THE WEAK AND STRONG LEFSCHETZ PROPERTIES FOR ARTINIAN K-ALGEBRAS

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ABSTRACT. Let $A = \bigoplus_{i \geq 0} A_i$ be a standard graded Artinian K-algebra, where char $K = 0$. Then A has the Weak Lefschetz property if there is an element ℓ of degree 1 such that the multiplication $\times \ell : A_i \to A_{i+1}$ has maximal rank, for every i, and A has the Strong Lefschetz property if $\times \ell^d : A_i \to A_{i+d}$ has maximal rank for every i and d.

The main results obtained in this paper are the following.

1. *Every* height three complete intersection has the Weak Lefschetz property. (Our method, surprisingly, uses rank two vector bundles on $\mathbb{P}^2$ and the Grauert-Mülich theorem.)
2. We give a complete characterization (including a concrete construction) of the Hilbert functions that can occur for K-algebras with the Weak or Strong Lefschetz property (and the characterization is the same one!).
3. We give a sharp bound on the graded Betti numbers (achieved by our construction) of Artinian K-algebras with the Weak or Strong Lefschetz property and fixed Hilbert function. This bound is again the same for both properties! Some Hilbert functions in fact *force* the algebra to have the maximal Betti numbers.
4. *Every* Artinian ideal in $K[x, y]$ possesses the Strong Lefschetz property. This is false in higher codimension.

1. Introduction

Let A be a graded Artinian algebra over some field K (which we will restrict shortly). Then A has the *Weak Lefschetz property* (sometimes called the Weak Stanley property) if there is an element ℓ of degree 1 such that the multiplication $\times \ell : A_i \to A_{i+1}$ has maximal rank, for every i. We say that A has the *Strong Lefschetz property* if there is an element ℓ of degree 1 such that the multiplication $\times \ell^d : A_i \to A_{i+d}$ has maximal rank for every i and d. If $A = R/I$, where R is a polynomial ring and I is a homogeneous ideal, then sometimes we will abuse notation and refer to the Weak or Strong Lefschetz properties for I rather than for A. These are both fundamental properties and have been studied by many authors, especially when A is Gorenstein (e.g. [4], [13], [15], [16], [19], [24], [26], [27], [28]).

Throughout this paper, unless specified otherwise, we assume that we work over a field of characteristic zero. This paper began with a study of the Weak Lefschetz property for complete intersections of height three, and grew to a study of Artinian ideals of arbitrary codimension. Our original interest in the subject was to try to get a handle on "how many" Artinian complete intersections possess this natural property. However, a further motivation comes from the fact that this property can be translated into (at least) two other natural questions.

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First, suppose that $F_1, F_2, \ldots, F_n$ is a homogeneous complete intersection in the n dimensional polynomial ring R. Then the minimal free resolution of the ideal $(F_1, \ldots, F_n)$ is well understood; namely it is obtained as the Koszul complex. However, the graded Betti numbers of the minimal free resolution of the ideal $(F_1, \ldots, F_n, L)$, where L is a generic linear form, does not seem to be well understood. For example, should they be all the same, depending only on the degrees of the generators and not on the generators themselves, as long as they are a regular sequence of given degrees plus a generic element? (We could also ask the same question for L^d in the place for L.) The connection between the Weak Lefschetz property and this question is discussed in the last part of section 2, and we give a complete answer (Corollary 2.7) when $n = 3$.

One other problem concerns the generic initial ideal, $\mathrm{gin}(I)$, of a complete intersection I, i.e. the initial ideal of I with respect to generic variables (cf. for instance [9]). It is well known that $\mathrm{gin}(I)$ is Borel-fixed. But if I is a complete intersection and if we fix a monomial order, is the Borel-fixed ideal $\mathrm{gin}(I)$ unique? Or are there two complete intersections I and J such that $\mathrm{gin}(I)$ and $\mathrm{gin}(J)$ are different Borel-fixed ideals with the same Hilbert function? These questions seem to be open since if $\mathrm{gin}(I)$ is unique with respect to the reverse lex order then it would imply the Strong Lefschetz property of all complete intersections of those degrees. Since a Borel-fixed ideal is unique in codimension two (for a fixed Hilbert function) the Strong Lefschetz property can be proved in this case (Proposition 4.4).

It should also be mentioned that Stanley and others have made deep connections between the Weak and Strong Lefschetz properties and questions in combinatorics [24], [25]. For example, the Weak Lefschetz property was the crucial ingredient in Stanley's part of the characterization of the f-vectors of simplicial polytopes. Thus, we are exploring in this paper also the restrictions on the possible Hilbert functions and graded Betti numbers imposed by the presence of the Weak or Strong Lefschetz property.

It was noticed by Stanley [25] and independently by the fourth author [27] that any monomial complete intersection (in any number of variables) has the Strong Lefschetz property, and the fourth author proved that in any codimension, "most" Artinian Gorenstein rings with fixed socle degree possess the Strong Lefschetz property ([27], Example 3.9). We remark (following [15]) that Stanley's proof used the idea of recognizing $A = R/I$ as the cohomology ring of a product X of projective spaces, and then using the hard Lefschetz theorem for the algebraic variety X. The fourth author noticed that if follows from the representaion theory of the Lie algebra $sl(2)$.

Yet even in codimension 3, we do not have a clear idea of which Artin Gorenstein rings possess this property, and in particular whether all of them do. The (apparently) simplest situation is for height 3 complete intersections in $R = K[x_1, x_2, x_3]$. Until now the most general result for this case is again due to the fourth author. Suppose that the generators of the complete intersection I have degrees $2 \le d_1 \le d_2 \le d_3$. Then it was proved in [28] that if $d_3 > d_1 + d_2 - 2$ then R/I has the Weak Lefschetz property. But for arbitrary complete intersections, even the case of three polynomials of degree 4 had been open.

The first main result of this paper (Theorem 2.3) is that *all* Artinian complete intersections in $K[x_1, x_2, x_3]$ have the Weak Lefschetz property. It is a somewhat surprising result. Indeed, it was known to be a very difficult problem among the experts, and at times it seemed more natural to seek a counter-example rather than to try to prove it! We are able to give a relatively simple proof by translating the problem to one of vector bundles on $\mathbb{P}^2$ and invoking a deep theorem due to Grauert and Mülich.

This part of the paper was inspired by [28], but as mentioned earlier, our techniques are completely different from those of the papers cited above. Because we apply the Grauert-Mülich theorem, we are forced to assume characteristic zero (as indeed was done in [28]). In fact, the Weak Lefschetz property does not hold for all complete intersections in characteristic p; see Remark 2.9.

As a further illustration of the power of our approach, we give a simple proof (Corollary 2.5) of the main result of [28].

In the third section of the paper we do not assume that char $K = 0$. We consider graded Artinian K-algebras which are not necessarily complete intersections. Here we produce (Construction 3.4) a particular graded Artinian K-algebra, which allows us to give a necessary and sufficient condition for a sequence of integers to be the Hilbert function of a graded Artinian K-algebra with the Weak Lefschetz property (Proposition 3.5). We also answer several natural questions about the minimal free resolutions of algebras with the Weak Lefschetz property.

Our second main result (Theorem 3.20) shows that if we fix an allowable Hilbert function then there is a sharp upper bound on the graded Betti numbers among K-algebras having the Weak Lefschetz property. Indeed, this bound is achieved by the algebra produced by Construction 3.4, once we refine the construction slightly. This result is analogous to the main result of [19], which proved it for Gorenstein ideals with the Weak Lefschetz property (see also [11]). As a corollary we show that there are Hilbert functions which occur for K-algebras with the Weak Lefschetz property and for which this property forces the graded Betti numbers to be the maximal ones.

In section 4 we again assume char $K = 0$. We consider the Strong Lefschetz property, namely that there exists a linear form ℓ such that for each d, the multiplication $\times \ell^d : A_i \to A_{i+d}$ has maximal rank, for every i. This condition implies the Weak Lefschetz property, but is not equivalent to it in general. We show that these conditions are both automatic in codimension two, however.

Since there are algebras with the Weak Lefschetz property but not the Strong Lefschetz property, one might guess that the imposition of the Strong Lefschetz property reduces the number of possible Hilbert functions. However, we are able to show that with another slight refinement of Construction 3.4, that algebra has the Strong Lefschetz property. This yields the surprising result that a Hilbert function occurs among algebras with the Weak Lefschetz property if and only if it occurs among algebras with the Strong Lefschetz property. Furthermore, the extremal graded Betti numbers for algebras with the Weak Lefschetz property also occur among algebras with the Strong Lefschetz property.

Our results have some consequences for the punctual Hilbert scheme. Since by semicontinuity the Weak and Strong Lefschetz properties are open properties, it follows that the general point of a component has the Strong (resp. Weak) Lefschetz property if and only the component has one point with the Strong (resp. Weak) Lefschetz property. Moreover, we know precisely the possible Hilbert functions of the K-algebras corresponding to such a general point.

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2. The Weak Lefschetz Property for height three complete intersections

Let $R = K[x_1, x_2, x_3]$, where K is a field of characteristic zero. Initially we will assume that K is algebraically closed, in order to freely use the results of [23]. However, we note in Corollary 2.4 and beyond that our results hold without that assumption.

Let I be a complete intersection ideal of R generated by homogeneous elements $F_1, F_2, F_3 \in R$ of degrees d_1, d_2, d_3 respectively, and $d_1 \le d_2 \le d_3$. The minimal free resolution for R/I has the form

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & R(-d_1-d_2-d_3) \arrow[r] & \mathbb{F}_2 \arrow[rr] \arrow[dr] & & \mathbb{F}_1 \arrow[r, "{[F_1,F_2,F_3]}"] & R \arrow[r] & R/I \arrow[r] & 0 \\ & & & E \arrow[ur] \arrow[dr] & & & & \\ & & 0 \arrow[ur] & & 0 & & & \end{tikzcd} \tag{2.1}$$

where $\mathbb{F}_2 = R(-d_2-d_3)\oplus R(-d_1-d_3)\oplus R(-d_1-d_2)$ and $\mathbb{F}_1 = R(-d_1)\oplus R(-d_2)\oplus R(-d_3)$. Sheafifying, we get the following two exact sequences:

$$0 \to \mathcal{E} \to \mathcal{F}_1 \xrightarrow{[F_1,F_2,F_3]} \mathcal{O}_{\mathbb{P}^2} \to 0 \tag{2.2}$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d_1-d_2-d_3) \to \mathcal{F}_2 \to \mathcal{E} \to 0, \tag{2.3}$$

where $\mathcal{E}$ is locally free (since I is Artinian) of rank two, $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_3)$ and $\mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^2}(-d_2-d_3) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_1-d_3) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_1-d_2)$.

We would like a condition which forces $\mathcal{E}$ to be semistable. We first consider the case where $d_1 + d_2 + d_3$ is even. Choose an integer d so that $2d = d_1 + d_2 + d_3$. Notice that $c_1(\mathcal{E}) = -d_1 - d_2 - d_3 = -2d$, so the normalized bundle $\mathcal{E}_{norm}$ is $\mathcal{E}(d)$ (an easy computation, or see for instance [23] page 165). Twisting the sequence (2.3) by $d-1$ we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d-1) \to \begin{array}{c} \mathcal{O}_{\mathbb{P}^2}(-d+d_1-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_2-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_3-1) \end{array} \to \mathcal{E}_{norm}(-1) \to 0. \tag{2.4}$$

We now consider the case where $d_1+d_2+d_3$ is odd. Choose d so that $2d = d_1+d_2+d_3-1$. Then again $\mathcal{E}_{norm} = \mathcal{E}(d)$ (again see [23] page 165). Now we have the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d-1) \to \begin{array}{c} \mathcal{O}_{\mathbb{P}^2}(-d+d_1-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_2-1) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(-d+d_3-1) \end{array} \to \mathcal{E}_{norm} \to 0. \tag{2.5}$$

Lemma 2.1. *Let $\mathcal{E}$ be the rank two locally free sheaf obtained above as the kernel of the map $[F_1, F_2, F_3]$.*

1. *Assume $d_1 + d_2 + d_3$ is even. If $d_3 < d_1 + d_2 + 2$ then $\mathcal{E}$ is semistable.*
2. *Assume $d_1 + d_2 + d_3$ is odd. If $d_3 < d_1 + d_2 + 1$ then $\mathcal{E}$ is semistable.*

Proof. When $c_1(\mathcal{E})$ is even and $\mathcal{E}$ has rank two, we know from [23] Lemma 1.2.5 that $\mathcal{E}$ is semistable if and only if $H^0(\mathbb{P}^n, \mathcal{E}_{norm}(-1)) = 0$ (since it has rank two). When $c_1(\mathcal{E})$ is odd and $\mathcal{E}$ has rank two, stability and semistability coincide ([23] page 166) and the condition for semistability is $H^0(\mathbb{P}^2, \mathcal{E}_{norm}) = 0$.

The two sequences (2.4) and (2.5) are exact on global sections. Hence semistability follows in either case if we have $-d + d_3 - 1 < 0$ (where d changes slightly depending on the parity of $d_1 + d_2 + d_3$). The lemma then follows from a simple computation. □

Let $\lambda \cong \mathbb{P}^1$ be a general line in $\mathbb{P}^2$. Recall that every vector bundle on $\mathbb{P}^1$ splits, so in particular $\mathcal{E}|_\lambda \cong \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2)$. The pair (e_1, e_2) is called the *splitting type* of $\mathcal{E}$.

Corollary 2.2. *Let $\mathcal{E}$ be the locally free sheaf obtained above, and assume that $d_3 < d_1 + d_2 + 1$. Then the splitting type of $\mathcal{E}$ is*

$$(e_1, e_2) = \begin{cases} (-d, -d) & \text{if } d_1 + d_2 + d_3 = 2d; \\ (-d, -d-1) & \text{if } d_1 + d_2 + d_3 - 1 = 2d. \end{cases}$$

Proof. By Lemma 2.1, $\mathcal{E}$ is semistable. The theorem of Grauert and Mülich ([23] page 206, [8], page 68) says that in characteristic zero the splitting type of a semistable normalized 2-bundle $\mathcal{E}_{norm}$ over $\mathbb{P}^n$ is

$$(e_1, e_2) = \begin{cases} (0, 0) & \text{if } c_1(\mathcal{E}_{norm}) = 0; \\ (0, -1) & \text{if } c_1(\mathcal{E}_{norm}) = -1. \end{cases}$$

In our case $\mathcal{E}_{norm} = \mathcal{E}(d)$, so a simple calculation gives the result. □

With this preparation, we now prove the main result of the paper. We continue to assume that K is algebraically closed of characteristic zero.

Theorem 2.3. *Every height three Artinian complete intersection has the Weak Lefschetz Property.*

Proof. It was shown in [28] Corollary 3 that if $d_3 \geq d_1 + d_2 - 3$ then R/I has the Weak Lefschetz property. So without loss of generality assume that $d_3 < d_1 + d_2 - 3$. Note that then Corollary 2.2 applies. To prove the Weak Lefschetz property it is enough to prove injectivity in the "first half," so we will focus on this.

Let L be a general linear form and let $\bar{R} = R/L$. We denote by $\bar{F}$ the restriction of a polynomial F to $\bar{R}$ and by $\bar{\mathbb{F}}_1$ the free $\bar{R}$-module $\bar{R}(-d_1) \oplus \bar{R}(-d_2) \oplus \bar{R}(-d_3)$. Consider the multiplication induced by L. From (2.1) we obtain a commutative diagram

$$(2.6)\qquad \begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & E(-1) & \to & \mathbb{F}_1(-1) & \xrightarrow{[F_1\ F_2\ F_3]} & R(-1) & \to & R/I(-1) & \to & 0 \\
 & & & & \downarrow M & & \downarrow (\times L) & & \downarrow (\times L) & & \\
0 & \to & E & \to & \mathbb{F}_1 & \xrightarrow{[F_1\ F_2\ F_3]} & R & \to & R/I & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \bar{\mathbb{F}}_1 & \xrightarrow{[\bar{F}_1\ \bar{F}_2\ \bar{F}_3]} & \bar{R} & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

where M is the matrix $\begin{bmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & L \end{bmatrix}$. Note that the first vertical exact sequence is the direct sum of three copies of the exact sequence

$$0 \to R(-1) \xrightarrow{\times L} R \to \bar{R} \to 0$$

twisted by $-d_1$, $-d_2$ and $-d_3$ respectively. The induced map on the kernels

$$E(-1) \to E$$

is just multiplication by L.

Let λ be the line in $\mathbb{P}^2$ defined by L. Invoking the Snake Lemma and using the fact that the sheafification of R/I is 0, the sheafified version of (2.6) is

$$(2.7)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}(-1) & \to & \mathcal{F}_1(-1) & \xrightarrow{[F_1\ F_2\ F_3]} & \mathcal{O}_{\mathbb{P}^2}(-1) & \to & 0 \\
 & & \downarrow (\times L) & & \downarrow M & & \downarrow (\times L) & & \\
0 & \to & \mathcal{E} & \to & \mathcal{F}_1 & \xrightarrow{[F_1\ F_2\ F_3]} & \mathcal{O}_{\mathbb{P}^2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}|_\lambda & \to & \bar{\mathcal{F}}_1 & \xrightarrow{[\bar{F}_1\ \bar{F}_2\ \bar{F}_3]} & \mathcal{O}_\lambda & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Corollary 2.2,

$$\mathcal{E}|_\lambda \cong \begin{cases} \mathcal{O}_\lambda(-d)^2, & \text{if } d_1+d_2+d_3 = 2d; \\ \mathcal{O}_\lambda(-d) \oplus \mathcal{O}_\lambda(-d-1), & \text{if } d_1+d_2+d_3-1 = 2d. \end{cases}$$

Let $\bar{I}$ be the ideal $(\bar{F}_1, \bar{F}_2, \bar{F}_3)$ in $\bar{R}$. Taking global sections on the last line of (2.7) gives

$$0 \to \bigoplus_{i=1}^{2} \bar{R}(-e_i) \to \bar{\mathbb{F}}_1 \to \bar{I} \to 0$$

where $|e_1 - e_2| = 0$ or 1 according to whether $d_1 + d_2 + d_3$ is even or odd, respectively. It was observed in [28] (Remark on page 3165) that this implies that R/I has the Weak Lefschetz property. However, for completeness we will sketch the argument. We will treat only the case $d_1 + d_2 + d_3$ even, leaving the other case to the reader.

We have the exact sequence

$$0 \to \bar{R}(-d)^2 \to \begin{array}{c} \bar{R}(-d_1) \\ \oplus \\ \bar{R}(-d_2) \\ \oplus \\ \bar{R}(-d_3) \end{array} \xrightarrow{[\bar{F}_1\ \bar{F}_2\ \bar{F}_3]} \bar{I} \to 0 \tag{2.8}$$

where $d = \frac{d_1+d_2+d_3}{2}$.

As noted earlier, we only have to show that multiplication by L is injective on the "first half" of R/I. The socle degree of R/I is $d_1 + d_2 + d_3 - 3$, so we have to show that the multiplication

$$(R/I)_j \xrightarrow{\times L} (R/I)_{j+1}$$

is injective for $j \leq \frac{d_1+d_2+d_3}{2} - 2 = d - 2$. We will show it to be true for $j = d - 2$, and from the form of the proof it will be clear that it holds also for smaller j.

The kernel of $(\times L)$ is $[I :_R L]$, so if $(\times L)$ is not injective we have an element $F \in R_{d-2}$, $F \notin I$, such that $LF \in I_{d-1}$. That is, we have forms A_i, $1 \leq i \leq 3$, with $\deg A_i = d-1-d_i$ and

$$LF - A_1F_1 - A_2F_2 - A_3F_3 = 0.$$

Restricting this syzygy to $\bar{R}$ gives

$$\bar{A}_1\bar{F}_1 + \bar{A}_2\bar{F}_2 + \bar{A}_3\bar{F}_3 = 0.$$

But (2.8) says that the smallest possible syzygies come from polynomials of degree $d - d_i$, $1 \leq i \leq 3$, so this is a contradiction. As noted, this works equally well to prove injectivity for all $j \leq d - 2$. $\square$

Corollary 2.4. *Let K be a field of characteristic zero which is not necessarily algebraically closed. Then every height three Artinian complete intersection in $K[x_1, x_2, x_3]$ has the Weak Lefschetz property.*

Proof. The Weak Lefschetz property for a graded Artinian K-algebra A is equivalent to the statement that for a general linear form ℓ, the Hilbert function of $A/\ell A$ is just the positive part of the first difference of the Hilbert function of A. But this does not change under extension of the base field, so the result follows from Theorem 2.3. $\square$

Using the same methods, we can also give a new proof of the main result of [28]. As above, we can assume that K is algebraically closed initially, but the rest of the results of this section do not need this assumption.

Corollary 2.5. *Let $R = K[x_1, x_2, x_3]$, $I = (F_1, F_2, F_3)$ a complete intersection in R, $d_i = \deg F_i$ for $i = 1, 2, 3$, L a general linear form, $\bar{R} = R/LR$ and $\bar{I} = (I + LR)/LR$. Then the following are equivalent:*

(i) $\mu(\bar{I}) = 3$, *where μ is the minimal number of generators;*
(ii) $d_3 \leq d_1 + d_2 - 2$.

Proof. For completeness we repeat the proof from [28] of the fact that (i) implies (ii). Since L is general, F_1, F_2 and L are a regular sequence, and the socle degree of $R/(F_1, F_2, L)$ is $d_1 + d_2 - 2$. If (ii) is not true then F_3 is contained in the ideal (F_1, F_2, L), so $\bar{F}_3$ is contained in $((F_1, F_2) + LR)/LR$, contradicting (i).

The hard part of the proof is the converse, which we prove using our approach. We have from Corollary 2.2 that the splitting type of $\mathcal{E}$ is

$$(e_1, e_2) = \begin{cases} (-d, -d) & \text{if } d_1 + d_2 + d_3 = 2d; \\ (-d, -d-1) & \text{if } d_1 + d_2 + d_3 - 1 = 2d. \end{cases}$$

With this definition of d, a simple calculation gives that

$$\begin{array}{lll} \text{If } d \text{ is even then} & d_3 < d \;\Leftrightarrow\; d_3 < d_1 + d_2; \\ \text{If } d \text{ is odd then} & d_3 < d \;\Leftrightarrow\; d_3 < d_1 + d_2 - 1. \end{array}$$

So in either case, if (ii) holds then $d_3 < d$. But the splitting type gives exactly the leftmost free module in the short exact sequence (2.8), and the fact that $d_3 < d$ means that no splitting can occur in the resolution. □

We now apply these ideas to the question of minimal free resolutions. In particular, suppose $I = (F_1, F_2, F_3)$ is a complete intersection in $R = K[x_1, x_2, x_3]$ and F is a general form of degree d. What can be the possible minimal free resolutions of the ideal (I, F)? Does it depend only on the degrees of the generators of I, or does the choice of the complete intersection itself play a role? We can answer this question when F has degree 1, which in any case was an open question. To be consistent with notation, we write L for this general linear polynomial. We begin with a lemma.

Lemma 2.6. *Let $I \subset R = K[x_1, x_2, x_3]$ be an Artinian ideal. Then there exists a Cohen-Macaulay height two ideal $J \subset R$ such that $J + (L) = I + (L)$. J can even be taken to be reduced.*

Proof. Let $I = (F_1, \dots, F_k)$. We know that $I + (L)/(L) = (\bar{F}_1, \dots, \bar{F}_k)$ is Artinian in $\bar{R} = R/(L)$, hence Cohen-Macaulay of height 2. After a change of coordinates, we can assume that $L = x_3$, hence we obtain polynomials $G_1, \dots, G_k \in K[x_1, x_2]$ by canceling all monomials in $F_1, \dots, F_k$ which are a multiple of x_3. Then viewing these polynomials in R gives the first result. This ideal is not reduced, however. But it has a Hilbert-Burch matrix, whose entries are all polynomials in x_1, x_2. Using standard lifting techniques one can obtain a reduced scheme J with the desired property. (A more geometric use of this trick may be found in [6].) □

Note that the preceding lemma trivially implies that all the graded Betti numbers (over $R/(L)$) of the reduction of J modulo L are the same as those of the reduction of I modulo L. However, in general we are not able to say what these Betti numbers are, or what the Betti numbers of the ideal $I + (L)$ are (over R), or even what the Hilbert function is.

Nevertheless, in the case of complete intersections we can say something much stronger, thanks to our results above.

Corollary 2.7. *Let $I = (F_1, F_2, F_3) \subset R$ be a complete intersection. Then there is a (reduced) arithmetically Cohen-Macaulay ideal $J = (G_1, G_2, G_3) \subset R$ such that $\deg G_i = \deg F_i = d_i$ for $i = 1, 2, 3$ and such that $J + (L) = I + (L)$. Furthermore:*

a. *If $d_3 \le d_1 + d_2 - 2$ then J is an almost complete intersection with minimal generators given by the G_i. Let d be defined by*

$$\begin{cases} d_1 + d_2 + d_3 = 2d & \text{if } d_1 + d_2 + d_3 \text{ is even} \\ d_1 + d_2 + d_3 - 1 = 2d & \text{if } d_1 + d_2 + d_3 \text{ is odd} \end{cases}$$

If $d_1 + d_2 + d_3$ is even then the minimal free resolution of $R/(I + (L))$ is given by

$$0 \to R(-d-1)^2 \to \begin{array}{c} R(-d_1-1) \\ \oplus \\ R(-d_2-1) \\ \oplus \\ R(-d_3-1) \\ \oplus \\ R(-d)^2 \end{array} \to \begin{array}{c} R(-1) \\ \oplus \\ R(-d_1) \\ \oplus \\ R(-d_2) \\ \oplus \\ R(-d_3) \end{array} \to R \to R/(I + (L)) \to 0$$

(The case where $d_1 + d_2 + d_3$ is odd is analogous.)

b. *If $d_3 > d_1 + d_2 - 2$ then $J = (G_1, G_2)$ is a complete intersection. In this case the minimal free resolution of $R/(I + (L))$ is given by*

$$0 \to R(-d_1 - d_2 - 1) \to \begin{array}{c} R(-d_1-1) \\ \oplus \\ R(-d_2-1) \\ \oplus \\ R(-d_1-d_2) \end{array} \to \begin{array}{c} R(-1) \\ \oplus \\ R(-d_1) \\ \oplus \\ R(-d_2) \end{array} \to R \to R/(I + (L)) \to 0$$

Proof. The first part of the corollary is immediate from Lemma 2.6.

For both (a) and (b) we know that $(I + (L)) = (J + (L))$ where J is arithmetically Cohen-Macaulay of depth 1. Hence $R/(I + (L))$ has the same resolution as $R/(J + (L))$, either over R or over $R/(L)$.

Let us consider (a). We know from Corollary 2.5 that $[I + (L)]/(L) = [J + (L)]/(L) \subset R/(L)$ is an almost complete intersection, so the same is true of $J \subset R$ since depth $R/J = 1$. Suppose that $d_1 + d_2 + d_3$ is even (the case where it is odd is completely analogous). We have a minimal free resolution (over $R/(L)$) for $R/(J + (L))$ given in Theorem 2.3, so we thus have a minimal free resolution over R for R/J given by

$$0 \to R(-d)^2 \to \begin{array}{c} R(-d_1) \\ \oplus \\ R(-d_2) \\ \oplus \\ R(-d_3) \end{array} \to R \to R/J \to 0.$$

Then the desired minimal free resolution for $R/(J+(L))$ (and hence $R/(I+(L))$) is given by the tensor product of this resolution with the resolution

$$0 \to R(-1) \to R \to R/(L) \to 0.$$

The proof of (b) is trivial ☐

Remark 2.8. It is possible that similar techniques can be used to prove the Strong Lefschetz property for height three complete intersections (see Definition 4.1), or to attack either the Weak or Strong Lefschetz properties for Artinian complete intersections in higher dimensional rings. However, a more subtle proof will be needed, as simple examples show that the *degrees* of the syzygies will not be enough to obtain a contradiction.

Nevertheless, *we conjecture that every Artinian complete intersection in* $K[x_0, x_1, x_2]$ *has the Strong Lefschetz property.*

Remark 2.9. What happens in characteristic p? We first note that we cannot expect a result as strong as the one given in Theorem 2.3. Indeed, let $A = K[x_1, x_2, x_3]/(x_1^2, x_2^2, x_3^2)$ where K has characteristic 2. Let $g = ax_1 + bx_2 + cx_3$ be a general linear form. Then $g : A_1 \to A_2$ is not injective; indeed, g is itself in the kernel! A similar observation can be made for $A = K[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^4)$, etc.

The main problem here is that the Grauert-Mülich theorem does not hold in characteristic p. There are weaker versions: a Theorem of Ein ([8] Theorem 4.1) bounds the splitting type of $\mathcal{E}$ by a function of $c_2(\mathcal{E})$. However, as we saw in the proof of Theorem 2.3, we need the full strength of Grauert-Mülich in order to prove our result. In the highest degree (at the "middle" of the h-vector), the contradiction from the degrees of the syzygies would not have occurred if this degree had been one greater. Hence a weaker version of Grauert-Mülich is not good enough with the present techniques.

For example, if I is the complete intersection of three polynomials of degree 10 in R, then one can compute that $c_2(\mathcal{E}_{norm}) = 75$, and then Ein's theorem gives that the splitting type is no worse than $(5, -5)$. However, that means that the restriction to $\bar{R} = R/(L)$ has resolution "no worse" than

$$0 \to \bar{R}(-10) \oplus \bar{R}(-20) \to \bar{R}(-10)^3 \to \bar{I} \to 0$$

In particular, it cannot even be excluded that the restriction of I to $\bar{R}$ is a complete intersection. In characteristic zero this is excluded immediately by our work above (applying the strong Grauert-Mülich theorem) and in fact it follows immediately also from the main theorem of [28].

Remark 2.10. 1. The Weak Lefschetz Property says that a general linear form induces a map of maximal rank on consecutive components. One might be interested in a description of the set of (special) linear forms which do *not* give maps of maximal rank. This is parameterized by the variety of jumping lines of the bundle $\mathcal{E}$.

It is interesting to combine the two techniques involved here. For any set of distinct lines $\lambda_1, \dots, \lambda_r$ in $\mathbb{P}^2$ one can easily construct bundles having the λ_i as jumping lines. For example, let $r = 3$ and consider complete intersections of type (4,4,4).

On λ_i, $i = 1, 2, 3$, choose general points $P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,1}, Q_{i,2}, R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4}$. Consider the 4-tuples

$$(P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,1}), (P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,2}), \text{ and } (R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4}).$$

Choose a general quartic curve $F_1 \in R_4$ containing the 12 points $(P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,1})$ $(i = 1, 2, 3)$. Choose a general quartic curve $F_2 \in R_4$ containing the 12 points $(P_{i,1}, P_{i,2}, P_{i,3}, Q_{i,2})$ $(i = 1, 2, 3)$. Choose a general quartic curve $F_3 \in R_4$ containing the 12 points $(R_{i,1}, R_{i,2}, R_{i,3}, R_{i,4})$ $(i = 1, 2, 3)$. (This is possible since the points were chosen generically.)

Then (F_1, F_2, F_3) is a complete intersection, but its restrictions to λ_1, λ_2 and λ_3 each have linear syzygies. Let $\mathcal{E}$ be the bundle constructed from this complete intersection. Since the restriction to a general line has no smaller than quadratic syzygies, λ_1, λ_2 and λ_3 are jumping lines.

2. The bundle $\mathcal{E}$ used in this section is a Buchsbaum-Rim sheaf. The interested reader can find a much more extensive treatment of such sheaves and their properties in [17], [18] and [20].

3. Hilbert functions and maximal Betti numbers of algebras with the Weak Lefschetz property

In this section we do not require that char $K = 0$ or that K be algebraically closed. We give a complete characterization of the possible Hilbert functions of algebras with the Weak Lefschetz property. Furthermore, we show that there is a sharp upper bound on all of the graded Betti numbers in the minimal free resolution of an algebra with the Weak Lefschetz property. For the remainder of this paper we write $R = K[x_0, \dots, x_n]$.

Notation 3.1. If $A = R/I$ is a graded K-algebra then we denote the Hilbert function of A by

$$h_A(t) := \dim_K [R/I]_t.$$

The main result of [13] was to characterize the Gorenstein sequences (i.e. the sequences of integers that can arise as the Hilbert function of an Artinian Gorenstein ideal) corresponding to Artinian Gorenstein ideals with the Weak Lefschetz property. These turned out to be the so-called Stanley-Iarrobino (SI)-sequences. As a consequence, since the height three Gorenstein ideals are well understood ([5], [7] among others), in $K[x_1, x_2, x_3]$ every Gorenstein sequence occurs as the Hilbert function of an Artinian ideal with the Weak Lefschetz property. We now consider the non-Gorenstein case.

Question 3.2. Which Hilbert functions (in any codimension) can occur for ideals whose coordinate rings have the Weak Lefschetz property?

We will give a complete answer to this question, giving a construction for an Artinian K-algebra with any allowable Hilbert function, having the Weak Lefschetz property. Later we will give a bound for the graded Betti numbers of an Artinian K-algebra with the Weak Lefschetz property (Theorem 3.20), and we will show that our construction achieves the bound. Of course this result includes as a special case the maximal possible socle type. However, we have the additional nice result that this maximal socle type can be read directly from the Hilbert function, so we will consider the socle type along with the Hilbert function.

Let A be an Artinian graded K-algebra with the Weak Lefschetz Property, and let g be a Lefschetz element of A. We make the following observations about the Hilbert function and the socle type of A.

Remark 3.3. (1) Let d be the smallest degree for which $\times g : A_d \to A_{d+1}$ is surjective. Then the map $\times g : A_j \to A_{j+1}$ is also surjective for all $j \geq d$. This is because we are considering the natural grading.

(2) Hence $\times g : A_j \to A_{j+1}$ is injective, but not surjective, for all $j < d$.

(3) Let $\underline{h} = (h_0, h_1, \dots, h_s)$ be the Hilbert function of A. From (1) and (2) it follows that

$$h_0 < h_1 < \cdots < h_d \geq h_{d+1} \geq \cdots \geq h_s.$$

In particular, $\underline{h}$ is unimodal and strictly increasing until it reaches its peak, which is called the *Sperner number* of the Hilbert function of A ([27]).

(4) Thus we see that there exist integers $u_1, u_2, \dots, u_\ell$ such that

$$\begin{aligned} h_0 < h_1 < \cdots < h_{u_1} = \cdots = h_{u_2-1} > h_{u_2} = \cdots = h_{u_3-1} \\ > h_{u_3} \cdots > h_{u_\ell} = \cdots = h_s > 0. \end{aligned}$$

In particular $u_1 = d$.

(5) Furthermore from (1) and (2) we have that the positive part of the first difference of $\underline{h}$, namely

$$1, \quad h_1 - h_0, \quad h_2 - h_1, \quad \cdots, \quad h_{u_1} - h_{u_1-1},$$

is the Hilbert function of B=A/(g). In particular, this is an O-sequence.

(6) Let $(a_0, \dots, a_s)$ be the h-vector of the socle of A. The Hilbert series of the socle is called the socle type $S(A, \lambda)$ of A, i.e.

$$S(A, \lambda) = \sum_{i=0}^{s} a_i \lambda^i.$$

We want to compare the socle type with the following polynomial

$$\Phi_{\underline{h}}(\lambda) := \sum_{i=u_1}^{s} (h_i - h_{i+1}) \lambda^i,$$

where $h_{s+1} = 0$. It can easily be checked from (1), (2) and (4) that $a_i = 0$ for all $i \notin \{u_2 - 1, u_3 - 1, \dots, u_\ell - 1, s\}$. Furthermore we have

$$a_i \leq h_i - h_{i+1}$$

for all $i \in \{u_2 - 1, u_3 - 1, \dots, u_\ell - 1, s\}$. This follows from

$$\operatorname{Soc}(A)_i \subset \ker(\times g : A_i \to A_{i+1}),$$

$\dim \operatorname{Soc}(A)_i = a_i$ and $\dim \ker(\times g : A_i \to A_{i+1}) = h_i - h_{i+1}$. An Artinian K-algebra for which $a_i = h_i - h_{i-1}$ will be said to have *maximal socle type.* Notice that the rank of the last free module in the minimal free resolution of A is equal to $\sum a_i$, the dimension of the socle, so for an algebra with maximal socle type, this rank is actually equal to the Sperner number of A (see (3) above).

Conditions (3), (4) and (5) give a necessary condition for a Hilbert function $\underline{h}$ to be the Hilbert function of an Artinian graded K-algebra with the Weak Lefschetz property. We now show that not only are these conditions also sufficient, thus characterizing the Hilbert functions of Artinian K-algebras with the Weak Lefschetz property, but in fact an example exists with the maximal possible socle type, as described in (6). We first give the basic construction.

Construction 3.4. Let $\underline{h} = (h_0, h_1, \dots, h_s, h_{s+1} = 0)$ be a finite sequence of integers satisfying the conditions of (3), (4) and (5) above. Define

$$\bar{h}(j) := \max\{h_j - h_{j-1}, 0\}.$$

Choose Artinian ideals

$$\bar{J}_1 \subset \bar{J}_2 \subset \dots \subset \bar{J}_\ell \subset \bar{R} := K[x_1, \dots, x_n]$$

such that $h_{\bar{R}/\bar{J}_1} = \bar{h}$ and $\deg \bar{J}_i = h(u_i)$ for all $i = 2, \dots, \ell$. Now put $J_i = \bar{J}_i R$ for all $i = 1, \dots, \ell$ and

$$I := J_1 + \sum_{i=2}^{\ell} [J_i]_{\geq u_i} + \mathfrak{m}^{s+1},$$

where $\mathfrak{m} = (x_0, \dots, x_n)$. Set $A := R/I$. Note that J_i is not reduced, but it is the saturated ideal of a zeroscheme $\mathbb{X}_i$. Furthermore, we have $\mathbb{X}_1 \supset \mathbb{X}_2 \supset \cdots \supset \mathbb{X}_\ell$.

Proposition 3.5. *Let $\underline{h} = (1, h_1, \dots, h_s)$ be a finite sequence of positive integers. Then $\underline{h}$ is the Hilbert function of a graded Artinian K-algebra R/J having the Weak Lefschetz property if and only if $\underline{h}$ is a unimodal O-sequence such that the positive part of the first difference is an O-sequence.*

Furthermore, let $u_1, \dots, u_\ell$ and $\Phi_{\underline{h}}(\lambda)$ be as in Remark 3.3. Then the K-algebra A of Construction 3.4 has the Weak Lefschetz property, Hilbert function $\underline{h}$ and maximal socle type $\Phi_{\underline{h}}(\lambda)$.

Proof. The necessity is proved in Remark 3.3. The sufficiency follows immediately from the claim about Construction 3.4, which we now prove.

(1) The Artinian K-algebra A has the Weak Lefschetz property: Let $B^{(j)} := R/J_j = \oplus [B^{(j)}]_i$. We may assume that x_0 is not zero divisor mod J_j for all j. Considering the following commutative diagram

$$\begin{tikzcd} {[B^{(j)}]_{u_{j+1}-1}} \arrow[r, "x_0"] \arrow[d, equal] & {[B^{(j)}]_{u_{j+1}}} \arrow[r] & {[B^{(j+1)}]_{u_{j+1}}} \arrow[d, equal] \\ A_{u_{j+1}-1} \arrow[rr, "x_0"] & & A_{u_{j+1}} \end{tikzcd}$$

we have, as the proof of Lemma 3.2 in [13], that A has the Weak Lefschetz property.

(2) The Hilbert function of A is $\underline{h}$: First we recall a basic property of the Hilbert function of a zeroscheme $\mathbb{Y}$ in $\mathbb{P}^n$. Set

$$\sigma(\mathbb{Y}) := \min\{i \mid \Delta h_{R/I_{\mathbb{Y}}}(i) = 0\},$$

where $\Delta h_{R/I_{\mathbb{Y}}}(i)$ is the first difference of $h_{R/I_{\mathbb{Y}}}(i)$. Then it follows that

$$h_{R/I_{\mathbb{Y}}}(0) < \cdots < h_{R/I_{\mathbb{Y}}}(\sigma(\mathbb{Y}) - 1) = h_{R/I_{\mathbb{Y}}}(\sigma(\mathbb{Y})) = \cdots = \deg \mathbb{Y},$$

and we see that if $\mathbb{Y}' \subset \mathbb{Y}$ then $\sigma(\mathbb{Y}') \leq \sigma(\mathbb{Y})$. Hence from this property we get

$$h_{B^{(j)}}(i) = h_{u_j}$$

for all $i \geq u_1$. Thus since $A_i = [B^{(1)}]_i$ for all $0 \leq i \leq u_2 - 1$, $A_i = [B^{(j)}]_i$ for all $u_j \leq i \leq u_{j+1} - 1$ and $A_i = (0)$ for all $i \geq s+1$, we have that the Hilbert function of A coincides with $\underline{h}$.

(3) The socle type of A is $\Phi_{\underline{h}}(\lambda)$: We note that

$$[\mathrm{Soc}(A)]_{u_{j+1}-1} = [I^{(j+1)}]_{u_{j+1}-1}/[I^{(j)}]_{u_{j+1}-1}.$$

Furthermore we see that

$$\begin{aligned} & \dim\{[I^{(j+1)}]_{u_{j+1}-1}/[I^{(j)}]_{u_{j+1}-1}\} \\ = \; & h_{B^{(j)}}(u_{j+1}-1) - h_{B^{(j+1)}}(u_{j+1}-1) \\ = \; & h_{u_j} - h_{u_{j+1}}. \end{aligned}$$

Thus it follows from part (6) of Remark 3.3 that

$$S(A,\lambda) = \Phi_{\underline{h}}(\lambda).$$

This completes the proof. □

Example 3.6. Not all Artinian ideals in R whose Hilbert functions satisfy the necessary and sufficient conditions given in Proposition 3.5 have the Weak Lefschetz property. Indeed, we give a simple example of one which even has the Hilbert function of a complete intersection but does not have the Weak Lefschetz property. We take

$$I = (x_1^2, x_1x_2, x_1x_3, x_2^3, x_2^2x_3, x_2x_3^2, x_3^4),$$

so R/I has Hilbert function (1 3 3 1). For any linear form L, the element $x_1 \in (R/I)_1$ is in the kernel of multiplication by L, hence the Weak Lefschetz property fails in passing from degree 1 to degree 2.

A finer invariant of an Artinian K-algebra is its minimal free resolution. It is probably not possible now to give a set of necessary and sufficient conditions on the graded Betti numbers for the existence of an ideal with the Weak Lefschetz property and that set of Betti numbers. Even in the Gorenstein case this is open. However, as in the Gorenstein case [19], we can give a sharp upper bound for the graded Betti numbers. We will do this shortly.

However, we begin with some natural questions, which are the analogs, for resolutions, of results which we know for Hilbert functions.

Question 3.7. 1. Is there a minimal free resolution (meaning only the graded Betti numbers, not the maps) corresponding to an Artinian ideal with a Hilbert function allowed by Proposition 3.5, which cannot occur for an ideal with the Weak Lefschetz property?

2. Are there two Artinian ideals, I_1 and I_2, which have the same graded Betti numbers, but one has the Weak Lefschetz property and the other not?

We answer both of these questions. First we recall some terminology.

Definition 3.8. Let $>$ denote the degree-lexicographic order on monomial ideals, i.e. $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ if the first nonzero coordinate of the vector

$$\left(\sum_{i=1}^{n} (a_i - b_i), a_1 - b_1, \ldots, a_n - b_n \right)$$

is positive. Let J be a monomial ideal. Let m_1, m_2 be monomials in S of the same degree such that $m_1 > m_2$. Then J is a *lex-segment ideal* if $m_2 \in J$ implies $m_1 \in J$. When $\mathrm{char}(K) = 0$, we say that J is a *Borel-fixed ideal* if

$$m = x_1^{a_1} \cdots x_n^{a_n} \in J,\ a_i > 0,\ \text{implies } \frac{x_j}{x_i} \cdot m \in J$$

for all $1 \le j < i \le n$.

Example 3.9. We first answer the first part of Question 3.7. Let $J \subset K[x_1, x_2, x_3]$ be the lex-segment ideal for the Hilbert function (1 3 3 1). Then its minimal free resolution is of the form

$$\begin{array}{ccccccccc} & & R(-4) & & R(-3)^3 & & R(-2)^3 & & & & \\ & & \oplus & & \oplus & & \oplus & & & & \\ 0 & \to & R(-5)^2 & \to & R(-4)^5 & \to & R(-3)^3 & \to & J & \to & 0 \\ & & \oplus & & \oplus & & \oplus & & & & \\ & & R(-6) & & R(-5)^2 & & R(-4) & & & & \end{array}$$

Now let I be any Artinian ideal in $K[x_1, x_2, x_3]$ with these graded Betti numbers. The generators of I in degree 2 have three linear syzygies. It is not hard to check (e.g. using methods of [3]) that this can only happen if they have a common linear factor (so in particular there is no regular sequence of length 2 among these three quadrics). But then after a change of variables we may assume that this common factor is x_1, and we are in the situation of Example 3.6. Hence R/I cannot have the Weak Lefschetz property. (As an alternative proof, note that the Socle type is $\lambda + 2\lambda^2 + \lambda^3$, so it also follows from Remark 3.3 (6) that it cannot have the Weak Lefschetz property.)

Example 3.10. We now give a (positive) answer to the second part of Question 3.7. H. Ikeda has shown ([16] Example 4.4) that there is a Gorenstein Artinian K-algebra $A = R/I$ with Hilbert function $(1, 4, 10, 10, 4, 1)$ and minimal free resolution

$$0 \to \mathbb{F}_4 \to \mathbb{F}_3 \to \mathbb{F}_2 \to \mathbb{F}_1 \to R \to R/I \to 0,$$

where

$$\begin{array}{rcl} \mathbb{F}_1 & = & R(-3)^{10} \oplus R(-4)^6, \\ \mathbb{F}_2 & = & R(-4)^{15} \oplus R(-5)^{15}, \\ \mathbb{F}_3 & = & R(-5)^6 \oplus R(-6)^{10}, \text{ and} \\ \mathbb{F}_4 & = & R(-9). \end{array}$$

and not possessing the Weak Lefschetz property. These graded Betti numbers are precisely the maximum possible for this Hilbert function among ideals with the Weak Lefschetz property, and an ideal exists with these graded Betti numbers and with the Weak Lefschetz property, thanks to [19] Theorem 8.13.

In Example 3.9 we saw that the resolution of the lex-segment ideal (which is known to be extremal among all possible resolutions with the given Hilbert function [2], [14], and [22] for char $K > 0$) cannot, in general, be the minimal free resolution of an ideal with the Weak Lefschetz property, and we gave a reason for this failure based on the beginning of the resolution, and a different reason based on the end of the resolution. This suggests the following question:

Question 3.11. Let $\underline{h} = (h_0, h_1, \ldots, h_s)$ be a Hilbert function which can occur for Artinian K-algebras with the Weak Lefschetz property (see Proposition 3.5). Is there a maximal possible resolution among Artinian ideals with the Weak Lefschetz property and Hilbert function $\underline{h}$?

We now answer Question 3.11 by establishing upper bounds for the graded Betti numbers of an artinian K-algebra with the Weak Lefschetz property and exhibiting examples where these bounds are attained. Note that such bounds were found for artinian Gorenstein algebras with the Weak Lefschetz property in [19]. We adapt the techniques developed there to our problem.

We begin by recalling [19], Lemma 8.3.

Lemma 3.12. *Let M be a graded R-module, $\ell \in R$ a linear form. Then there is an exact sequence of graded R-modules (where $\bar{R} := R/\ell R$):*

$$\begin{aligned}\cdots \to \operatorname{Tor}_{i-1}^{\bar{R}}((0 :_M \ell), K)(-1) \to \operatorname{Tor}_i^R(M, K) \to \operatorname{Tor}_i^{\bar{R}}(M/\ell M, K) \to \cdots \\ \cdots \to \operatorname{Tor}_1^R(M, K) \to \operatorname{Tor}_1^{\bar{R}}(M/\ell M, K) \to 0.\end{aligned}$$

Notation 3.13. Now let $A = R/I$ be an artinian graded K-algebra with the Weak Lefschetz property, and let $g \in [R]_1$ be a Lefschetz element of A. Denote by d the end of A/gA and by a the initial degree of $0 : g := 0 :_A g$, i.e.

$$d := \max\{j \in \mathbb{Z} \mid [A/gA]_j \neq 0\}$$

$$a := \min\{j \in \mathbb{Z} \mid [0 : g]_j \neq 0\}.$$

Observe that $d \leq a$. Using the notation of Remark 3.3 we have $d = u_1, a = u_2 - 1$.

Moreover, we put $\bar{R} := R/gR$ and define

$$\left[\operatorname{tor}_i^R(M, K)\right]_j := \operatorname{rank}\left[\operatorname{Tor}_i^R(M, K)\right]_j.$$

Now we can state the next result.

Proposition 3.14. *We have for all integers i, j:*

$$\left[\operatorname{tor}_i^R(A, K)\right]_{i+j} =$$

$$\begin{cases} \left[\operatorname{tor}_i^{\bar{R}}(A/gA, K)\right]_{i+j} & \text{if } j \leq a-2 \\ \leq \left[\operatorname{tor}_i^{\bar{R}}(A/gA, K)\right]_{i+j} & \text{if } j = a-1 \\ \leq \left[\operatorname{tor}_{i-1}^{\bar{R}}(0 : g, K)\right]_{i+j-1} + \left[\operatorname{tor}_i^{\bar{R}}(A/gA, K)\right]_{i+j} & \text{if } a \leq j \leq d \\ \leq \left[\operatorname{tor}_{i-1}^{\bar{R}}(0 : g, K)\right]_{i+j-1} & \text{if } j = d+1 \\ \left[\operatorname{tor}_{i-1}^{\bar{R}}(0 : g, K)\right]_{i+j-1} & \text{if } j \geq d+2 \end{cases}$$

Furthermore, $\operatorname{Tor}_{n+1}^R(A, K) \cong \operatorname{Tor}_n^{\bar{R}}(0 : g, K)(-1)$.

Proof. Using $[\operatorname{Tor}_i^{\bar{R}}((0 :_A g), K)]_{i+j} = 0$ if $j < a$ and $[\operatorname{Tor}_i^{\bar{R}}(A/gA, K)]_{i+j} = 0$ if $j > d$ the claim follows by analyzing the exact sequence given in Lemma 3.12. ◻

Observe that the condition $a \leq j \leq d$ can only be satisfied if $a = d$.

Next, we need an elementary estimate.

Lemma 3.15. *Let M be a graded R-module. Then we have for all integers i, j:*

$$\left[\operatorname{tor}_i^R(M, K)\right]_{i+j} \leq h_M(j) \cdot \binom{n+1}{i}.$$

Proof. Put $P := R^{n+1}(-1)$. Then the Koszul complex gives the following minimal free resolution of $R/\mathfrak{m} \cong K$:

$$0 \to \wedge^{n+1} P \to \ldots \to \wedge^{i+1} P \to \wedge^i P \to \ldots \to P \to R \to K \to 0.$$

Thus, $[\operatorname{Tor}_i^R(M, K)]_{i+j}$ is the homology of the complex

$$\left[\wedge^{i+1} P \otimes M\right]_{i+j} \to \left[\wedge^i P \otimes M\right]_{i+j} \to \left[\wedge^{i-1} P \otimes M\right]_{i+j}.$$

Since $\operatorname{rank}[\wedge^i P \otimes M]_{i+j} = h_M(j) \cdot \binom{n+1}{i}$, the claim follows. □

Notation 3.16. Let h be the Hilbert function of an artinian K-algebra R/I. Then there is a uniquely determined lex-segment ideal $J \subset R$ such that R/J has h as its Hilbert function. We define

$$\beta_{i,j}(h, R) := \left[\operatorname{tor}_i^R(R/J, K)\right]_{i+j}.$$

Remark 3.17. The numbers $\beta_{i,j}(h, R)$ can be computed numerically without considering lex-segment ideals. Explicit formulas can be found in [10].

Theorem 3.18 ([2], [14], [22]). *If $A = R/I$ is an artinian algebra then we have for all $i, j \in \mathbb{Z}$*

$$\left[\operatorname{tor}_i^R(A, K)\right]_{i+j} \leq \beta_{i,j}(h_A, R).$$

In order to construct algebras with the Weak Lefschetz property and maximal Betti numbers we need one more technical result. In the following lemma, for a graded module M of finite length we denote by $e(M)$ the last degree in which M is non-zero.

Lemma 3.19. *Let $\bar{I} \subset \bar{J} \subset \bar{R} := K[x_1, \ldots, x_n]$ be artinian ideals. Put $d := e(\bar{R}/\bar{I})$, $I = \bar{I}R$, $J := \bar{J}R$ and $\mathfrak{a} := I + [J]_{\geq d+1}$. Then $\mathfrak{a} + x_0 R = I + x_0 R$ and we have for the graded Betti numbers of $A := R/\mathfrak{a}$:*

$$[\operatorname{tor}_i^R(A, K)]_j = \begin{cases} [\operatorname{tor}_i^{\bar{R}}(A/x_0A, K)]_j & \text{if } j \neq i + d \\ [\operatorname{tor}_i^{\bar{R}}(A/x_0A, K)]_j + k \cdot \binom{n}{i-1} & \text{if } j = i + d \end{cases}$$

where $k := \deg I - \deg J$.

Proof. We proceed in several steps.

(I) Since $\bar{I} \subset \bar{J}$, we get $e(\bar{R}/\bar{J}) \leq e(\bar{R}/\bar{I}) = d$. Hence, $\bar{I}$ and $\bar{J}$ are generated by forms of degree $\leq d+1$. In particular, $[J]_{\geq d+1}$ is generated by forms of degree $d+1$.

The ideals $I + x_0 R$ and $\mathfrak{a} + x_0 R$ differ at most in degrees $\geq d+1$. Thus, the Hilbert functions of $A/x_0 A$ and $\bar{R}/\bar{I}$ agree. It follows that $I + x_0 R = \mathfrak{a} + x_0 R$. In particular, we can write

$$\mathfrak{a} = I + x_0 \cdot (f_1, \ldots, f_k)$$

where $f_1, \ldots, f_k \in [J]_d$ because $J : x_0 = J$.

(II) Put $\mathfrak{b} := (f_1, \dots , f_k)R$, i.e. $\mathfrak{a} = I + x_0 \cdot \mathfrak{b}$. For $j \le d$, multiplication by x_0 factors through two maps of maximal rank:

$$\begin{array}{ccccc} [A]_j & & \xrightarrow{x_0} & & [A]_{j+1} \\ \| & & & & \| \\ [R/I]_j & \xrightarrow{x_0} & [R/I]_{j+1} & \to & [R/\mathfrak{a}]_{j+1}. \end{array}$$

It follows that

$$0 :_A x_0 \cong [\mathfrak{a}/I]_d \cong K^k(-d)$$

and, in particular, $0 :_A x_0 \cong \operatorname{Soc} A$.

(III) Denote by $g_1, \dots , g_t$ the minimal generators of I. Let $(r_1, \dots , r_t, s_1, \dots , s_k)^t$ be a syzygy of $\mathfrak{a}$, i.e.

$$\sum_{i=1}^{t} r_i g_i + \sum_{j=1}^{k} s_j x_0 f_j = 0.$$

We can write $r_i = \bar{r}_i + x_0 \tilde{r}_i$ where $\bar{r}_i \in \bar{R}$ and $\tilde{r}_i \in R$. It follows that

$$\sum_{i=1}^{t} \bar{r}_i g_i + x_0 \left[\sum_{i=1}^{t} \tilde{r}_i g_i + \sum_{j=1}^{k} s_j f_j \right] = 0.$$

Comparing coefficients we obtain $\sum_{i=1}^t \bar{r}_i g_i = 0$ and $\sum_{i=1}^t \tilde{r}_i g_i + \sum_{i=1}^k s_j f_j = 0$. Thus, we see that $(\bar{r}_1, \dots , \bar{r}_t, 0, \dots , 0)^t + (x_0\tilde{r}_1, \dots , x_0\tilde{r}_t, s_1, \dots , s_k)^t$ is a syzygy of $\mathfrak{a}$ if and only if $(\bar{r}_1, \dots , \bar{r}_t)^t$ is a syzygy of I and $(\tilde{r}_1, \dots , \tilde{r}_t, s_1, \dots , s_k)^t$ is a syzygy of $I + \mathfrak{b}$.

(IV) Let

$$0 \to \bar{G}_n \to \dots \bar{G}_2 \xrightarrow{\bar{\alpha}} \bar{G}_1 \xrightarrow{\bar{\beta}} \bar{R} \to \bar{R}/\bar{I} \to 0$$

be a minimal free resolution of $\bar{R}/\bar{I}$ as $\bar{R}$-module and let

$$0 \to F_n \to \dots \to F_1 \to R \to A \to 0$$

be a minimal free resolution of A as R-module. Tensoring by $\bar{R}$ gives the complex (with $\bar{F}_i := F_i \otimes_R \bar{R}$)

$$0 \to \bar{F}_n \to \dots \to \bar{F}_2 \xrightarrow{\alpha} \bar{F}_1 \xrightarrow{\beta} \bar{F}_0 \to R/\mathfrak{a} + x_0 R \to 0.$$

Since $\mathfrak{a} + x_0 R = I + x_0 R$ we get

$$\ker \beta \cong \ker \bar{\beta} \oplus \bar{R}^k(-d-1).$$

Step (III) shows that $\operatorname{im} \alpha$ splits as

$$(*) \qquad \operatorname{im} \alpha \cong \operatorname{im} \bar{\alpha} \oplus M$$

for some $\bar{R}$-module M such that

$$\ker \beta / \operatorname{im} \alpha \cong \bar{R}^k(-d-1)/M.$$

The proof of [19], Lemma 8.3 shows $\ker \beta / \operatorname{im} \alpha \cong 0 :_A x_0(-1)$. Using step (II) we obtain the exact sequence of $\bar{R}$-modules

$$0 \to M \to \bar{R}^k(-d-1) \to K^k(-d-1) \to 0.$$

It implies for all integers $i \geq 0$:

$$\operatorname{Tor}_i^{\bar{R}}(M, K) \cong K^{k\binom{n}{i+1}}(-d-2-i).$$

From the proof of [19], Lemma 8.3 we also have for $i \geq 0$:

$$\operatorname{Tor}_{i+2}^{R}(A, K) \cong \operatorname{Tor}_i^{\bar{R}}(\operatorname{im}\alpha, K).$$

Hence, the sequence $(*)$ implies our claim. $\square$

We are now ready for the announced result.

Theorem 3.20.

(a) *Let $A = R/I$ be a K-algebra with the Weak Lefschetz property and denote by $\bar{h} : \mathbb{Z} \to \mathbb{Z}$ the function defined by*

$$\bar{h}(j) := \max\{\Delta h_A(j), 0\}.$$

Then the graded Betti numbers of A satisfy

$$\left[\operatorname{tor}_i^R(A, K)\right]_{i+j} \leq \begin{cases} \beta_{i,j}(\bar{h}, \bar{R}) & \text{if } j \leq a-1 \\ \beta_{i,j}(\bar{h}, \bar{R}) + \max\{0, -\Delta h_A(j+1)\} \cdot \binom{n}{i-1} & \text{if } a \leq j \leq d \\ \max\{0, -\Delta h_A(j+1)\} \cdot \binom{n}{i-1} & \text{if } j \geq d+1 \end{cases}$$

(b) *Let $h : \mathbb{Z} \to \mathbb{Z}$ be a numerical function such that there is an artinian algebra R/J having the Weak Lefschetz property and h as Hilbert function. Then there is an artinian algebra $A = R/I$ having the Weak Lefschetz property and h as Hilbert function such that equality is true in* (a) *for all integers i, j.*

Proof. We first prove (a). Since g is a Lefschetz element of A, the Hilbert function of A/gA is $\bar{h}$ and the Hilbert function of $0 :_A g$ is given by

$$h_{0:_A g}(j) = \max\{0, -\Delta h_A(j+1)\}.$$

Thus, our claim is a consequence of Proposition 3.14, Lemma 3.15 and Theorem 3.18 (using [22] for the case char $K > 0$).

Now we show (b). We use the notation of Remark 3.3. Consider the ideal I of Construction 3.4, and assume furthermore that

$$[\operatorname{tor}_i^{\bar{R}}(\bar{R}/\bar{J}_1, K)]_{i+j} = \beta_{i,j}(\bar{h}, \bar{R}) \quad \text{for all integers } i, j.$$

Such an ideal $\bar{J}_1$ certainly exists: for example, we can choose it as a lex-segment ideal.

As in step (I) of the proof of Lemma 3.19 we see that $I + x_0R = J_1 + x_0R$. An argument as in step (II) of that proof shows that

$$0 :_A x_0 = \operatorname{Soc} A$$

and

$$\operatorname{rank}[0 :_A x_0]_j = \max\{0, -\Delta h(j+1)\}.$$

It follows that A has the Weak Lefschetz property, x_0 is a Lefschetz element for A and

$$[\operatorname{tor}_i^{\bar{R}}(0 :_A x_0, K)]_{i+j} = \max\{0, -\Delta h(j+1)\} \cdot \binom{n}{i}.$$

Moreover, since $A/x_0A \cong \bar{R}/\bar{J}_1$ we have

$$[\operatorname{tor}_i^{\bar{R}}(A/x_0A, K)]_{i+j} = \beta_{i,j}(\bar{h}, \bar{R}).$$

Observe again that $d = u_1$ and $a := u_2 - 1 \geq d$. If $a \geq d+1$ all Betti numbers $[\operatorname{tor}_i^R(A, K)]_{i+j}$ are determined by Proposition 3.14 if $j \geq d+2$. Since $[A]_j = [R/J_1]_j$ for $j \leq a$ we get

$$[\operatorname{tor}_i^R(A, K)]_{i+j} = [\operatorname{tor}_i^R(R/J_1, K)]_{i+j} = [\operatorname{tor}_i^{\bar{R}}(A/x_0A, K)]_{i+j} \quad \text{if } j \leq d.$$

The remaining graded Betti numbers $[\operatorname{tor}_i^R(A, K)]_{i+d+1}$ can be computed recursively from the Hilbert function of A. (A similar computation can be found on page 4386 of [21].)

Now let $a = d$. From the definition of I we immediately obtain

$$[\operatorname{tor}_i^R(A, K)]_{i+j} = [tor_i^R(R/(J_1 + [J_2]_{\geq a}), K)]_{i+j} \quad \text{for all } j \leq d.$$

Thus, we know these graded Betti numbers by Lemma 3.19. If $j \geq d+2$ we know $[\operatorname{tor}_i^R(A, K)]_{i+j}$ by Proposition 3.14. Thus, the remaining Betti numbers can be computed as in the previous case.

In any case, we can compute all graded Betti numbers of A. The result shows our claim. □

We would also like to point out that there are Hilbert functions such that all algebras with that Hilbert function and the Weak Lefschetz property have the same (maximal) graded Betti numbers. A similar phenomenon is true for Gorenstein algebras with the Weak Lefschetz property (cf. [19], Corollary 8.14).

Corollary 3.21. *Let $I \subset R$ be an artinian ideal such that $A := R/I$ has the Weak Lefschetz property and its Hilbert function satisfies*

$$h_A(j) = \binom{n+j}{n} \quad \textit{for all } j \leq d = u_1 \leq u_2 - 3$$

and $u_k + 2 \leq u_{k+1}$ for all k with $2 \leq k < \ell$. Then the graded Betti numbers of A are

$$\left[\operatorname{tor}_i^R(A, K)\right]_{i+j} = \begin{cases} \binom{n+d}{i+d} & \textit{if } j = d \\ -\Delta h_A(u_k) \cdot \binom{n}{i-1} & \textit{if } j = u_k - 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By assumption we have $a \geq d+2$. Thus, Lemma 3.12 provides

$$\left[\operatorname{tor}_i^R(A, K)\right]_{i+j} = \begin{cases} \left[\operatorname{tor}_i^{\bar{R}}(A/gA, K)\right]_{i+j} & \text{if } j \leq d \\ 0 & \text{if } j = d+1 \\ \left[\operatorname{tor}_{i-1}^{\bar{R}}(0 : g, K)\right]_{i+j-1} & \text{if } j \geq d+2 \end{cases}$$

We may assume that $g = x_0$. Then we get $A/x_0A \cong \bar{R}/(x_1, \ldots, x_n)^{d+1}$. Thus, the graded Betti numbers of A/gA are known (cf., e.g., the proof of [19], Corollary 8.14). This shows our claim for $j \leq d+1$.

Since A has the Weak Lefschetz property we have

$$\operatorname{rank}[0 :_A x_0]_j = \max\{0, -\Delta h(j+1)\}.$$

This implies

$$0 :_A x_0 = \operatorname{Soc} A.$$

Our claim follows. □

4. Hilbert functions and maximal Betti numbers of algebras with the Strong Lefschetz property

In this section we give some results about a more stringent condition, namely the Strong Lefschetz property. Several of our results require that char $K = 0$, (e.g. Proposition 4.4), and we make this assumption throughout this section.

Not all algebras with the Weak Lefschetz property possess the Strong Lefschetz property in codimension ≥ 3. We show that nevertheless this *does* hold in codimension two. Furthermore, we give the surprising result that the *same* characterization of Hilbert functions and maximal graded Betti numbers that we gave in the last section for the Weak Lefschetz property continues to hold for the Strong Lefschetz property.

The conditions for the Hilbert function given in the statement of Proposition 3.5 are automatic in codimension two. In this case, interestingly, something much stronger than Proposition 3.5 holds. We first recall the notion of the Strong Lefschetz property.

Definition 4.1. An Artinian ideal $I \subset R$ has the *Strong Lefschetz property* if, for a general linear form L and any $d > 0, i \geq 0$, the map

$$\times L^d : (R/I)_i \to (R/I)_{i+d}$$

has maximal rank.

Clearly if R/I has the Strong Lefschetz property then it has the Weak Lefschetz property. However, there are examples of ideals with the Weak Lefschetz property which do not have the Strong Lefschetz property.

Example 4.2. We first give a simple example of an ideal with the Weak Lefschetz property but not the Strong Lefschetz property. Let I be the lex-segment ideal with generators

$$x_1^2,\ x_1x_2,\ x_1x_3^2,\ x_2^3,\ x_2^2x_3^2,\ x_2x_3^3,\ x_3^5.$$

This has Hilbert function (1 3 4 3 1), and one can check that for multiplication by a general linear form L we have maximal rank between consecutive components, while L^2 has the element x_1 in the kernel of the multiplication from degree 1 to degree 3.

Of much greater interest is the fact that there exist examples of *Gorenstein* ideals with the Weak Lefschetz property but not the Strong Lefschetz property. One uses the theory of inverse systems.

Example 4.3. Let R be the ring $K[u,v,x,y,z]$ and let $f = xu^2 + yuv + zv^2$. The dual of f gives a Gorenstein algebra with h-vector (1 5 5 1) (this can be checked, for instance, with the computer program Macaulay [1] using the script `<l_from_dual`). This algebra has neither the Weak Lefschetz property nor the Strong Lefschetz property.

However, now take the polynomial $g = uf$. It gives an algebra with h-vector (1 5 6 5 1). It has the Weak Lefschetz property but *not* the Strong Lefschetz property.

More generally, choose an element $g \in S = [u,v][f]$ which is, in particular, homogeneous in the variables x, y, z, u, v. Let A be the algebra obtained from such a form. Then for a general linear form L, the map $\times L^{s-2} : A_1 \to A_{s-1}$ is not bijective. The key to this goes back to P. Gordan and M. Noether [12]. They showed that if the Hessian of a form is identically zero then one of the variables can be eliminated by means of a linear change of the variables, as long as the number of variable is at most *four*. In dimension 5 or more it is not true, and they gave the above example. In dimension 5 they claimed that these types of forms are the only cases, where you have zero Hessian and still all variables are essentially involved. Then the fourth author [29] showed that the zero Hessian of a form is equivalent to the condition that the map $g^{s-2} : A_1 \to A_{s-1}$ does not have full rank.

We believe that in general a polynomial of the above form does give rise to an Artinian algebra with the Weak Lefschetz property, but have not confirmed it.

We saw in Example 3.6 that for a given Hilbert function in codimension ≥ 3 it is possible to find two ideals with that Hilbert function, one possessing the Weak Lefschetz property and the other not. In codimension two we have the following, generalizing some results in [15]:

Proposition 4.4. *Every Artinian ideal in $K[x, y]$ (char $K = 0$) has the Strong Lefschetz property (and consequently also the Weak Lefschetz property).*

Proof. First suppose that I is a Borel-fixed ideal in $R = K[x, y]$. Since char $K = 0$, I_d consists of consecutive monomials from the first (each d). (Say x^d is the first monomial and y^d the last.) So the vector space R/I_d is spanned by the consecutive monomials from the last.

Let $(h_0, h_1, ..., h_s)$ be the Hilbert function of $A = R/I$. Then it is well known (and easy to see) that it is unimodal. Assume first that $h_i \leq h_{i+d}$. Then $y^d : (R/I)_i \to (R/I)_{i+d}$ is injective, because if a monomial M is in $(R/I)_i$ then $y^d M$ is in $(R/I)_{i+d}$. (The point here is that if M is the t-th monomial of $(R/I)_i$ from the last then $y^d M$ is also the t-th monomial of $(R/I)_{i+d}$ from the last.)

Now assume that $h_i \geq h_{i+d}$. Suppose that a monomial M is in $(R/I)_{i+d}$. Say M is the t-th monomial from the last. Then the t-th monomial of $(R/I)_i$ from the last exists since $h_i > h_{i+d}$. Let it be N. Then we have $y^d N = M$. Thus the map $y^d : (R/I)_i \to (R/I)_{i+d}$ is surjective. Hence we have proved that if I is Borel-fixed in characteristic 0, then R/I has the Strong Lefschetz property.

In the general case we have the fact that $\mathrm{gin}(I)$ is Borel-fixed, where $\mathrm{gin}(I)$ denotes the generic initial ideal of I. It is easy to see and well known (or see Proposition 15.12, Eisenbud [9]) that $\mathrm{In}(I : y^d) = \mathrm{In}(I) : y^d$ for $d = 1, 2, 3 \ldots$, where y is the last variable with respect to the reverse lexicographic order. Since the Hilbert function does not change by passing to $\mathrm{gin}(I)$ and since the Strong Lefschetz property is characterized by the Hilbert function of $A/(y^d))$, $d = 1, 2, 3 \ldots$, the general case reduces to the case of Borel-fixed ideals. $\square$

We have seen that the Strong Lefschetz property is (naturally) a stronger condition than the Weak Lefschetz property, in the sense that there exist ideals whose coordinate ring has the Weak Lefschetz property but not the Strong Lefschetz property. One would naturally expect that the imposition of this extra condition would be accompanied by a further restriction on the possible Hilbert functions (Proposition 3.5) or on the upper bounds on the graded Betti numbers (Theorem 3.20).

We now show that any Hilbert function that occurs for ideals with the Weak Lefschetz property also occurs for ideals with the Strong Lefschetz property. The following two results do not require char $K = 0$.

Proposition 4.5. *Let K be any field. Let I be the ideal obtained in Construction 3.4, with the further assumption that $\bar{J}_2, \dots, \bar{J}_\ell$ satisfy*

$$h_{\bar{R}/\bar{J}_i}(t) = \Delta h^{(i)}(t)$$

for all $i = 2, \dots, \ell$, where

$$h^{(i)}(t) := \begin{cases} \min\{h_t, h_{u_i}\} & \text{if } t < u_i, \\ h_{u_i} & \text{otherwise.} \end{cases}$$

(Such ideals certainly exist. For example, we can choose those as lex-segment ideals.) Then $A = R/I$ has the Strong Lefschetz property.

Proof. We maintain the notation of Construction 3.4 and Proposition 3.5. We may assume that x_0 is not a zero divisor mod J_j. First suppose that $i + d < u_2$. Then from the proof of Proposition 3.5, we see that (A, x_0) has the Weak Lefschetz Property. Hence it follows that the map $\times x_0^d : A_i \to A_{i+d}$ is injective.

So without loss of generality we may assume that $u_j \leq i+d \leq u_{j+1}-1$ (where $2 \leq j \leq \ell$ and $u_{\ell+1} := s + 1$). We note that

$$h_{B^{(j)}}(t) = \begin{cases} h_t & \text{if } \ 0 \leq t \leq \sigma(\mathbb{X}_j) - 2, \\ h_{u_j} & \text{otherwise .} \end{cases}$$

Hence we see that

$$\text{the natural map } A_i \to B_i^{(j)} \text{ is } \begin{cases} \text{bijective} & \text{if } 0 \leq i \leq \sigma(\mathbb{X}_j) - 2, \\ \text{surjective} & \text{if } \sigma(\mathbb{X}_j) - 1 \leq i \leq u_j - 1, \\ \text{bijective} & \text{if } u_j \leq i \leq u_{j+1} - 1. \end{cases}$$

Also we note that

$$x_0^d : B_i^{(j)} \to B_{i+d}^{(j)} \text{ is } \begin{cases} \text{injective} & \text{if } i \leq \sigma(\mathbb{X}_j) - 2, \\ \text{bijective} & \text{otherwise.} \end{cases}$$

Thus, considering the following commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow{x_0^d} & A_{i+d} \\ \downarrow & & \| \\ B_i^{(j)} & \xrightarrow{x_0^d} & B_{i+d}^{(j)} \end{array}$$

we have

$$x_0^d : A_i \to A_{i+d} \text{ is } \begin{cases} \text{injective} & \text{if } i \leq \sigma(\mathbb{X}_j) - 2, \\ \text{surjective} & \text{otherwise.} \end{cases}$$

□

The next result shows that the bounds on the graded Betti numbers that were given in Theorem 3.20 are also achieved by an ideal with the Strong Lefschetz property.

Corollary 4.6. *Let K be any field. A Hilbert function $\underline{h}$ occurs for some graded Artinian K-algebra with the Weak Lefschetz property if and only if it occurs for one with the Strong Lefschetz property, and these Hilbert functions are characterized in Proposition 3.5. Furthermore, the bound on the graded Betti numbers obtained in Theorem 3.20 is achieved by an algebra with the Strong Lefschetz property.*

Proof. The only thing that needs to be observed is that the extra condition on $\bar{J}_1$ imposed in Theorem 3.20, namely

$$[\operatorname{tor}_i^{\bar{R}}(\bar{R}/\bar{J}_1, K)]_{i+j} = \beta_{i,j}(\bar{h}, \bar{R}) \quad \text{for all integers } i, j,$$

can be imposed in the context of Proposition 4.5: simply take $\bar{J}_1$ to be a lex-segment ideal. $\square$

We end with a natural question.

Question 4.7. Is there a set of graded Betti numbers that occurs for algebras with the Weak Lefschetz property but not the Strong Lefschetz property?

We conjecture the answer to this question to be "no."

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