MA123, Chapter 1: Equations, functions, and graphs (pp. 1-15, Gootman)

`Chapter Goals:`

- Solve an equation for one variable in terms of another.
- What is a function?
- Find inverse functions.
- What is a graph?
- Understand linear and quadratic functions.
- Find the intersection point(s) of two graphs.
- Learn basic strategies for solving word problems.
- Understand piecewise defined functions.

`Assignments:` `Assignment 01`

► **Equations and solution(s) to equations:** One way in which humanity increases its understanding of the universe is by discovering relationships between various objects, concepts, quantities, and so on. Our understanding of a relationship between two quantities is sharpest when this relationship can be completely quantified and expressed in an equation. Roughly speaking, an **equation** is a statement that two mathematical expressions are equal. For instance, $x^3 - 2xy + y^2 = 5$ is an equation relating x and y. A set of numbers that can be substituted for the variables in an equation so that the equality is true is a **solution** for the equation. A solution is said to **satisfy** the equation.

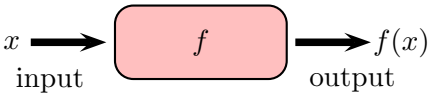
► **Equations into functions:** An equation in two (or more) variables can sometimes be solved in terms of one of the variables. This type of equation is closely related to the notion of a **function**.

Example 1: Solve the equation $x^3 + 2xy + 5y = 7$ for y in terms of x.

Observe that in the equation $y = \dfrac{7 - x^3}{2x + 5}$, the expression on the right-hand side can be viewed as a recipe that associates to any given value of x precisely one corresponding value for y.

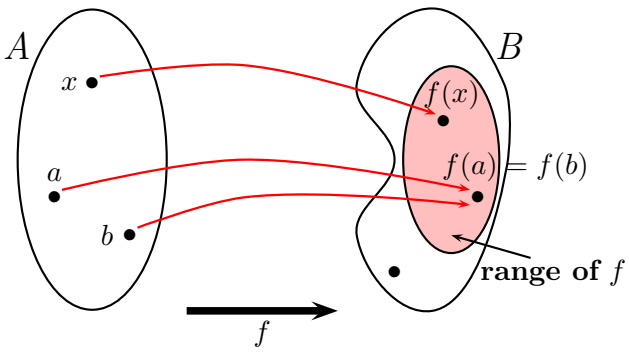
Definition of function:

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of f whereas the set B is called the **codomain** of f; $f(x)$ is called the **value of** f **at** x, or the **image of** x **under** f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain: range of $f = \{f(x) \,|\, x \in A\}$.



Machine diagram of f

Arrow diagram of f



Evaluating a function: The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. In the definition of a function the independent variable plays the role of a "placeholder". For example, the function $f(x) = 2x^2 - 3x + 1$ can be thought of as

$$f(\square) = 2 \cdot \square^2 - 3 \cdot \square + 1.$$

To evaluate f at a number (expression), we substitute the number (expression) for the placeholder.

Note: If f is a function of x, then $y = f(x)$ is a special kind of equation, in which the variable y appears alone on the left side of the equal sign and the expression on the right side of the equal sign involves only the other variable x. **Conversely**, when we have this special kind of equation, such as $y = e^x + x^3 - 3x + 5$, it is common to think of the right hand side as defining a function $f(x)$, and of the equation as being simply $y = f(x)$.

Example 2: Find the domain of the following functions:

$f(x) = \sqrt{3 - x}$ $\qquad$ $g(x) = \dfrac{1}{x^2 - 4}$ $\qquad$ $h(x) = \dfrac{1}{x} + \sqrt{x + 2}$

Example 3: If $f(x) = \sqrt{6x + 4}$, write an expression for: $\Big[f(1 + h) - f(1)\Big] \cdot \Big[f(1 + h) + f(1)\Big]$.

Example 4: If $P(x) = 3x^3 + 2x^2 + x + 11$ and we rewrite $P(x)$ in the form

$$P(x) = A + B(x - 1) + C(x - 1)(x - 2) + D(x - 1)(x - 2)(x - 3),$$

what are the values of A and B?

Example 5: If we rewrite the function $f(x) = \dfrac{3}{x(x-1)(x-2)}$ in the form:

$$f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2},$$

what are the values of A, B, and C?

► **Inverse of a function:** Recall that two functions $f(x)$ and $g(x)$ are said to be inverse of each other if

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$

Intuitively, inverse function pairs are in some sense "opposites." The three most familiar inverse function pairs are: "addition and subtraction", "multiplication and division", and "square and square root:"

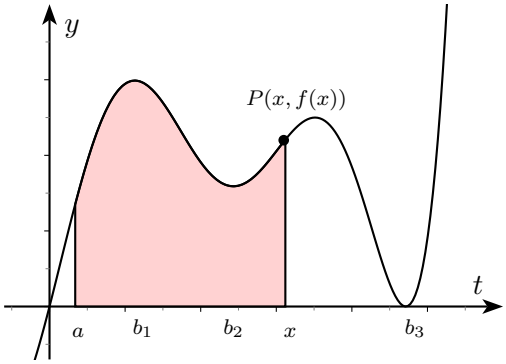
- $g(t) = t - c$ and $h(t) = t + c$ are inverse to each other, for any real number c.
- $g(t) = t/k$ and $h(t) = k \cdot t$ are inverse to each other, for any real number $k \neq 0$.
- $g(t) = \sqrt{t}$ and $h(t) = t^2$ are inverse to each other, provided $t \geq 0$.

Example 6: If $h(t) = 3t + 7$, find a function $g(t)$ such that $h(g(t)) = t$.

Example 7: If $F(x) = \sqrt{4x + 9} - 7$, find the inverse function $F^{-1}(x)$.

► **Function versus Algebraic Function:**

You are probably used to thinking of functions in terms of algebraic formulas, like $f(x) = x^2 + 3x + 8$ or $g(t) = \ln(t - 9)$. However, the definition of function only requires some notion in which objects in the domain are mapped to objects in the range. We will focus primarily on functions that can are described by algebraic formulas, but you should be aware that many functions in applications are not described algebraically. For example, at each age, t, in your life, you have a well defined height, H, so your height is a function of your age, $H = h(t)$. However, it is unlikely that you would be able to find an algebraic formula describing your height at any given age. Here are few other examples of functions that are not (directly) described in terms of algebraic formulas:



(a) The steepness of the curve $y = f(t)$ at $P(x, f(x))$ is a function of x, say $y = s(x)$. This is called the derivative of $f(t)$.

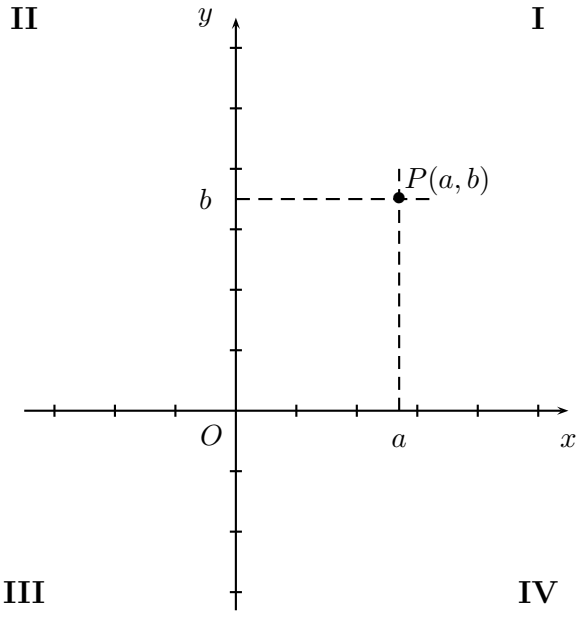
(b) The degree of curvature of the curve $y = f(t)$ at $P(x, f(x))$ is a function of x, say $y = c(x)$. This is called the second derivative of $f(t)$.

(c) The area of the shaded region, bounded between $t = a$ and $t = x$, below $y = f(t)$ and above the t-axis is a function of x, say $y = A(x)$. This is called the definite integral of $f(t)$.

In due time, we will find algebraic formulas describing the above types of functions. Even without algebraic formulas, we can still make qualitative statements regarding these three functions. For instance, $s(x)$ should be near zero when x is near each of the points b_1, b_2, and b_3 since $y = f(t)$ is flat at each of these points. On the other hand, $s(x)$ should be very large for $x > b_3$ since the graph is very steep in that region. Likewise, $A(x)$ will be an increasing function of x, since moving x further to the right adds more area under the curve. Not only is $A(x)$ increasing, we can also see that $y = A(x)$ increases rather quickly if x is near b_1 ($y = f(t)$ is very tall at that point so new area will be added quite rapidly) whereas $y = A(x)$ is barely increasing when x is near b_3 ($y = f(b_3) = 0$ so very little area is being added at that point.)

► **Cartesian plane and the graph of a function:**

Points in a plane can be identified with ordered pairs of numbers to form the *coordinate plane*. To do this, we draw two perpendicular oriented lines (one horizontal and the other vertical) that intersect at 0 on each line. The horizontal line with positive direction to the right is called the *x-axis*; the other line with positive direction upward is called the *y-axis*. The point of intersection of the two axes is the *origin O*. The two axes divide the plane into four *quadrants*, labeled I, II, III, and IV. The coordinate plane is also called *Cartesian plane* in honor of the French mathematician/philosopher René Descartes (1596-1650). Any point P in the coordinate plane can be located by a unique ordered pair of numbers (a, b) as shown in the picture. The first number a is called the *x-coordinate* of P; the second number b is called the *y-coordinate* of P.

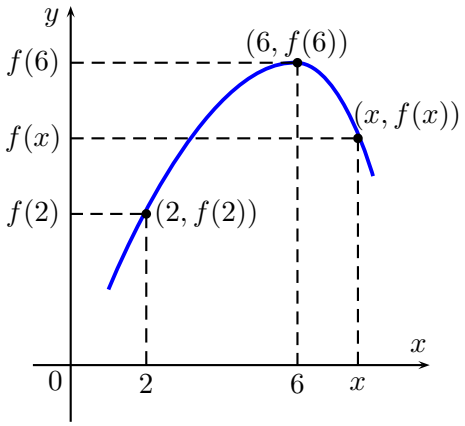


Graphing functions:

If f is a function with domain A, then the graph of f is the set of ordered pairs

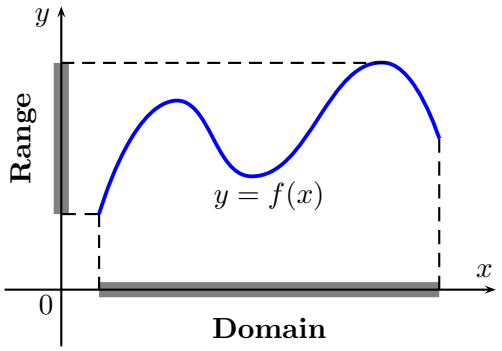
$$\textbf{graph of } f = \{(x, f(x)) \mid x \in A\}.$$

In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.



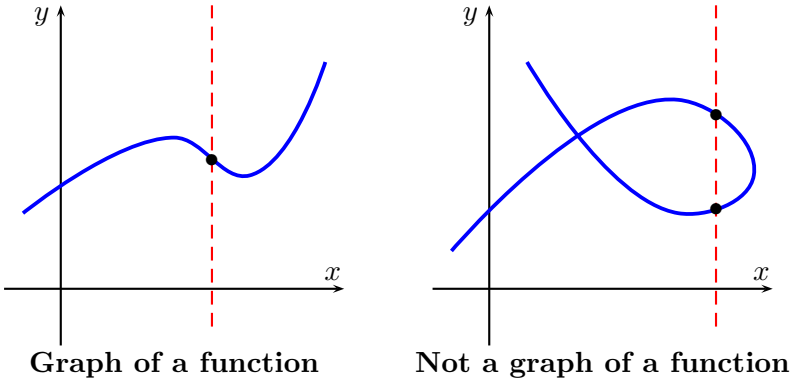
Obtaining information from the graph of a function:

The values of a function are represented by the height of its graph above the x-axis. So, we can read off the values of a function from its graph.
In addition, the graph of a function helps us picture the domain and range of the function on the x-axis and y-axis as shown in the picture:



The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions?

The vertical line test:
A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

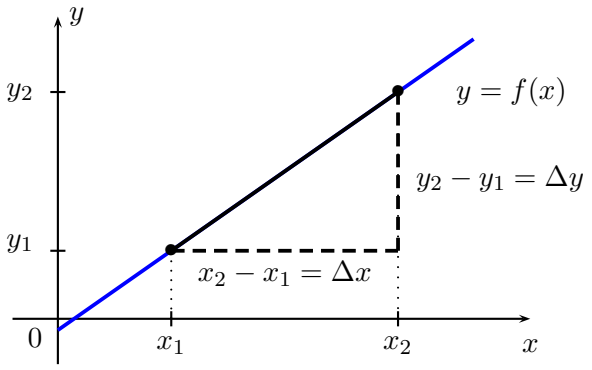


Graph of a function **Not a graph of a function**

► **Lines and Linear Functions:** A **linear function** is a function whose graph is a straight line.

Slope of a Line: The slope of a (non-vertical) line can be determined by any two distinct points, (x_1, y_1) and (x_2, y_2), on the line:

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}.$$



Point-Slope Form of a Line:

If a line has slope m and passes through the point (x_0, y_0) and (x, y) is another point on the line, then

$$m = \frac{y - y_0}{x - x_0} \quad \Longrightarrow \quad y - y_0 = m \cdot (x - x_0)$$

Slope-Intercept Form of a Line:

A linear function can be written in the form

$$f(x) = y = mx + b,$$

where m is the slope and b is the **y-intercept**.

You will need to be comfortable using both the point-slope and the slope-interecept forms of lines.

Example 8: Suppose a line passes throught the points $(3, 4)$ and $(-1, 6)$. Determine the values of A and B if the equation of the line is written in the following forms:

(a) $y = A + B(x + 1)$

(b) $y = A + Bx$

(c) $y = A + B(x + 2)$

Example 9: Suppose the linear function, f, satisfies $f(1.5) = 2$ and $f(3) = 5$. Determine $f(4)$.

► **Parabolas and Quadratic Functions:**

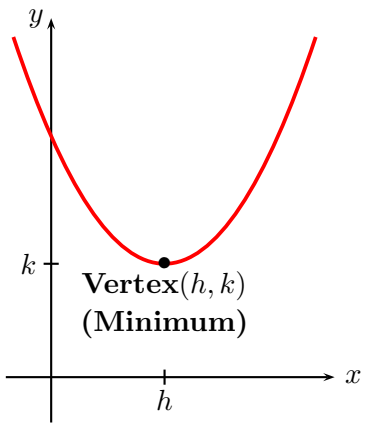
A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c,$$

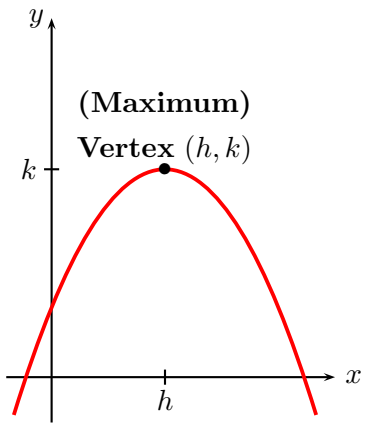
where a, b, and c are real numbers and $a \neq 0$.
The graph of any quadratic function is a parabola; it can be obtained from the graph of $f(x) = x^2$ by using shifting, reflecting, and stretching transformations.
Indeed, by completing the square a quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form** or **vertex form**

$$f(x) = a(x - h)^2 + k.$$

The graph of f is a parabola with vertex (h, k); the parabola opens upward if $a > 0$, or downward if $a < 0$.



$f(x) = a(x - h)^2 + k,$
$a > 0$



$f(x) = a(x - h)^2 + k,$
$a < 0$

Example 10:

(a) The parabola $y = x^2 - 15x + 54$ intersects the x-axis at the two points P and Q. What is the distance from P to Q?

(b) If we rewrite the inequality $x^2 - 15x + 54 < 0$ in the form $A < x < B$, what are the values of A and B?

► **Intersection points:** The graphs of two equations intersect at a point if and only if that point is a solution for both equations.

Example 11: Find the point(s) of intersection between the graph of the equation $4x^2 + 9y^2 = 36$ and

(a) the line with equation $y = -2$;

(b) the line with equation $y = 1$;

(c) the x-axis.

Example 12: Find all points where the graph of $(y-1)^2 - 2 = x$ crosses

(a) the y-axis;

(b) the line $y = x$.

► **Word Problems:** While there is no single routine that will solve every word problem, the following rough guideline should help you get started:

1. Read through the problem quickly: Try to get a general feeling for the type of problem. Ignore details in this first reading.
2. Read the problem more carefully: Identify all of the quantities involved. Which quantities are given? Which quantities need to be found?
3. Define names for all of the variable quantities: Draw and label a picture, if relevant. Fill in values for the given quanities.
4. Find relationships between the given quantities and the unknown quantities: This is where the outline needs to be abandoned, as this part will vary greatly from problem to problem. For now, the relationship will be an algebraic formula directly relating the knowns to the unknowns. Later in the course, the relationship may be more like "Now solve a standard calculus problem using the knowns and unknowns as inputs".
5. Solve the mathematical problem that you found in the previous step. For now, this will usually require solving an algebraic equation for a single unknown.
6. Interpret your answer: Does your answer make sense in the context of the original problem? In hindsight, could you have seen the answer more directly? How would your answer (or even method of solution) change if the known quantities took on other values? How would your answer (or even method of solution) change if some of the knowns and unknowns were swapped?

Example 13: The owner of a coffee shop decides to sell a blend of her two most popular types of coffee. The premium roast costs \$2.50 per pound and the classic roast costs \$1.75 per pound. How many pounds of the premium roast should she include in the blend if she wants 20 pounds of the blend coffee, and she wants to sell the blend at \$1.95 per pound?

Example 14: Suppose a fuel mixture is 4% ethanol and 96% gasoline. How much ethanol (in gallons) must you add to one gallon of fuel so that the new fuel mixture is 10% ethanol?

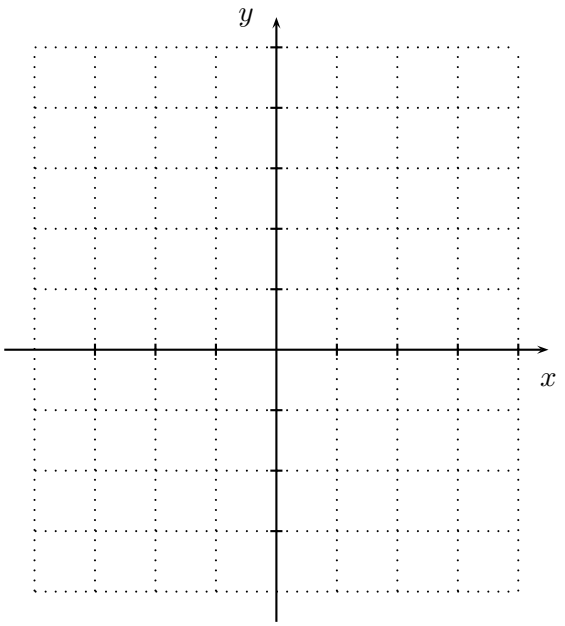
Example 15: The area of a right triangle is 7. The sum of the lengths of the two sides adjacent to the right angle of the triangle is 11. What is the length of the hypotenuse of the triangle?

► **Piecewise Defined Functions:** A **piecewise defined function** is a function given by several different rules. In order to evaluate a piecewise defined, you first need to decide which rule applies to the given value of the independent variable.

Example 16: Suppose

$$f(x) = \begin{cases} 2x+1, & \text{for } x \leq -1; \\ x^2, & \text{for } -1 < x < 2; \\ -3, & \text{for } 2 \leq x. \end{cases}$$

(a) Find each of $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(3)$, $f(4)$



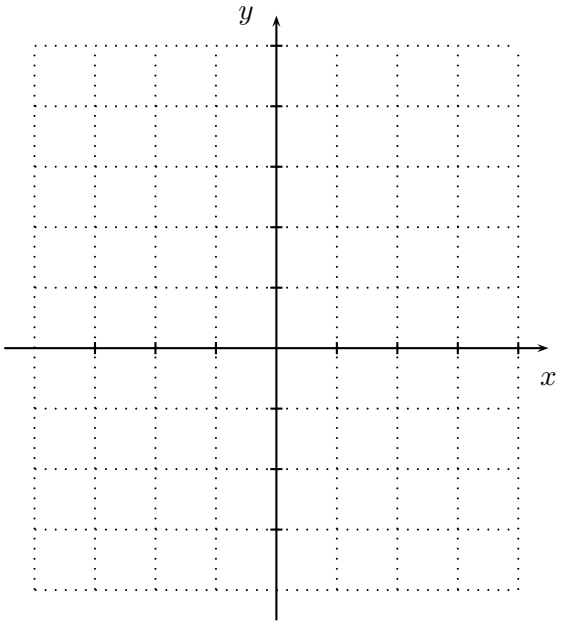
(b) Sketch the graph of $f(x)$.

Example 17: Sam's cell phone provider charges him \$35.00 per month for basic service, which includes 150 "anytime minutes". Sam is charged and extra \$0.75 for each minute beyond 150 minutes. Let t denote the number of minutes that Sam used in a given month and $B(t)$ denote the amount of Sam's cell phone bill. Write a piecewise defined function for $B(t)$.

Example 18 (Greatest Integer Function):

The **greatest integer function** (a.k.a, **unit step function**), denoted $[[x]]$, associates to each real number x the greatest integer less than or equal to x.

(a) Find each of $[[0]]$, $[[1]]$, $[[2]]$, $[[1.2]]$, $[[1.97]]$, $[[-1.8]]$



(b) Sketch the graph of $f(x)$.

The greatest integer function will appear several times througout the course. You may want to commit its graph to memory.