More precisely, if the curve C is defined by $(a_1, a_2, a_3, a_4, a_5, a_6)$ and $a_6 = \max\{a_i\} > 0$ then its minimal free resolution has the form

$$0 \to R^{\beta_3}(-a_1 - a_6 - 2) \to R^{\beta_2}(-a_1 - a_6 - 1) \to R^{\beta_1}(-a_1 - a_6) \to I_C \to 0$$

where

$$\begin{aligned}
\beta_1 &= (a_1 + 1)(a_6 + 1) - \sum_{i=2}^{5} \frac{a_i(a_i + 1)}{2} \\
\beta_2 &= 2a_1a_6 + a_1 + a_6 - \sum_{i=2}^{5} a_i(a_i + 1) \\
\beta_3 &= a_1a_6 - \sum_{i=2}^{5} \frac{a_i(a_i + 1)}{2}.
\end{aligned}$$

Proof. Let C be an S-minimal tetrahedral curve, and assume that C is the curve defined by the tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$. Without loss of generality assume that a_6 is the largest entry. Note that $(a, b)^{a_1}$ and $(c, d)^{a_6}$ are disjoint ACM curves, so their intersection is given by their product (cf. [21], Corollaire, p. 143 or [15], Theorem 1.2). In particular, the curve $(a_1, 0, 0, 0, 0, a_6)$ is given by the entries of the following $(a_1 + 1) \times (a_6 + 1)$ matrix:

$$
\begin{bmatrix}
a^{a_1}c^{a_6} & a^{a_1}c^{a_6-1}d & \dots & a^{a_1}cd^{a_6-1} & a^{a_1}d^{a_6} \\
a^{a_1-1}bc^{a_6} & a^{a_1-1}bc^{a_6-1}d & \dots & a^{a_1-1}bcd^{a_6-1} & a^{a_1-1}bd^{a_6} \\
\vdots & \vdots & & \vdots & \vdots \\
ab^{a_1-1}c^{a_6} & ab^{a_1-1}c^{a_6-1}d & \dots & ab^{a_1-1}cd^{a_6-1} & ab^{a_1-1}d^{a_6} \\
b^{a_1}c^{a_6} & b^{a_1}c^{a_6-1}d & \dots & b^{a_1}cd^{a_6-1} & b^{a_1}d^{a_6}
\end{bmatrix}
\tag{4.1}
$$

By Corollary 3.5, we have the inequalities

$$
\begin{array}{rcl}
a_1 & > & a_3 + a_5 \\
a_1 & > & a_2 + a_4 \\
a_6 & > & a_4 + a_5 \\
a_6 & > & a_2 + a_3
\end{array}
\tag{4.2}
$$

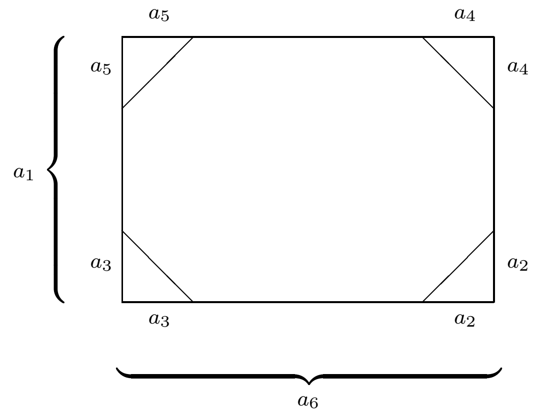
In particular, $a_1 > 0$.

Now we want to describe the ideal $(a_1, a_2, a_3, a_4, a_5, a_6)$. Note that as ideals,

$$(a_1, a_2, a_3, a_4, a_5, a_6) \subset (a_1, 0, 0, 0, 0, a_6).$$

Consider the monomials obtained by deleting the following from the matrix (4.1): a_2 diagonals from the "Southeast corner," a_3 diagonals from the "Southwest corner," a_4 diagonals from the "Northeast corner," and a_5 diagonals from the "Northwest corner." We obtain the following shape:

(4.3)



Note that the a_i in the diagram measure the length, and not the number of vertices. Note also that the minimality, and in particular the inequalities (4.2), guarantee that these "cuts" do not overlap. In particular, any monomial that is removed falls unambiguously into exactly one of the removed regions.

Now let us define $I(a_1, a_2, a_3, a_4, a_5, a_6)$ to be the ideal generated by the remaining monomials, after removing the corners as described above. We first make the following claim:

Claim: $I(a_1, a_2, a_3, a_4, a_5, a_6) = (a_1, a_2, a_3, a_4, a_5, a_6)$, *that is that* $I(a_1, a_2, a_3, a_4, a_5, a_6)$ *is precisely the ideal of the corresponding tetrahedral curve.*

It is clear that the monomial generators that we have removed from $(a_1,0,0,0,0,a_6)$ are precisely those generators that are not in $(a_1,a_2,a_3,a_4,a_5,a_6)$. This proves $\subseteq$.

We now prove the reverse inclusion. Since $(a_1,a_2,a_3,a_4,a_5,a_6)$ is a monomial ideal and $(a_1,a_2,a_3,a_4,a_5,a_6) \subset (a_1,0,0,0,0,a_6)$, we see that the two ideals in the statement of the claim agree in degrees $\leq a_1+a_6$. The danger is that $(a_1,a_2,a_3,a_4,a_5,a_6)$ could have a monomial minimal generator of larger degree. We now show that this does not occur.

Suppose that M were such a generator, of degree $> a_1+a_6$. M cannot be a multiple of a minimal generator of $I(a_1,a_2,a_3,a_4,a_5,a_6)$, by the first inclusion. But on the other hand, M is contained in $(a_1,0,0,0,0,a_6)$. Therefore M is a multiple of one of the monomials that we have removed, say N.

We will consider the case where N is in the removed "Northwest corner;" the other cases are identical. We have removed a_5 diagonals from this corner, where a_5 is the exponent of the component (b,d). Clearly if we multiply N by either a or c, we do not produce a monomial that is in $(a_1,a_2,a_3,a_4,a_5,a_6)$. (The problem is in the component (b,d).)

More generally, suppose that N lies on the k-th diagonal from the "Northwest corner" $(k \leq a_5)$. Then N has the form $a^{\ell_1}b^{\ell_2}c^{\ell_3}d^{\ell_4}$ where

$$\begin{array}{rcl} \ell_1+\ell_2 &=& a_1, \\ \ell_3+\ell_4 &=& a_6, \\ \ell_2+\ell_4 &=& k-1 \ (< a_5) \end{array} \tag{4.4}$$

Suppose that we multiply N by a monomial $a^{k_1}b^{k_2}c^{k_3}d^{k_4}$. The result is, of course, $a^{k_1+\ell_1}b^{k_2+\ell_2}c^{k_3+\ell_3}d^{k_4+\ell_4}$. This will be in $(a_1,a_2,a_3,a_4,a_5,a_6)$ if and only if

$$\begin{array}{rcl} k_1+\ell_1+k_2+\ell_2 &\geq& a_1 \\ k_1+\ell_1+k_3+\ell_3 &\geq& a_2 \\ k_1+\ell_1+k_4+\ell_4 &\geq& a_3 \\ k_2+\ell_2+k_3+\ell_3 &\geq& a_4 \\ k_2+\ell_2+k_4+\ell_4 &\geq& a_5 \\ k_3+\ell_3+k_4+\ell_4 &\geq& a_6. \end{array} \tag{4.5}$$

We make the subclaim that (4.5) holds if and only if $k_2+k_4 \geq a_5-(k-1)$. If (4.5) holds then from (4.5) and (4.4) respectively we have

$$k_2+k_4 \geq a_5-\ell_2-\ell_4 = a_5-(k-1)$$

as desired. Conversely, assume that $k_2+k_4 \geq a_5-(k-1)$. The first and last inequalities of (4.5) are immediate from (4.4), without needing our hypothesis. Similarly, none of the other inequalities apart from the fifth one need our hypothesis. Consider for instance the second inequality. We have

$$\begin{array}{rcll} k_1+\ell_1+k_3+\ell_3 &=& k_1+k_3+a_1+a_6-(k-1) & \text{(from (4.4))} \\ &>& k_1+k_3+a_2+a_4+a_4+a_5-(k-1) & \text{(from (4.2))} \\ &>& k_1+k_3+a_2+2a_4 & \text{(since } k \leq a_5\text{)} \\ &\geq& a_2. & \end{array}$$

We have only to show the fifth inequality, and this is immediate using (4.4) and our hypothesis. This completes the proof of our subclaim.

We return to our monomial $M = a^{k_1+\ell_1}b^{k_2+\ell_2}c^{k_3+\ell_3}d^{k_4+\ell_4}$. By our subclaim, $M \in (a_1,a_2,a_3,a_4,a_5,a_6)$ if and only if $k_2+k_4 \geq a_5-(k-1)$. In order for M to have a chance to be a minimal generator of $(a_1, a_2, a_3, a_4, a_5, a_6)$, we have to choose k_1, k_2, k_3, k_4 as small as possible. This means that we may assume that M has the form

$$a^{\ell_1} b^{\ell_2+k_2} c^{\ell_3} d^{\ell_4+k_4}$$

where $k_2+k_4 = a_5-(k-1)$. We have to show that this is in fact a multiple of an element, N, of $I(a_1, a_2, a_3, a_4, a_5, a_6)$. We assert that the following is the desired element:

$$N' = a^{\ell_1-k_2} b^{\ell_2+k_2} c^{\ell_3-k_4} d^{\ell_4+k_4}.$$

To see this, consider the matrix (4.1). Recall that the monomial $N = a^{\ell_1} b^{\ell_2} c^{\ell_3} d^{\ell_4}$ lies in the removed Northwest corner (see diagram (4.3)), on the k-th diagonal. Each step South represents a decrease of ℓ_1 by 1 and an increase of ℓ_2 by 1, while each step East represents a decrease of ℓ_3 by 1 and an increase of ℓ_4 by 1. The monomial N' (accepting temporarily that the exponents are non-negative) represents a move from N of k_2 steps South and k_4 steps East. Each such step moves one to the next diagonal. But $k_2 + k_4 = a_5 - (k-1)$, so the result of the $k_2 + k_4$ steps shows that N' lies on the border of (4.3), on the diagonal in the Northwest corner. Hence $N' \in I(a_1, a_2, a_3, a_4, a_5, a_6)$ and so also $M \in I(a_1, a_2, a_3, a_4, a_5, a_6)$. This completes the proof of the claim. In particular, we have shown that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is generated in degree $a_1 + a_6$, and the value of β_1 is clear from the construction (see the diagram (4.3)).

Having found the minimal generators of our S-minimal curve we can use the theory of cellular resolutions as developed in [2]. Hence, we will first define an appropriate regular cell complex X. Together with a choice of an incidence function, the cell complex determines a complex $\mathbb{F}_X$ of free R-modules. Finally, we will show that it is acyclic and resolves the ideal of our curve.

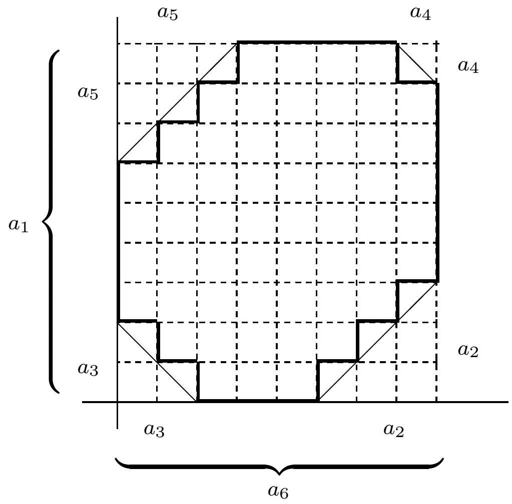
As above we begin by describing the cell complex X_0 for the curve $(a_1, 0, 0, 0, 0, a_6)$. Its geometric realization will be a rectangle in the (i,j)-plane. To this end we will identify the integer point (i,j) with the monomial $a^j b^{a_1-j} c^{a_6-i} d^i$. The vertices of X_0 are all the integer points (i,j) satisfying $0 \le i \le a_6$ and $0 \le j \le a_1$. The edges of X_0 are the relative interiors of all "horizontal" line segments with endpoints $(i-1,j)$ and (i,j) and all "vertical" line segments with endpoints $(i,j-1)$ and (i,j) where all the endpoints are vertices of X_0. The 2-dimensional faces of X_0 are the relative interiors of all squares with vertices $(i-1,j-1), (i,j-1), (i,j), (i-1,j)$, all being vertices of X_0. Together with the empty set, X_0 is clearly a finite regular cell complex (cf., e.g., [6], Section 6.2 for the definition). The empty set has dimension -1.

The cell complex $X := X(a_1, a_2, a_3, a_4, a_5, a_6)$ of the curve $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a subcomplex of X_0 obtained by deleting corners of X_0. More precisely, the vertices of X are the points (i,j) satisfying

$$\begin{array}{ccccc} a_3 & \le & i+j & \le & a_1+a_6-a_4 \\ a_5-a_1 & \le & i-j & \le & a_1-a_2 \end{array} \tag{4.6}$$

The edges and 2-dimensional faces of X are by definition the corresponding faces of X_0 whose edges all belong to X.

(4.7)



In the above diagram, the full rectangle indicates the cell complex X_0, while the bold face segments are the boundary of the cell complex X. Again it is clear that X is a finite regular cell complex. For a face F of X we set m_F the least common multiple of the vertices of F. The exponent vector of the monomial m_F in $\mathbb{Z}^4$ is called the degree of the face F.

Now we fix an incidence function $\varepsilon(F, F')$ on pairs of faces of X (see for instance [2]). Then the cellular complex $\mathbb{F}_X$ is the $\mathbb{Z}$-graded complex

$$\mathbb{F}_X : \qquad 0 \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} 0$$

where

$$F_i = \bigoplus_{F \in X, \dim F = i} R(-\deg m_F) = \bigoplus_{F \in X, \dim F = i} Re_F$$

and the differential is given by

$$\varphi_i(F) = \sum_{F' \in X, \dim F' = i-1} \varepsilon(F, F') \frac{m_F}{m_{F'}} e_F. \tag{4.8}$$

By the construction of this complex, the image of φ_0 is the ideal $(a_1, a_2, a_3, a_4, a_5, a_6)$, thanks to the computation above of the minimal generators. We want to show that $\mathbb{F}_X$ provides a minimal free resolution of this ideal. To this end denote for $b = (b_0, \ldots, b_3) \in \mathbb{Z}^4$ by $X_{\leq b}$ the sub-complex of X on the vertices of degree $\leq b$. Here, we use the partial ordering on $\mathbb{Z}^3$ induced by comparing componentwise. It is easy to see that for all degrees $b \in \mathbb{Z}^4$ the geometric realization of the cell complex $X_{\leq b}$ is contractible, thus $X_{\leq b}$ is acyclic. Hence, [2], Proposition 1.2, shows that $\mathbb{F}_X$ is a free resolution of the ideal $(a_1, a_2, a_3, a_4, a_5, a_6)$.

Next, we observe for every i-dimensional face F of X that m_F has degree $a_1 + a_6 + i$. It follows immediately that $\mathbb{F}_X$ is a minimal free resolution. Moreover, it is easy to see that X contains $(a_1+1)(a_6+1) - \sum_{i=2}^5 \frac{a_i(a_i+1)}{2}$ vertices, $2a_1a_6 + a_1 + a_6 - \sum_{i=2}^5 a_i(a_i+1)$ edges, and $a_1a_6 - \sum_{i=2}^5 \frac{a_i(a_i+1)}{2}$ 2-dimensional faces. Hence the complex $\mathbb{F}_X$ giving the free resolution of C is of the form as claimed. $\square$

Combining this result with the Auslander-Buchsbaum formula we get:

Corollary 4.3. *An S-minimal tetrahedral curve is arithmetically Cohen-Macaulay if and only if it is trivial.*

Remark 4.4. (i) In [2] Bayer and Sturmfels introduce the hull resolution. They show that the hull resolution of a monomial ideal is a free resolution which is not necessarily minimal. The hull resolution contains as sub-complex the so-called Scarf complex which is easy to compute. For generic monomial ideals the Scarf complex and the hull resolution agree and provide a minimal free resolution. However, the defining ideal of a tetrahedral curve is typically not generic (in the sense of Bayer and Sturmfels). For example, the ideal $(1, 0, 0, 0, 0, 1)$ corresponding to a pair of skew lines has a Scarf complex of length 2 while its minimal free resolution has length 3 (cf. [23], Example 3.4.20). Tappe has also computed the hull resolution of the ideals $(a_1, 0, 0, 0, 0, a_6)$ for $a_1, a_6 \leq 3$. In these cases the hull resolution does give the minimal free resolution. However, it is not clear (to the authors) if this is true for all S-minimal curves.

Note that the computation of the hull resolution requires one to compute the convex hull of as many points as the ideal has minimal generators. This is a rather non-trivial task if the number of points is large. On the other hand our description of the minimal free resolution of S-minimal curves uses a very simple cell complex which gives a more direct approach.

(ii) After we had found the cell complex that determines the minimal free resolution of an S-minimal curve we realized that Schwartau [20] uses a similar description in order to compute the minimal free resolution of any tetrahedral curve with $a_2 = a_5 = 0$. However, much of the theory that we have used here was not available during the writing of [20], and we found that proof to be less "clean" even in the restricted setting.

Remark 4.5. We have seen that any tetrahedral curve can be obtained from an S-minimal one by a sequence of basic double links – this is just the reverse of our reduction procedure. Using this fact, it is possible in principle to write (by hand) the minimal free resolution of any tetrahedral curve. The main tool is Corollary 4.5 of [16]. Briefly, this says that if $F \in (I_C)_{d_2}$ and $G \in R_{d_1}$, with (F, G) a regular sequence, and if $I_Y = GI_C + (F)$ is the basic double link obtained with F and G (in our case G is always a, b, c or d so $d_1 = 1$), then we have the following. Suppose that I_C is minimally generated by $G_1, \dots, G_s$, of degrees $e_1, \dots, e_s$ respectively. Then I_Y is clearly generated by $\{GG_1, \dots, GG_s, F\}$. We note immediately that F has to be a minimal generator of I_Y (to obtain codimension two). The question is whether the others are minimal generators. Then it is shown in [16] that

(a) If F is not part of a minimal generating set of I_C then $\{GG_1, \dots, GG_s, F\}$ is a minimal generating set for I_Y.
(b) If F is part of a minimal generating set of I_C then exactly one of the generators GG_i is redundant.

In particular, in case (a) the generator degrees for I_Y are $e_1 + d_1, \dots, e_s + d_1, d_2$ and in case (b) one of these (but not d_2) is removed. It is also shown in [16] Corollary 4.5 that past the first syzygies, the minimal free resolutions of I_C and I_Y are identical. So one can write a free resolution $\mathbb{F}_\bullet$ for I_Y from knowledge of a minimal free resolution of I_C, and the minimality of $\mathbb{F}_\bullet$ is automatic except possibly for the cancellation of a summand of the form $R(-e_i - d_1)$; furthermore this cancellation is completely determined by the question of whether F is a minimal generator of I_C or not.

In our case, everything is governed by Proposition 3.1 and Corollary 3.5. Given a tetrahedral curve Y, we can tell immediately from Corollary 3.5 if it is S-minimal or not. If it is, we have the minimal free resolution from Theorem 4.2. If it is not, Proposition 3.1 tells how to obtain it by a very explicit basic double link from a smaller tetrahedral curve C. Since the ideals are monomial, it is easy to determine if the polynomial F is a minimal generator of I_C or not. Since we ultimately arrive at an S-minimal (or trivial) curve, whose resolution we know, in principle we know the minimal free resolution of any S-minimal curve. However, an explicit description of this resolution (even the graded Betti numbers) just in terms of the exponents seems to be difficult to find.

As another consequence of Theorem 4.2 we get some information about the Hartshorne-Rao module of any tetrahedral curve. Recall that the Hartshorne-Rao module of a curve C is the graded module

$$M(C) := \oplus_{j \in \mathbb{Z}} H^1(\mathcal{I}_C(j)).$$

Corollary 4.6. *The K-dual of the Hartshorne-Rao module of a tetrahedral curve is generated in one degree.*

Proof. Since the Hartshorne-Rao module is up to degree shift invariant in an even liaison class, it suffices to show the claim for a non-trivial S-minimal curve C. By local duality we have the following graded isomorphism

$$M(C)^\vee \cong \operatorname{Ext}^3(R/I_C, R)(-4).$$

The latter module can be computed from the minimal free resolution of I_C. Since it is linear the claim follows. $\square$

Remark 4.7. (i) The conclusion of the last result is in general not true for the Hartshorne-Rao module itself. Indeed, the Hartshorne-Rao module of the curve $(5, 2, 2, 1, 1, 5)$ has minimal generators in two different degrees, namely 4 and 5.

(ii) The curve in (i) also shows that in general we cannot link a tetrahedral curve in an odd number of steps to a tetrahedral curve.

From a cohomological point of view the simplest curves that are not arithmetically Cohen-Macaulay are the arithmetically Buchsbaum curves. The tetrahedral curves among them have special properties.

Corollary 4.8. *A tetrahedral curve is arithmetically Buchsbaum if and only if its Hartshorne-Rao module satisfies*

$$M(C) \cong K^m(t) \quad \textit{for some integers } m, t \textit{ where } m \geq 0.$$

Proof. This follows immediately by Corollary 4.6. $\square$

Remark 4.9. The fact that such arithmetically Buchsbaum curves exist was known already to Schwartau, who constructed them with liaison addition. The fact that they are the *only* arithmetically Buchsbaum curves among the tetrahedral curves is new.

5. Minimality in the Even Liaison Class, and Applications

Now we want to show that for tetrahedral curves the concept of S-minimality and minimality in its even liaison class agree. This is the second main result of this paper.

Theorem 5.1. *Let C be a tetrahedral curve which is not arithmetically Cohen-Macaulay. Then C is S-minimal if and only if it is minimal in its even liaison class.*

Proof. It suffices to show that every S-minimal curve is minimal in its even liaison class. Denote by $s(C)$ the initial degree of C,

$$s(C) := \min\{j \in \mathbb{Z} \mid [I_C]_j \neq 0\},$$

and by $e(C)$ its index of speciality,

$$e(C) := \max\{j \in \mathbb{Z} \mid H^2(\mathcal{I}_C(j)) \neq 0\}.$$

A result of Lazarsfeld and Rao in [10] says that C is minimal if

$$s(C) \geq e(C) + 4.$$

We will show that this criterion applies to our tetrahedral curves. We may again assume that $a_6 = \max\{a_i\} > 0$. The minimal free resolution of C is

$$\begin{array}{ccccccccc} 0 & \rightarrow & R^{\beta_3}(-a_1-a_6-2) & \stackrel{\phi_2}{\rightarrow} & R^{\beta_2}(-a_1-a_6-1) & \rightarrow & R^{\beta_1}(-a_1-a_6) & \rightarrow & I_C \rightarrow 0 \\ & & & & & \searrow & \nearrow & & \\ & & & & & K & & & \\ & & & & & \nearrow & \searrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where K splits the resolution into two short exact sequences. We know that $s(C) = a_1 + a_6$, so we have to show that $h^2(\mathcal{I}_C(a_1+a_6-3)) = 0$. (See Remark 5.5 (ii).) Letting $\mathcal{K}$ be the sheafification of K, it is enough to show that $h^3(\mathcal{K}(a_1+a_6-3)) = 0$. The leftmost short exact sequence gives

$$0 \rightarrow H^2(\mathcal{K}(a_1+a_6-3)) \rightarrow H^3(\mathcal{O}_{\mathbb{P}^3}^{\beta_3}(-5)) \rightarrow H^3(\mathcal{O}_{\mathbb{P}^3}^{\beta_2}(-4)) \rightarrow H^3(\mathcal{K}(a_1+a_6-3)) \rightarrow 0.$$

Since the rightmost short exact sequence gives $h^1(\mathcal{I}_C(t)) = h^2(\mathcal{K}(t))$ for all t, the above long exact sequence gives

$$h^3(\mathcal{K}(a_1+a_6-3)) = \beta_2 - 4\beta_3 + h^1(\mathcal{I}_C(a_1+a_6-3)). \tag{5.1}$$

Now, recall from [19], Theorem 2.5, that if C is a curve and if $M(C)$ has minimal free resolution

$$0 \rightarrow L_4 \xrightarrow{\sigma_4} L_3 \rightarrow L_2 \rightarrow L_1 \rightarrow L_0 \rightarrow M(C) \rightarrow 0$$

then I_C has minimal free resolution

$$0 \rightarrow L_4 \xrightarrow{(\sigma_4,0)} L_3 \oplus F \rightarrow F_1 \rightarrow I_C \rightarrow 0.$$

In our case we know the minimal free resolution of I_C. We first claim that $F = 0$, i.e. the last two free modules (and the map between them) in the minimal free resolution of I_C exactly coincide with the corresponding ones for $M(C)$. In the proof of Theorem 4.2 we have described the maps in the minimal free resolution of C. Denote by M the matrix describing the map ϕ_2 after choosing canonical bases for the free modules F_i, where we think of the columns of M as second syzygies. We want to show that M does not have a row of zeros. But if it did, this means that there is a first syzygy which does not "contribute" to any second syzygy. This gives a contradiction since the first syzygies correspond to edges of the cell complex X, but every edge of X is in the boundary of a facet of X that corresponds to a second syzygy. (This is a consequence of the inequalities (4.2).)

Thus we know the end of the minimal free resolution for $M(C)$ has the form

$$0 \to R^{\beta_3}(-a_1 - a_6 - 2) \xrightarrow{\phi_2} R^{\beta_2}(-a_1 - a_6 - 1) \to \dots,$$

and as a result the minimal free resolution for the dual module (over the field K) $M(C)^\vee$ has the form

$$\cdots \to \mathbb{F} \to R^{\beta_2}(a_1 + a_6 - 3) \xrightarrow{\phi_2^\vee} R^{\beta_3}(a_1 + a_6 - 2) \to M(C)^\vee \to 0.$$

It is a basic property of minimal free resolutions that $\mathbb{F}$ has no summand of the form $R(a_1 + a_6 - 3)$. Hence $\dim M(C)^\vee_{-a_1-a_6+3} = 4\beta_3 - \beta_2$, so (5.1) gives $h^3(\mathcal{K}(a_1 + a_6 - 3)) = 0$ as desired. □

Corollary 5.2. *Assume that $a_6 = \max\{a_1, \dots, a_6\}$. Then the degree of an S-minimal curve of type $(a_1, \dots, a_6)$ is $\sum_{i=1}^6 \binom{a_i+1}{2}$, and the arithmetic genus is*

$$\left[\sum_{i=1}^6 \binom{a_i + 1}{2}\right] (a_1 + a_6 - 1) + 1 - \binom{a_1 + a_6 + 2}{3}.$$

Proof. The degree statement is obvious. As for the arithmetic genus, let us denote it by g. It follows easily from the calculations in the proof of Theorem 5.1 that

$$h^0(\mathcal{I}_C(a_1 + a_6 - 1)) = h^1(\mathcal{I}_C(a_1 + a_6 - 1)) = h^2(\mathcal{I}_C(a_1 + a_2 - 1)) = 0.$$

From the cohomology of the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0$$

(twisted by $a_1 + a_6 - 1$) it follows that

$$h^0(\mathcal{O}_C(a_1 + a_6 - 1)) = \binom{a_1 + a_6 + 2}{3}.$$

We also have $h^1(\mathcal{O}_C(t)) = h^2(\mathcal{I}_C(t))$ for all t. But the Riemann-Roch theorem gives that

$$h^0(\mathcal{O}_C(a_1 + a_6 - 1)) = (\deg C)(a_1 + a_6 - 1) - g + 1 + h^1(\mathcal{O}_C(a_1 + a_6 - 1)).$$

Combining the above calculations gives the result. □

The following theorem can be viewed as a clarification of Schwartau's theorem (see Theorem 2.4 and Remark 2.5) and its proof is new and uses the methods of our paper. The main tool is Lemma 3.8, but we have put it after Theorem 5.1 because we use the fact that an S-minimal curve is not ACM, and this follows from the fact that a curve that is minimal in its even liaison class is not ACM.

Theorem 5.3 (invariant Schwartau). *The ideal*

$$(a,b)^{a_1} \cap (a,c)^0 \cap (a,d)^{a_3} \cap (b,c)^{a_4} \cap (b,d)^0 \cap (c,d)^{a_6}$$

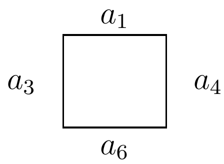
defines an arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$ *if and only if*

Case 1. $a_1, a_3, a_4, a_6 > 0$: $\quad a_1 + a_6 = a_3 + a_4 + \epsilon$, *where* $\epsilon \in \{-1, 0, 1\}$.

Case 2. *Exactly one of* a_1, a_3, a_4, a_6 *is zero, say* a_i: $\quad a_{7-i} + 1 \geq$ *the sum of the weights of the lines meeting the line* l_i.

Case 3. *At least two of* a_1, a_3, a_4, a_6 *are zero:* *the curve is connected.*

Proof. For convenience we will call a curve of the form given in Theorem 5.3 a *Schwartau curve.* If C is a Schwartau curve then its components can be represented by a square:



If three of the integers a_i are zero then the ideal is a power of a complete intersection, and is automatically ACM and connected. Suppose that two of the integers a_i are zero. If C is ACM then it is a general fact that it must be connected. Conversely, suppose that C is connected. This means that the two non-zero values of a_i represent adjacent sides of the square. Hence the ideal involves only three variables. The fourth variable is thus a non zero-divisor, and reducing modulo this variable gives the saturated ideal of a zeroscheme in $\mathbb{P}^2$. Hence in this case the Schwartau curve is ACM. This proves Case 3.

Now suppose that one of the integers is zero, say a_i. (Note that Schwartau took $a_3 = 0$.) As long as C is not S-minimal, we can reduce any maximal facet, by Lemma 3.8. But with $a_i = 0$, there remain only two facets, whose weights are the sums of two consecutive sides of the square. (For example, if $a_3 = 0$ then the weights of the facets are $a_4 + a_6$ and $a_1 + a_4$.)

If C is ACM then no reduction will ever be S-minimal. We repeatedly reduce the facet of maximal weight until we obtain another zero for the weight of an edge, and the fact that C is ACM means that after obtaining this zero, the remaining two edges must be connected (by Case 3 above). This means that the "middle" of the three non-zero edges, a_{7-i}, must not be the first to reach 0 unless there is a tie. But this middle edge is involved in every reduction! Note that since we are always reducing the facet of maximal weight, we eventually reach the point where the weights of the non-middle edges differ by at most 1. It follows that $a_{7-i} + 1 \geq$ the sum of the weights of the two "non-middle" edges, as claimed.

Conversely, suppose that $a_{7-i} + 1 \geq$ the sum of the weights of the two "non-middle" edges. Then clearly a_{7-i} is the edge of maximal weight. The condition (b) of Lemma 3.8 is trivial to check, so C is not minimal and we can reduce the facet of maximal weight. In doing so, both sides of the inequality in Case 2 are reduced by 1, so the inequality still holds for the new curve. Again, reducing the facet of maximal weight each time eventually leads to a balance in the two non-middle edges, and the given inequality guarantees that the curve resulting when a second 0 is obtained, will be connected, hence ACM. So C is ACM.

We now consider the case where none of the a_i are 0. Again without loss of generality assume that $a_6 = \max\{a_1, a_3, a_4, a_6\}$. One can check that then $w = \max\{a_3 + a_6, a_4 + a_6\}$. It follows that when we reduce a facet of maximal weight (if it is possible), we simultaneously reduce $\max\{a_1, a_6\}$ and $\max\{a_3, a_4\}$ by 1. In particular, we simultaneously reduce $(a_1 + a_6)$ by 1 and $(a_3 + a_4)$ by 1.

If C is ACM then it is not S-minimal, so by Lemma 3.8 we have

$$a_1 + a_6 \leq \max\{a_3 + a_6, a_4 + a_6\}, \text{ i.e. } a_1 \leq \max\{a_3, a_4\}.$$

We conclude that if C is ACM then the largest two weights are on consecutive edges. That is, the facet of maximal weight is one of the two facets involving a_6, and we can reduce that facet. Since C is ACM, we will never reach an S-minimal curve in this process, and the result of performing a reduction will again be an ACM curve. Hence after each step in the reduction, the two edges of largest weight are consecutive and each can be reduced by 1 by a new reduction.

Suppose that we have reached the point where the application of Proposition 3.1 reduces an a_i to zero. If two edges are reduced to 0 simultaneously, the remaining two edges are consecutive and have weight 1 each (at this point). So because of our procedure, it follows that $a_1 + a_6 = a_3 + a_4$.

If one edge is reduced to 0, suppose that it is supported on ℓ_i. Then at this point ℓ_{7-i} has weight 1, and the resulting curve is ACM. But ℓ_{7-i} is then the middle edge, so by Case 2 we have the other two edges having weight 1 each. It follows that $a_1 + a_6 = a_3 + a_4 + \epsilon$ as claimed.

Conversely, assume that $a_1 + a_6 = a_3 + a_4 + \epsilon$ for $\epsilon = -1, 0, 1$. We want to show that a reduction can be performed, i.e. C is not minimal. Without loss of generality we still assume that a_6 is the largest weight. We claim that $a_1 \leq \max\{a_3, a_4\}$. Indeed, if $a_6 \geq a_1 > \max\{a_3, a_4\}$ then $a_1 + a_6 \geq a_3 + a_4 + 2$, contradicting the hypothesis. But then if w is the maximal weight of a facet, we have $w = a_6 + \max\{a_3, a_4\}$ and $a_1 + a_6 \leq w$. Therefore, by Lemma 3.8, a reduction can be performed.

Hence our numerical assumption guarantees that at least as long as the four integers are positive, we can always perform a reduction, reducing the larger of a_1 and a_6 by 1 and the larger of a_3 and a_4 by 1. Suppose that a_3 is the first to be reduced to zero. Then at that point $a_4 = 1$ and $a_1 + a_6$ is either 1 or 2. Either way, we are in the previously studied cases and C is ACM. □

We can also characterize the S-minimal arithmetically Buchsbaum curves.

Corollary 5.4. *Let C be an S-minimal tetrahedral curve defined by $(a_1, a_2, a_3, a_4, a_5, a_6)$. Then C is arithmetically Buchsbaum and not arithmetically Cohen-Macaulay if and only if there are integers $i \neq j$ in $\{1, 2, 3\}$ such that*

$$a_i + 1 = a_j = a_{7-j} = a_{7-i} + 1$$

and the two remaining weights are zero.

Proof. According to Corollary 4.3 we may assume that $a_6 = \max\{a_i\} > 0$. We will use the notation of Theorem 4.2 and its proof. Since C is minimal, a result of Martin-Deschamps and Perrin says that the transpose of the matrix describing the map φ_2 is a minimal presentation matrix of $\operatorname{Ext}^3(R/I, R)$. (See [19] Theorem 2.5 and [11] Proposition IV.4.4.) We also saw this directly, in the case of tetrahedral curves, in the proof above. By Corollary 4.8, C is arithmetically Buchsbaum if and only if the latter module is a (shifted) direct sum of copies of K. Using the cell complex X that governs the minimal free resolution of C, this is equivalent to the fact that every edge of X belongs to the boundary of exactly one facet. Now the claim follows from the description of X. Indeed, after removing diagonals from the corners of (4.1) and translating to the notation of the cell complex X, we must be left with a diagonal of facets and no additional edges, and this can only happen if we only remove diagonals from two opposite corners of (4.1) in the prescribed way. □

Remark 5.5. (i) The arguments in the proof of Theorem 5.1, coupled with a recent result of Strano [22], Theorem 1, show the following fact:

If a non-arithmetically Cohen-Macaulay curve $C \subset \mathbb{P}^3$ has a linear resolution then it is minimal in its even liaison class if and only of $s(C) \geq e(C) + 4$.

(ii) There are tetrahedral curves that have a linear resolution, but are not minimal in their even liaison class. An example is given by the curve with weights $(5, 1, 3, 2, 2, 5)$. This shows that our task in the proof of Theorem 5.1, of showing that $h^2(\mathcal{I}_C(a_1 + a_6 - 3)) = 0$, cannot be shown merely by invoking the linearity of the resolution. Indeed, this only guarantees $h^2(\mathcal{I}_C(a_1 + a_6 - 2)) = 0$.

(iii) Sometimes a non-trivial S-minimal curve is the unique minimal curve in its even liaison class. This is true if $s(C) \geq e(C) + 5$ (thanks to [10]). Examples of such curves have weights $(m, 0, 0, 0, 0, k)$ where $m, k \geq 2$.

6. Unobstructedness of some curves

We will show that some of the minimal tetrahedral curves correspond to smooth points of a component of the corresponding Hilbert scheme that has the "expected" dimension.

One starting point is the following result whose proof is an adaptation of Dolcetti's method in [7]. Note that if d_1 is the degree of the first non-zero component of a graded module, M, of finite length and d_2 is the degree of the last non-zero component of M, then the *diameter* of M is $d_2 - d_1 + 1$.

Proposition 6.1. *Let $C \subset \mathbb{P}^3$ be a curve with a linear resolution. If the diameter of its Hartshorne-Rao module is at most two then its normal sheaf $\mathcal{N}_C$ satisfies*

$$H^1(\mathcal{N}_C) = 0.$$

Therefore, the corresponding component of the Hilbert scheme is generically smooth of dimension $4 \cdot \deg C$.

Proof. By assumption the minimal free resolution of C is of the form

$$0 \to R^{\beta_3}(-s-2) \to R^{\beta_2}(-s-1) \to R^{\beta_1}(-s) \to I_C \to 0.$$

It implies

$$H^1(\mathcal{I}_C(s-2)) \neq 0 \quad \text{and} \quad H^1(\mathcal{I}_C(j)) = 0 \ \text{ if } j \geq s-1$$

as well as

$$H^2(\mathcal{I}_C(j)) = 0 \quad \text{for all } j \geq s-2.$$

Thus, the assumption on the diameter provides

$$H^1(\mathcal{I}_C(j)) = 0 \quad \text{for all } j \leq s-4.$$

Sheafifying the resolution above we define the vector bundle $\mathcal{E}$ as follows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{\beta_3}(-s-2) \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{\beta_2}(-s-1) \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^3}^{\beta_1}(-s) \arrow[r] & \mathcal{I}_C \arrow[r] & 0 \\
& & & \mathcal{E} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Thus, we have

$$H^1_*(\mathcal{E}) = 0 \quad \text{and} \quad H^2_*(\mathcal{E}) = H^1_*(\mathcal{I}_C).$$

Furthermore, dualizing the exact sequence on the left-hand side and then tensoring by $\mathcal{E}$ we get the exact sequence

$$0 \to \mathcal{E}^* \otimes \mathcal{E} \to \mathcal{E}^{\beta_2}(s+1) \to \mathcal{E}^{\beta_3}(s+2) \to 0.$$

Taking cohomology we obtain

$$H^2(\mathcal{E}^* \otimes \mathcal{E}) = 0.$$

Now, the cohomology of the exact sequence

$$0 \to \mathcal{E}^* \otimes \mathcal{E} \to (\mathcal{E}^*)^{\beta_1}(-s) \to \mathcal{E}^* \otimes \mathcal{I}_C \to 0$$

provides

$$(H^1(\mathcal{E}^*(-s)))^{\beta_1} \to H^1(\mathcal{E}^* \otimes \mathcal{I}_C) \to H^2(\mathcal{E}^* \otimes \mathcal{E}) = 0.$$

Since we have by duality

$$H^1(\mathcal{E}^*(-s)) \cong [\operatorname{Ext}(R/I_C, R)]_{-s} \cong H^1(\mathcal{I}_C(s-4)) = 0$$

we conclude

$$H^1(\mathcal{E}^* \otimes \mathcal{I}_C) = 0.$$

The sequence defining $\mathcal{E}$ also gives the exact sequence

$$\operatorname{Ext}^1(\mathcal{E}, \mathcal{I}_C) \to \operatorname{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) \to \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}^{\beta_1}(-s), \mathcal{I}_C).$$

Using

$$\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}^{\beta_1}(-s), \mathcal{I}_C) \cong (H^2(\mathcal{I}_C(s)))^{\beta_1} = 0$$

and

$$\operatorname{Ext}^1(\mathcal{E}, \mathcal{I}_C) \cong H^1(\mathcal{E}^* \otimes \mathcal{I}_C) = 0$$

we get

$$0 = \operatorname{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) \cong H^1(\mathcal{N}_C),$$

as claimed. □

Since minimal tetrahedral curves have a linear resolution we would like to know the ones whose Hartshorne-Rao module has diameter at most two. The curves with diameter one have already been characterized in Corollary 5.4. The diameter two case is described below:

Lemma 6.2. *Let C be a minimal tetrahedral curve. Then its Hartshorne-Rao module has diameter two if and only if C is isomorphic to one of the curves*

$$(k, k-1, 0, 0, k-1, k+1) \quad \textit{where} \quad k \geq 1, \quad \textit{or}$$
$$(k, k-2, 0, 0, k-1, k) \quad \textit{where} \quad k \geq 2.$$

Proof. Let C be the minimal curve defined by the tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$. Without loss of generality assume that a_6 is the largest entry. We will use the description of the minimal free resolution of C given in Theorem 4.2. Denote by ψ the dual of the last map in this resolution, i.e. $\psi = \varphi_2^*$. Thus, we have the minimal presentation

$$\longrightarrow R^{\beta_3}(a_1 + a_6 + 1) \xrightarrow{\psi} R^{\beta_2}(a_1 + a_6 + 2) \longrightarrow \operatorname{Ext}^3(R/I_C, R) \longrightarrow 0$$

Since $\operatorname{Ext}^3(R/I_C, R)$ is the K-dual of the Hartshorne-Rao module M_C of C, the diameter of the latter is at most two if and only if

$$[\operatorname{Ext}^3(R/I_C, R)]_{-a_1-a_6} = 0. \tag{6.1}$$

Denote by $X := X(a_1, a_2, a_3, a_4, a_5, a_6)$ the cell complex of the curve C. Recall from the proof of Theorem 4.2 that the facets of X correspond to minimal generators of $\operatorname{Ext}^3(R/I_C, R)$ and that $\operatorname{im}\psi$ has a system of minimal generators consisting of binomials and monomials (cf. formula (4.8)). These generators correspond to the edges of X. Below, we will always refer to this system of generators. Moreover, we will use Diagram (4.7).

We begin with deriving necessary conditions for Condition (6.1) being true. In order to rule out curves we show two claims.

Claim 1: If X contains "3 facets in a row" then the diameter of M_C is greater than two.

Here the assumption means that X contains a subcomplex that is isomorphic to $X(3, 0, 0, 0, 0, 1)$ (a "vertical row") or $X(1, 0, 0, 0, 0, 3)$ (a "horizontal row").

Now, assume that X contains 3 facets in a horizontal row. Denote by e_1, e_2, e_3 these facets from left to right. Then, the only minimal generators of $\operatorname{im}\psi$ involving ce_1, ce_2, de_2, or de_3 are (up to sign)

$$-ce_1 + de_2 \quad \text{and} \quad -ce_2 + de_3.$$

It follows easily that all the monomials c^2e_1, cde_2, d^2e_3 do not belong to $\operatorname{im}\psi$. Thus, condition (6.1) is not satisfied.

If X contains 3 facets in a vertical row we argue similarly and Claim 1 is shown.

Claim 2: If X contains a "square of 4 facets" then the diameter of M_C is greater than two.

More precisely, the assumption means that X contains a subcomplex that is isomorphic to $X(2, 0, 0, 0, 0, 2)$. Enumerate its facets counterclockwise by $e_1, \dots, e_4$ beginning with e_1 in the "Southwest" corner. Then, the only minimal generators of $\operatorname{im}\psi$ involving $be_1, ce_1, be_2, de_2, ae_3, de_3, ae_4$, or ce_4 are (up to sign)

$$-ce_1 + de_2, \quad -be_2 + ae_3$$
$$-ce_4 + de_3, \quad -be_1 + ae_4.$$

Hence, all the monomials $bce_1, bde_2, ade_3, ace_4$ are not in $\operatorname{im}\psi$. Thus, condition (6.1) is not satisfied and Claim 2 is established.

As the next step we look for the minimal tetrahedral curves whose cell complex contains neither 3 facets in a row nor a square of 4 facets. Let C be such a curve. Looking at the boundaries of its cell complex X we get in conjunction with Corollary 3.5

$$\begin{aligned} 1 \leq a_6 - a_4 - a_5 \leq 2 \\ 1 \leq a_6 - a_2 - a_3 \leq 2 \\ 1 \leq a_1 - a_3 - a_5 \leq 2 \\ 1 \leq a_1 - a_2 - a_4 \leq 2. \end{aligned}$$

Now, we look at a horizontal or vertical boundary of X a bit more carefully. If there are true "cuts" on both ends then the row of X next to it has length at least 3, a contradiction. Thus, we may assume without loss of generality that

$$a_3 = 0.$$

Next, we distinguish two cases.

Case 1: Assume $a_2 = 0$. Then we get $a_6 \leq 2$ and it is easy to see that C can be any curve satisfying these conditions except the curve defined by $(2, 0, 0, 0, 0, 2)$.

Case 2: Assume $a_2 > 0$. Using the argument above for the Eastern boundary of X we conclude

$$a_4 = 0.$$

We are left with two possibilities for the value of a_5.

Case 2.1: Assume $a_5 = a_6 - 1$. Since $a_5 < a_1 \leq a_6$ we obtain $a_1 = a_6$. We conclude that either $a_2 = a_6 - 1$, i.e. C is an arithmetically Buchsbaum curve of type

$$(k, k-1, 0, 0, k-1, k),$$

or $a_2 = a_6 - 2$, i.e. C is of type

$$(k, k-2, 0, 0, k-1, k).$$

Case 2.2: Assume $a_5 = a_6 - 2$. Since the length of the row next to the Northern boundary of X is at most 3 we get $a_2 = a_1 - 1$. It follows that either $a_1 = a_6$, i.e. C is isomorphic to the second kind of curve in Case 2.1, or $a_1 = a_6 - 1$, i.e. C is of type

$$(k, k-1, 0, 0, k-1, k+1).$$

Summing up. we have shown that there are at most three types of minimal tetrahedral curves satisfying Condition (6.1). Since we already know all the curves with diameter one (by Corollary 4.8), it remains to show that the diameter of the curves specified in the statement is at most two.

Let C be the curve defined by

$$(k, k-1, 0, 0, k-1, k+1).$$

Enumerate the facets of the cell complex X of C by $e_1, \ldots, e_{2k}$ such that e_1 is the facet in the Southeast corner and the facets e_i and e_{i+1} are neighbours. This enumeration exists and is uniquely determined due to the shape of X. We will show that

$$(a, b, c, d)^2 \cdot e_i \in \operatorname{im} \psi \quad \text{for all } i = 1, \ldots, 2k. \tag{6.2}$$

First, we consider e_1. Since $(a, b, d) \cdot e_1 \in \operatorname{im} \psi$ it suffices to show $c^2 e_1 \in \operatorname{im} \psi$. But this follows easily using

$$c \cdot (-ce_1 + de_2) \in \operatorname{im} \psi$$

and

$$ce_2 \in \operatorname{im} \psi.$$

Next, we consider e_2. Since also $ae_2 \in \operatorname{im} \psi$ it remains to show $(b, d)^2 \cdot e_2 \in \operatorname{im} \psi$. Using the minimal generators $-ce_1 + de_2, be_1, de_1$ we see that

$$(bd, d^2)e_2 \in \operatorname{im} \psi.$$

We also have

$$b^2 e_2 \in \operatorname{im} \psi$$

because $-be_2 + ae_3, be_3 \in \operatorname{im} \psi$.

Continuing similarly in this fashion we can show the relations (6.2) that imply Condition (6.1). This completes the argument. $\square$

Corollary 6.3. *Let C be a curve that is isomorphic to one of the curves*

$$\begin{aligned} &(k, k-1, 0, 0, k-1, k) \quad \textit{where} \quad k \geq 1, \\ &(k, k-1, 0, 0, k-1, k+1) \quad \textit{where} \quad k \geq 1, \quad \textit{or} \\ &(k, k-2, 0, 0, k-1, k) \quad \textit{where} \quad k \geq 2. \end{aligned}$$

Then C is unobstructed and the corresponding component of the Hilbert scheme is generically smooth and has dimension $4 \cdot \deg C$.

Proof. This follows by combining Proposition 6.1, Lemma 6.2, and Theorem 4.2. $\square$

Remark 6.4. It was proved in [4] that if C is a curve in $\mathbb{P}^3$ (not necessarily tetrahedral) with *natural cohomology* and if the diameter of $M(C)$ is two, then C is unobstructed and its Hilbert scheme has the expected dimension. "Natural cohomology" means that for any t, no two of $h^0(\mathcal{I}_C(t))$, $h^1(\mathcal{I}_C(t))$ and $h^2(\mathcal{I}_C(t))$ are non-zero. Note that minimal tetrahedral curves whose Hartshorne-Rao modules have diameter two have natural cohomology.

Note also that our Proposition 6.1, while similar, is independent of that result. For example, a non-minimal tetrahedral curve with linear resolution (e.g. $(6, 5, 1, 0, 4, 6)$) does not have natural cohomology, but is unobstructed by Proposition 6.1. On the other hand, a minimal arithmetically Buchsbaum curve C with $M(C)$ of diameter two does not have a linear resolution (since the dual module would then be generated in more than one degree), but it does have natural cohomology (cf. [3], for instance Corollary 2.5 with $t = 2$ and $h = 0$) and so is unobstructed.

We also remark that Miró-Roig has given sufficient conditions on the numerical character of an *irreducible* arithmetically Buchsbaum curve C of maximal rank with $M(C)$ of diameter one or two, for C to be unobstructed (cf. [17]).

The condition on the diameter of the minimal tetrahedral curves in Proposition 6.1 is sufficient for unobstructedness, but not necessary. This is shown by the curves $(a_1, 0, 0, 0, 0, a_6)$ because their diameter is $a_1 + a_6 - 1$, but all these curves are unobstructed. This follows from the following result.

Proposition 6.5. *The global sections of the normal sheaf of the curve C defined by $(a_1, 0, 0, 0, 0, a_6)$ are given by*

$$H^0_*(\mathcal{N}_C) \cong (R/(a,b))^{a_1(a_1+1)}(1) \oplus (R/(c,d))^{a_6(a_6+1)}(1).$$

Before turning to the proof of this statement we record its announced consequence.

Corollary 6.6. *Let $C \subset \mathbb{P}^3$ be the curve defined by $(a_1, 0, 0, 0, 0, a_6)$. Then its normal sheaf satisfies*

$$H^1(\mathcal{N}_C(-2)) = 0.$$

Therefore, the corresponding component of the Hilbert scheme is generically smooth of dimension $4 \cdot \deg C$.

Proof. The claim easily follows by Proposition 6.5 and duality. □

The proof of Proposition 6.5 is based on the computation of the normal module of infinitesimal neighbourhoods of a line. Recall that the normal module of the curve C is

$$N_C = \operatorname{Hom}(I_C, R/I_C).$$

Lemma 6.7. *For any positive integer k there is a graded isomorphism*

$$\operatorname{Hom}((a,b)^k, R/(a,b)^k) \cong (R/(a,b))^{k(k+1)}(1).$$

Proof. Put $I = (a,b)^k$ and $A = R/I$. The minimal free resolution of I is well-known

$$0 \to R^k(-k-1) \to R^{k+1}(-k) \to I \to 0.$$

Dualizing it with respect to R provides the minimal free resolution of the canonical module K_A of A

$$0 \to R \to R^{k+1}(k) \to R^k(k+1) \to K_A(4) \to 0.$$

Dualizing with respect to A we get the exact sequence

$$0 \to \operatorname{Hom}(I,A) \to A^{k+1}(k) \xrightarrow{\psi} A^k(k+1) \to \operatorname{Ext}^1(I,A) \to 0.$$

Thus, we see in particular that $\operatorname{Ext}^1(I,A) \cong K_A \otimes A(4)$ which allows to compute its Hilbert function. Using the last exact sequence above we obtain

$$\operatorname{rank}_K[\operatorname{Hom}(I,A)]_j = \begin{cases} 0 & \text{if } j \le -2 \\ k(k+1)(j+2) & \text{if } j \ge -1 \end{cases}$$

Now, let $\{e_1, \ldots, e_{k+1}\}$ be the canonical basis of the free A-module A^{k+1}. Using $(a,b) \cdot (a,b)^{k-1} = I$ we see that

$$(a,b)^{k-1} e_i \subset [\ker \psi]_{-1} \quad \text{for all } i = 1, \ldots, k+1.$$

Thus, the R-module M that is generated by

$$G := \{a^j b^{k-1-j} e_i \mid j = 0, \ldots, k-1,\ i = 1, \ldots, k+1\}$$

is a submodule of $\operatorname{Hom}(I,A)$. It is not too difficult to see that the minimal generators of M are annihilated by the ideal (a,b) and, thus, G is a basis of the free $R/(a,b)$-module M. Comparing Hilbert functions we conclude that $M = \operatorname{Hom}(I,A)$ completing the proof. □

We are ready for the proof of Proposition 6.5.

Proof of Proposition 6.5. Let C_1, C_2 be the curves defined by $(a,b)^{a_1}$ and $(c,d)^{a_6}$, respectively. Using the exact sequence

$$0 \to I_C \to I_{C_1} \oplus I_{C_2} \to I_{C_1} + I_{C_2} \to 0$$

we get

$$H^0_*(\mathcal{O}_C) \cong H^0_*(\mathcal{O}_{C_1}) \oplus H^0_*(\mathcal{O}_{C_2}).$$

Since $H^0_*(\mathcal{N}_C) \cong \operatorname{Hom}(I_C, H^0_*(\mathcal{O}_C))$ it is not too difficult to see that the claim follows by Lemma 6.7. □

7. Remarks and problems

In this section we collect some observations that do not quite fit into earlier sections of this paper, as well as some natural questions that arise from this work. The list of questions is rather long, highlighting the richness of this line of inquiry. The authors plan to continue investigating these questions.

Remark 7.1. It is natural to ask, among all tetrahedral curves, "how many" are minimal in their even liaison class. That is, can we describe the density of minimal tetrahedral curves among all tetrahedral curves? The first answer is "almost none," since of course every such even liaison class has infinitely many tetrahedral curves, but essentially one minimal one (but see question 1. below). But paradoxically, we can take a different point of view that shows that there are more than one might think.

We want to investigate how many minimal tetrahedral curves are in a finite set of tetrahedral curves. To this end it seems reasonable to fix the maximum entry of the 6-tuple. After a change of coordinates we may assume that this entry is the last one. Then we get:

Lemma 7.2. *The number of minimal tetrahedral curves $(a_1, \dots, a_6)$ such that $a_6 = max\{a_1, \dots, a_6\}$ is*

$$N(a_6) := \sum_{a_1=0}^{a_6} \sum_{a_2=0}^{a_1-1} \sum_{a_5=0}^{a_1-1} \min\{a_1 - a_5, a_6 - a_2\} \cdot \min\{a_1 - a_2, a_6 - a_5\}.$$

Proof. This is a consequence of Corollary 3.5. If $(a_1, \dots, a_6)$ is minimal then we have $0 \le a_1 \le a_6$ and $0 \le a_2, a_5 < a_1$. Moreover, having chosen a_1, a_2, a_5, the only condition for a_3 and a_4 is $0 \le a_3 < \min\{a_1 - a_5, a_6 - a_2\}$ and $0 \le a_4 < \min\{a_1 - a_2, a_6 - a_5\}$, respectively. The claim follows. □

The number of tetrahedral curves with fixed $a_6 = \max\{a_1, \dots, a_6\}$ is $(a_6 + 1)^5$. The lemma shows that the number of minimal curves among them is also of order a_6^5. To see this it suffices to find a lower estimate of the number $N(a_6)$. Since both minima in the formula in Lemma 7.2 are at least $a_1 - \min\{a_2, a_5\}$ we get

$$\begin{aligned} N(a_6) &\geq \sum_{a_1=0}^{a_6} \sum_{a_2=0}^{a_1-1} \sum_{a_5=0}^{a_1-1} [a_1 - \min\{a_2, a_5\}]^2 \\ &= \sum_{a_1=0}^{a_6} \sum_{a_2=0}^{a_1-1} \left[\sum_{a_5=0}^{a_2-1} (a_1 - a_5)^2 + \sum_{a_5=a_2}^{a_1-1} (a_1 - a_2)^2 \right] \\ &= \sum_{a_1=0}^{a_6} \sum_{a_2=0}^{a_1-1} \left[\sum_{k=a_1-a_2+1}^{a_1} k^2 \ + \ (a_1 - a_2)^3 \right] \end{aligned}$$

Now it is easy to see that this lower estimate of $N(a_6)$ is a polynomial in a_6 of order 5. In other words, we get the somewhat surprising result: The probability to find a minimal curve among the set of tetrahedral curves $(a_1, \ldots, a_6)$ such that $a_6 = \max\{a_1, \ldots, a_6\}$ is positive, even as $a_6 \to \infty$.

Remark 7.3. The results in this paper can be extended as follows. Let (F_1, F_2, F_3, F_4) be a regular sequence. Define curves by $I_{C_1} = (F_1, F_2), I_{C_2} = (F_1, F_3), \ldots, I_{C_6} = (F_3, F_4)$, and let $I_C = I_{C_1} \cap \cdots \cap I_{C_6}$. Taking $(F_1, F_2, F_3, F_4) = (a, b, c, d)$ gives the context of this paper. As before, we denote by $(a_1, a_2, a_3, a_4, a_5, a_6)$ the ideal I_C. Note that no two of the curves C_i can have a common component.

Many of our results extend to this situation. The key observation is that we still have

$$F_i \cdot (F_i, F_j)^{n-1} + (F_j^n) = (F_i, F_j)^n,$$

as we noted at the beginning of the proof of Proposition 3.1. Then Proposition 3.1 and Corollary 3.5 (and indeed virtually all of section 3) are still true, as stated, in this more general context. Turning to Section 4, Theorem 4.2 is clearly not true as stated. If the F_i all have the same degree, it is fairly easy to modify the statement, and in fact the linearity of the resolution is preserved. If the F_i have different degrees, it can still be modified, but the resolution is no longer linear. The

other results are less obviously generalizable to this context.

We end with some open questions.

Question 7.4. (1) Fix an even liaison class containing tetrahedral curves. We know that among the minimal curves there is at least one that is tetrahedral.

tetrahedral? This is not the same as asking when minimal

Are the minimal tetrahedral unique up to isomorphism? It is clear that at the very integer $a_1 + a_6$ is an invariant among minimal curves in even tetrahedral curves be minimal in the curves within an even liaison class?

When is a minimal tetrahedral curve the *unique* minimal curve in the even liaison class (see Remark 5.5, (iii))? Note that sometimes two curves that are projectively equivalent are linked, and sometimes they are not!

(2) Is it possible to characterize the tetrahedral curves with linear resolutions? Recall that this set of curves contains more than just the minimal tetrahedral curves (Remark 5.5 (ii)). See Remark 4.5 for a description of how to control the information

about the resolution.

(3) We have seen that several of the minimal tetrahedral curves are unobstructed. Computer experiments show that there are more of them than the ones described above. Are all minimal tetrahedral curves unobstructed? Are all tetrahedral curves with linear resolution unobstructed? Are all tetrahedral curves unobstructed?

(4) How "dense" are our minimal curves that give rise to nice components of the Hilbert scheme $H_{d,g}$, i.e. components that are generically smooth of dimension $4d$? In other words: How big are the "gaps" in the sets of pairs (d, g) such we cannot find such a minimal curve with that degree and genus?

(5) Can the tetrahedral curves in $\mathbb{P}^3$ that are arithmetically Cohen-Macaulay be identified by explicitly giving the 6-tuples (as Schwartau does for 4-tuples)?

(6) Is there a combinatorial description of the minimal free resolution of the deficiency module of a tetrahedral curve? Can we at least express the dimensions of the components using the entries of the 6-tuple?

(7) Can the same kind of program be carried out in higher projective space? Now the new question of local Cohen-Macaulayness arises. This includes codimension two cases and higher codimension cases. The former can still rely on complete intersection liaison techniques, but the latter will require Gorenstein liaison techniques.

(8) On a computer algebra program such as `macaulay` [1] one can make experiments, and one notices a great deal of structure in the Betti diagram. In particular, non-minimal tetrahedral curves have linear strands in the resolution. Certainly one can follow the development of these linear strands via basic double links (see Remark 4.5), but it should be done in a systematic way. Can one relate the "degree of non-minimality" to the number of linear strands? Can we describe the entire Betti diagram purely in terms of the exponents $a_1, \ldots, a_6$?

(9) Is it possible to use similar techniques to study more general monomial ideals?

Appendix: The algorithm to find S-minimal curves

Below we record a crude MAPLE implementation of Algorithm 3.9. It is a MAPLE work sheet that can be downloaded at

http://www.ms.uky.edu/∼uwenagel/.

```
>   # algorithm to compute the S-minimal curve of a tetrahedral curve
    # Input: weight vector of the given curve

>   smin := proc(aa,bb,c,d,e,f)
```

```
    local a, b, m, i, j, k, s, t, w, W, x, y, z, output1, output2;

    a := [aa,bb,c,d,e,f];
    b:= a; s := 0;

    if (s > -1) then do

    # 1. computation of the maximal weight m of a line
    m := a[1]; j := 1;
    for i from 2 to 6 do
                         if (a[i]  > m) then m := a[i];
                                             j := i;
                         end if;
                      od;

    # 2. avoid unnecessary computations
    if (m = 0) then
         output1 := matrix(1,6,[[‘ ‘,‘Minimal curve to‘, b, ‘is‘, a,‘ ‘]]):
         output2:= matrix(1,3,[[‘It is obtained after‘, s,‘
reduction(s).‘]]):
         print(output1); print(output2);
         return(a);
    end if;

    # 3. computation of facet of maximal weight W
    x := a[1] + a[2] + a[3];
    y := a[1] + a[4] + a[5];
    z := a[2] + a[4] + a[6];
    t := a[3] + a[5] + a[6];

    w := [x, y, z, t];

    W:= w[1]; k := 1;

    for i from 2 to 4 do
                         if (w[i]  > W) then W := w[i];
                                             k := i;
                         end if;
                      od;


    # 4. test if curve is S-minimal
    if (a[j] + a[7-j] > W) then
       output1 := matrix(1,6,[[‘ ‘,‘Minimal curve to‘, b, ‘is‘, a,‘ ‘]]):
```

```
      output2:= matrix(1,3,[[‘It is obtained after‘, s,‘ reduction(s).‘]]):
      print(output1); print(output2);
      return(a);
   end if;

   # 5. reduction of the non-minimal curve
   s := s+1;

   if (k = 1) then a[1] := max(0, a[1] - 1);
                   a[2] := max(0, a[2] - 1);
                   a[3] := max(0, a[3] - 1);

   end if;

   if (k = 2) then a[1] := max(0, a[1] - 1);
                   a[4] := max(0, a[4] - 1);
                   a[5] := max(0, a[5] - 1);

   end if;

   if (k = 3) then a[2] := max(0, a[2] - 1);
                   a[4] := max(0, a[4] - 1);
                   a[6] := max(0, a[6] - 1);

   end if;

   if (k = 4) then a[3] := max(0, a[3] - 1);
                   a[5] := max(0, a[5] - 1);
                   a[6] := max(0, a[6] - 1);

   end if;

   end do;

   end if;

   end:
```

In order to use the procedure above in a different work sheet one could write it into a text file named, for example, proc.txt. Then it can be used as follows.

```
>  read ‘C:\\suitable path\\proc.txt‘;
>  smin(5,1,3,2,2,5):
```

```
        [  , Minimal curve to , [5, 1, 3, 2, 2, 5] , is ,

        [5, 1, 2, 2, 1, 4] ,  ]


            [It is obtained after    1     reduction(s).]

>   smin(6,0,8,1,0,4):

        [  , Minimal curve to , [6, 0, 8, 1, 0, 4] , is ,

        [0, 0, 0, 0, 0, 0] ,  ]


            [It is obtained after    10     reduction(s).]
```

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