Exponential Function Word Problems (pages 16-17), Solutions

Exponential growth is modelled by

$$y = y_0 e^{kt}$$

There are four variables, the initial amount, y_0, the time t, the growth factor k, and the current amount y. You should be comfortable with finding any one of these four, given the other three. You should also try to understand how changing any one of these affects the others. (For instance, if k is increased, and y_0 and t are both fixed, will y increase, decrease, stay the same, etc?)

1. \$10,000 is invested at an annual interest rate of 5% compounded continuously.

(a) How long will it take for this initial investment to double in value?

(b) How long will it take for this initial investment to triple?

(c) How long will it take for this initial investment to quadruple?

(d) Can you see the pattern? How long to reach 5 times the initial amount? 7 times the initial amount? etc.

Solution:
Let $P(t)$ denote the amount of money after t years. Then $P(t) = 10000\, e^{0.05t}$.
(a) How long to double? We wish to find t so that $P(t) = 20000$, so

$$20000 = 10000 e^{0.05t} \implies 2 = e^{0.05t}$$

so

$$\ln 2 = 0.05t$$

therefore, $t = \ln 2 / 0.05$.
(b) How long to triple? We wish to find t so that $P(t) = 30000$, so

$$30000 = 10000 e^{0.05t} \implies 3 = e^{0.05t}$$

so

$$\ln 3 = 0.05t$$

therefore, $t = \ln 3 / 0.05$.
(c) How long to quadruple? We wish to find t so that $P(t) = 40000$, so

$$40000 = 10000 e^{0.05t} \implies 4 = e^{0.05t}$$

so

$$\ln 4 = 0.05t$$

therefore, $t = \ln 4/0.05$.
(d) The above cases lead us to believe that the time required for the invest to grow by a factor of k will be $t = \ln k/0.05$. Indeed, we wish to find t so that $P(t) = 10000 \cdot k$, so

$$10000 \cdot k = 10000e^{0.05t} \Longrightarrow k = e^{0.05t}$$

so

$$\ln k = 0.05t$$

therefore, $t = \ln k/0.05$.

2. Repeat the previous problem, but this time with an initial investment of \$500. Also, do it with an initial investment of \$250,000. How do your answers change? Can you see why?
Solution:
The answers do not change at all. To see why, lets look at the doubling case, and let P_0 denote the initial value, so that $P(t) = P_0\, e^{0.05t}$. We wish to find t so that $P(t) = 2P_0$. Thus,

$$2P_0 = P_0\, e^{0.05t} \Longrightarrow 2 = e^{0.05t}$$

Notice that the initial value drops out. Thus, we find the doubling time is $t = \ln 2/0.05$, regardless of how much we started with. The only thing that affects the doubling time is the interest rate.

3. A recent college graduate decides he would like to have \$20,000 in five years to make a down payment on a home.

(a) How much money will he need to invest today in order to have \$20,000 in five years, given that he can invest at an annual interest rate of 4% compounded continuously?

(b) Suppose instead he can invest at an annual interest rate of 6%. How much does he need to invest? Will he need to make a larger or a smaller initial investment than in part (a)? (First think through this without calculations. Then find the exact answer. Be sure to check that these are consistent.)

(c) Suppose the interest rate is 4% again, but now he would like to have the \$20,000 in *four* years. How much does he need to invest? Will he need to make a larger or a smaller initial investment than in part (a)? (Again, first do this without calculations, then find the exact answer.)

(d) Suppose the interest rate is 4% again, but now he would like to have \$30,000 in five years. How much does he need to invest? Will he need to make a larger or a smaller initial investment than in part (a)? (Again, first do this without calculations, then find the exact answer.)

Solution:
(a) Here, $P(t) = P_0\, e^{0.04t}$. We are not given P_0, but we know that $P(5) = 20000$, so

$$20000 = P_0 e^{0.04 \cdot 5} = P_0 e^{0.2} \Longrightarrow P_0 = 20000/e^{0.2} = 16374.62$$

(Calculations with money should always be taken to the cents digit.)
(b) The investment earns at a higher interest rate, so an initial investment at 6% will grow faster than

an initial investment at 4%, so less money is required today to guarantee \$20,000 in five years. Based on our answer to (a), we guess that around \$14,000 should suffice. To find the exact value, we use $P(t) = P_0\, e^{0.06t}$ and $P(5) = 20,000$, so

$$20000 = P_0 e^{0.06 \cdot 5} = P_0 e^{0.3} \Longrightarrow P_0 = 20000/e^{0.3} = 14816.36$$

(Our guess was a little low, but roughly in the correct range)
(c) The investment has less time to grow, so we need to start with a larger initial investment to guarantee \$20,000 in four years. Based on our answer to (a), we guess that around \$17,000 should suffice. To find the exact value, we use $P(t) = P_0\, e^{0.04t}$ and $P(4) = 20,000$, so

$$20000 = P_0 e^{0.04 \cdot 4} = P_0 e^{0.16} \Longrightarrow P_0 = 20000/e^{0.16} = 17042.88$$

(Our guess was very good this time!)
(d) The investment has to grow to a larger amount, so we need to start with a larger initial investment to guarantee \$30,000 in five years. Based on our answer to (a), we guess that around \$25000 should suffice. To find the exact value, we use $P(t) = P_0\, e^{0.04t}$ and $P(5) = 30,000$, so

$$30000 = P_0 e^{0.04 \cdot 5} = P_0 e^{0.20} \Longrightarrow P_0 = 30000/e^{0.20} = 24561.92$$

(Our guess was pretty good this time,)

4. The half-life of caffeine is 5 hours. This means the amount of caffeine in your bloodstream is reduced by 50% every 5 hours. A grande French Roast has 330 milligrams of caffeine. Let $Q(t)$ denote the amount of caffeine in your system t hours after consuming your grande French Roast. For simplicity, assume the entire grande French Roast is consumed instantly.

(a) How many milligrams of caffeine will be in your system after 5 hours? After 10 hours? After 15 hours? (Think! This part should not require a lot of work.)

(b) $Q(t) = Q_0 e^{-kt}$. Find Q_0 and k.

(c) How many milligrams of caffeine will be in your system after 2 hours?

Solution:
(a) The amount of caffeine is cut in half every 5 hours, so after 5 hours, there will be $330/2 = 165$ milligrams. In another 5 hours the amount is cut in half again, so after a total of 10 hours, $165/2 = 82.5$ milligrams remain. In another 5 hours the amount is cut in half again, so after a total of 15 hours, $82.5/2 = 41.25$ milligrams will remain.
(b) Q_0 is the initial amount, so $Q_0 = 330$. To find k, we use $Q(5) = 330/2 = 165$, so

$$330e^{-5k} = 165 \Longrightarrow 330 = 165e^{5k} \Longrightarrow 2 = e^{5k}$$

so $k = \ln(2)/5$.
(c) The trick in part (a) only works if the time is an integer multiple of the doubling time, so we'll have to do a calculation for part (c). From (b) we know

$$Q(t) = 330e^{-\frac{\ln 2}{5}t}$$

so

$$Q(2) = 330e^{-\frac{\ln 2}{5} \cdot 2} = 250.09\ldots$$

5. A bacteria culture triples in size every 7 hours. Three hours from now, the culture has 8,000 bacteria. If $Q(t)$ denotes the number of bacteria, then $Q(t) = Q_0 e^{kt}$ for some number Q_0 and some number k.

(a) Determine Q_0 and k.

(b) How many bacteria are there at time $t = 0$?

(c) How many bacteria are there after ten hours? Do you see the "easy" way to solve this part?

Solution:
(a) Whatever the initial amount is, it will be tripled in 7 hours, so

$$3Q_0 = Q_0\, e^{7k} \Longrightarrow 3 = e^{7k} \Longrightarrow k = \frac{\ln 3}{7}$$

so

$$Q(t) = Q_0 e^{\frac{\ln 3}{7}\, t}$$

Now, $Q(3) = 8000$ so

$$8000 = Q_0 e^{\frac{\ln 3}{7} \cdot 3}$$

so $Q_0 = 8000/e^{\frac{3 \ln 3}{7}} = 4995.85\ldots$ We can't have a fractional number of bacteria, so we'll round to get $Q_0 = 4996$.
(b) The number of bacteria at time $t = 0$ is $Q_0 = 4996$.
(c) Since we've gone through the trouble of finding k and Q_0, we can just plug in to get $Q(10)$:

$$Q(10) = 4996e^{\frac{\ln 3}{7} \cdot 10} = 24000$$

However, it is perhaps simpler to note that since the population triples every 7 hours, then the population at $t = 10$ is three times the population at $t = 3$, which is therefore $3 \cdot 8000 = 24000$.