MA 113 — Calculus I Fall 2011
Exam 1 Solutions September 20, 2011

- Each question is followed by space to write your answer. Write your solutions neatly in the space below the question.
- Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may not receive credit.*
- You must give exact answers, rather than decimal approximations. *Approximations of the answers will not receive credit.*
- You may use a calculator, but not one which has symbolic manipulation capabilities.
- Turn off your cell phones, and do not wear ear-plugs during the exam.
- No books or notes may be used.
- Additional paper for scratch work is available upon request.

Name: ______________________________

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		10=5+5
2		12=8+2+2
3		10=5+5
4		16=8+8
5		14=3+5+6
6		6
7		4
8		14=6+8
9		14=6+8
		100

1. Consider the function $f(x) = \dfrac{8}{x-9}$ and $g(x) = x^2 - 16$.

   (a) Evaluate $g(f(1))$.

$$g(f(1)) = g(-1) = (-1)^2 - 16 = -15$$

   (b) What is the domain of the composite function $h(x) = f(g(x))$? Justify your answer.

$$h(x) = \frac{8}{(x^2-16)-9} = \frac{8}{x^2-25} = \frac{8}{(x-5)(x+5)} \; .$$

Since division by zero is the only thing we must avoid in this case, the domain of f is given by

$$D(f) = \{x \in \mathbb{R} : \; x \neq 5 \text{ and } x \neq -5\} \; .$$

2. Let $f(x) = \dfrac{2x-1}{3x+2}$.

(a) If f^{-1} is the inverse function of f, give a formula for $f^{-1}(x)$.

$$\begin{aligned} y &= \frac{2x-1}{3x+2} \\ y(3x+2) &= 2x-1 \\ 3yx+2y &= 2x-1 \\ 3yx-2x &= -2y-1 \\ x(3y-2) &= -2y-1 \\ x &= \frac{-2y-1}{3y-2} \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{-2x-1}{3x-2} = -\dfrac{2x+1}{3x-2}$

(b) State the domain of f^{-1}.

The domain of f^{-1} is $D(f^{-1}) = \{x \in \mathbb{R} : \ x \neq 2/3\}$.

(c) State the range of f^{-1} , using the relationship between f and f^{-1}.

The range of f^{-1} coincides with the domain of f, so

$$R(f^{-1}) = \{y \in \mathbb{R} : \ y \neq -2/3\} \ .$$

3. Let f and g be two functions satisfying the following conditions:

$$\lim_{x\to -3} f(x) = -6 \quad , \quad \lim_{x\to -3} g(x) = 9$$

Use the limit laws to compute the following limits.

(a) $\displaystyle\lim_{x\to -3}\left(\frac{f(x)}{2} + g(x)\,\log_3(x^2)\right)$.

$$\lim_{x\to -3}\left(\frac{f(x)}{2} + g(x)\,\log_3(x^2)\right) = \frac{-6}{2} + 9\cdot\log_3(9) = -3 + 18 = 15 \ .$$

(b) $\displaystyle\lim_{x\to -3}\sqrt{2xf(x)}$.

Since $\displaystyle\lim_{x\to -3} 2xf(x) = 2(-3)(-6) = 36 > 0$,

$$\lim_{x\to -3}\sqrt{2xf(x)} = \sqrt{36} = 6 \ .$$

4. Calculate each of the following limits, if they exist. If a limit does not exist, write "DNE" in the space provided, together with an explanation of why the limit does not exist.

(a) $\lim_{x \to 0} \dfrac{\sqrt{25-x}-5}{x}$

For $x \neq 0$, it holds that

$$\begin{aligned} \frac{\sqrt{25-x}-5}{x} &= \frac{\sqrt{25-x}-5}{x} \, \frac{\sqrt{25-x}+5}{\sqrt{25-x}+5} \\ &= \frac{(25-x)-25}{x(\sqrt{25-x}+5)} \\ &= \frac{-1}{\sqrt{25-x}+5} \, . \end{aligned}$$

Therefore,

$$\lim_{x \to 0} \frac{\sqrt{25-x}-5}{x} = \lim_{x \to 0} \frac{-1}{\sqrt{25-x}+5} = \frac{-1}{\sqrt{25}+5} = -\frac{1}{10}$$

(b) $\lim_{h \to 0} \dfrac{\frac{5}{x+h}-\frac{5}{x}}{h}$

For $h \neq 0$, we have

$$\frac{\frac{5}{x+h}-\frac{5}{x}}{h} = \frac{\frac{5x-5(x+h)}{x(x+h)}}{h} = \frac{-5h}{x(x+h)h} = \frac{-5}{x(x+h)} \, .$$

Therefore,

$$\lim_{h \to 0} \frac{\frac{5}{x+h}-\frac{5}{x}}{h} = \frac{-5}{x(x+h)} = -\frac{5}{x^2} \, .$$

5. Suppose that the height (in meters) of a ball, t seconds after it is launched from the ground, is given by $H(t) = -5t^2 + 15t$.

   (a) Find the average velocity of the ball over the time interval $2 \leq t \leq 3$.

$$\frac{H(3) - H(2)}{3 - 2} = \frac{0 - 10}{1} = -10$$

   The average velocity is -10 meters per second.

   (b) Find the average velocity of the ball over the time interval $[2, t]$, where $t > 2$. **Simplify your answer.**

   When $t \neq 2$,

$$\begin{aligned} \frac{H(t) - H(2)}{t - 2} &= \frac{-5t^2 + 15t - 10}{t - 2} = \frac{-5(t^2 - 3t + 2)}{t - 2} \\ &= \frac{-5(t - 2)(t - 1)}{t - 2} = -5(t - 1) \ . \end{aligned}$$

   (c) Use your answer in (b) to find the instantaneous velocity of the ball 2 seconds into its flight.

   We have

$$\lim_{t \to 2} \frac{H(t) - H(2)}{t - 2} = \lim_{t \to 2} -5(t - 1) = -5 \ ,$$

   so the instantaneous velocity is -5 meters per second.

6. Use the Intermediate Value Theorem to argue that the equation $5^x - 6x = 0$ has a solution in the interval $(0, 1)$. Make sure to confirm that the conditions of this theorem are satisfied.

   Letting $f(x) = 5^x - 6x$, we see that f is continuous on $\mathbb{R}$ (and, hence, on $[0, 4]$) because it is the difference of an exponential function and a polynomial, both of which are continuous on $\mathbb{R}$. We also see that $f(0) = 1 - 0 = 1 > 0$ and $f(1) = 5 - 6 = -1 < 0$. Since $-1 < 0 < 1$, the Intermediate Value Theorem guarantees a solution x of $f(x) = 5^x - 6x = 0$ in the interval $(0, 1)$.

7. Suppose that g is a function such that $g'(1) = -5$, and the graph of g contains the point $(1, 3)$. Give an equation of the tangent line to the graph of g at the point $(1, 3)$.

   The equation of the tangent line, first in point-slope form and then in slope-intercept form, is
   $$y = -5(x - 1) + 3 = -5x + 8 \ .$$

8. (a) State the formal limit definition of the derivative of a function at a point a.

A function f has a derivative at a number a provided the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If this limit exists, we denote it by $f'(a)$.

Alternatively, we can define $f'(a)$ by $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, provided this limit exists.

(b) Let $f(x) = \begin{cases} x^2 & , \ x < 0 \\ 2x & , \ x \geq 0 \end{cases}$.

Using the definition of a derivative, explain why $f'(0)$ does not exist.

We have the following one-sided limits:

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0$$
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{2x}{x} = \lim_{x \to 0^-} 2 = 2 \ .$$

Because these two limits disagree, f is not differentiable at 0.

9. (a) Suppose that the domain of the function f is $\mathbb{R}$. Formally define what it means for f to be continuous at a.

A function f is continuous at a number a provided $\lim_{x \to a} f(x) = f(a)$.

(b) Let

$$f(x) = \begin{cases} bx + 9, & \text{if } x < 5, \\ c, & \text{if } x = 5, \\ x^2 + b, & \text{if } x > 5. \end{cases}$$

Give all pairs of numbers, b and c, so that f is continuous at $a = 5$. If no such pairs exist, state that "f cannot be made continuous at $a = 5$".

It holds that $f(5) = c$. We also have the two one-sided limits

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} bx + 9 = 5b + 9$$
$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} x^2 + b = 25 + b$$

If f is to be continuous at 5, these two limits must be equal, and agree with the value of f at 5.

$$5b + 9 = 25 + b \quad , \quad 4b = 16 \quad , \quad b = 4$$

With $b = 4$, the common limit is 29. Therefore, $b = 4$, $c = 29$ is the only pair of numbers for which f is continuous at 5.