Exam 1

Form A Solutions

Multiple Choice Questions

1. Find the exact value of the expression

$$\log_5 100 + \log_5 25 - 2\log_5 2.$$

A. 4

B. 5

C. 6

D. 7

E. 8

Solution:

$$\begin{aligned}\log_5 100 + \log_5 25 - 2\log_5 2 &= \log_5(25 \times 4) + \log_5 5^2 - \log_5 2^2 \\ &= \log_5 25 + \log_5 4 + 2 - \log_5 4 \\ &= 2 + 2 \\ &= 4\end{aligned}$$

2. If $f(x) = x + 5$ and $h(x) = 4x - 10$, find a function $g(x)$ so that $g(f(x)) = h(x)$.
   A. $g(x) = 4x + 30$
   B. $g(x) = 4x$
   C. $g(x) = x - 30$
   D. $g(x) = 4x - 30$
   E. $g(x) = x + 30$

Solution: $g(f(x)) = g(x+5) = 4x - 10$. The simplest way to get $4x$ is to multiply by 4. If we just multiply by 4, i.e. if $g(x) = 4x$, then $g(f(x)) = 4(x+5) = 4x + 20$. We need $4x - 10$ so we need to subtract 30 and take $g(x) = 4x - 30$. Then

$$g(f(x)) = g(x+5) = 4(x+5) - 30 = 4x - 10 = h(x).$$

3. Find the inverse function of $f(x) = \frac{x+1}{4x+1}$.
   A. $f^{-1}(x) = -\frac{4x+1}{x-1}$
   B. $f^{-1}(x) = \frac{x}{4x-1}$
   C. $f^{-1}(x) = \frac{4x+1}{x+1}$
   D. $f^{-1}(x) = \frac{x+1}{\frac{1}{4}x+1}$
   E. $f^{-1}(x) = -\frac{x-1}{4x-1}$

Solution: To find the inverse function set $y = \frac{x+1}{4x+1}$ and solve for x.

$$\begin{aligned} y &= \frac{x+1}{4x+1} \\ y(4x+1) &= x+1 \\ 4xy - x &= -y+1 \\ x(4y-1) &= -y+1 \\ x &= \frac{-y+1}{4y-1} = -\frac{y-1}{4y-1} \end{aligned}$$

Thus, the inverse function is the function $f^{-1}(x) = -\frac{x-1}{4x-1}$.

4. Evaluate the limit

$$\lim_{x \to 1} (x+5)^3 \left(x^2 - 6\right)$$

A. -1090

B. -1080

C. -1070

D. -448

E. 320

Solution: To find this limit, simply use substitution:

$$\lim_{x \to 1} (x+5)^3 \left(x^2 - 6\right) = (1+5)^3(1^2 - 6) = -1080$$

5. Given that $\lim_{x \to a} f(x) = -3$, $\lim_{x \to a} g(x) = -4$, and $\lim_{x \to a} h(x) = 2$, find

$$\lim_{x \to a} \left((h(x))^2 - f(x)g(x)\right).$$

A. 16

B. 17

C. 22

D. -8

E. 0

Solution: We use the basic laws of limits to compute this limit:

$$\begin{aligned}
\lim_{x \to a} \left((h(x))^2 - f(x)g(x)\right) &= \lim_{x \to a}(h(x))^2 - \lim_{x \to a}(f(x)g(x)) \\
&= \left(\lim_{x \to a} h(x)\right)^2 - \lim_{x \to a} f(x) \lim_{x \to a} g(x) \\
&= (2)^2 - (-3)(-4) \\
&= -8
\end{aligned}$$

6. If $1 \leq f(x) \leq x^2 + 2x + 2$, for all x, find $\lim_{x \to -1} f(x)$.

   A. $-1/8$

   B. $-1/16$

   C. 1

   D. 8

   E. Does not exist

Solution: This looks like a situation where we can use the Squeeze Theorem, but we need to know $\lim_{x \to -1} x^2 + 2x + 2$. By substitution, this limit is 1, so by the Squeeze Theorem

$$1 = \lim_{x \to -1} 1 \leq \lim_{x \to -1} f(x) \leq \lim_{x \to -1} x^2 + 2x + 2 = 1$$

Therefore, $\lim_{x \to -1} f(x) = 1$

7. Simplify the following: $\sin(2 \arctan(x)) = \sin(2 \tan^{-1}(x))$.

   A. $\frac{2x}{1 + x^2}$

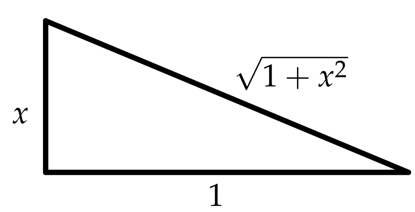
   B. $\frac{2x}{\sqrt{1 + x^2}}$

   C. $\frac{x}{\sqrt{1 + x^2}}$

   D. $\frac{x}{1 + x^2}$

   E. $\frac{2}{1 + x^2}$

Solution:



Since the tangent is $\frac{x}{1}$, we have that the two legs are x units and 1 unit. The hypotenuse is then $\sqrt{1 + x^2}$.

$$\begin{aligned} \sin(2\arctan(x)) &= 2\sin(\arctan(x))\cos(\arctan(x)) \\ &= 2\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right) \\ &= \frac{2x}{1+x^2} \end{aligned}$$

8. Find the equation of the line passing through the points $(-1, 3)$ and $(2, 9)$.

   A. $y = 2x + 1$

   B. $y = 2x + 4$

   C. $y = 2x + 5$

   D. $y = \frac{1}{2}x + \frac{7}{2}$

   E. $y = \frac{1}{2}x + \frac{5}{2}$

Solution: First, find the slope of the line connecting the two points:

$$m = \frac{9-3}{2-(-1)} = \frac{6}{3} = 2.$$

Now, use one of the points and the slope-point form of the equation of a line we get

$$\begin{aligned} y - 9 &= 2(x-2) \\ y &= 2x - 4 + 9 \\ y &= 2x + 5 \end{aligned}$$

9. Find $\arcsin\left(\sin\left(\frac{7\pi}{6}\right)\right)$.

   A. $\frac{7\pi}{6}$
   B. $-\frac{\pi}{6}$
   C. $\frac{\pi}{6}$
   D. $\frac{5\pi}{6}$
   E. $-\frac{5\pi}{6}$

Solution: $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$. The range of the arcsine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so

$$\arcsin\left(\sin\left(\frac{7\pi}{6}\right)\right) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

10. Solve the equation for x:

$$3^{x^2-3x} = 9^{x+7}$$

   A. $x = -7$ and $x = 2$
   B. $x = 2$ and $x = 7$
   C. $x = -2$ and $x = 7$
   D. $x = 2 \pm \sqrt{11}$
   E. There is no solution.

Solution:

$$3^{x^2-3x} = 9^{x+7} = 3^{2^{x+7}}$$
$$3^{x^2-3x} = 3^{2x+14}$$
$$\log_3 3^{x^2-3x} = \log_3 2^{2x+14}$$
$$x^2 - 3x = 2x + 14$$
$$x^2 - 5x - 14 = 0$$
$$(x+2)(x-7) = 0$$

Thus, the solutions are $x = -2$ and $x = 7$.

11. The population of a city at time t is $P(t) = 500e^{.075t}$. When will the population be four times larger than $P(0)$?

A. $\dfrac{\ln(0.075)}{4}$

B. $\dfrac{\ln(4)}{0.075}$

C. $0.075\ln(4)$

D. $500\ln(4)$

E. None of the above

Solution: Substituting in $t = 0$, we find that $P(0) = 500$. Thus, we must find t so that $P(t) = 2000$.

$$\begin{aligned} 500e^{.075t} &= 2000 \\ e^{.075t} &= 4 \\ 0.075t &= \ln 4 \\ t &= \frac{\ln 4}{0.075} \end{aligned}$$

12. A stone is tossed in the air from ground level. Its height at time t is $h(t) = 45t - 4.9t^2$ meters. Compute the average velocity of the stone over the time interval $[1.5, 3.5]$.

A. 41 m/s

B. 30.3 m/s

C. 20.5 m/s

D. 10.7 m/s

E. None of the above

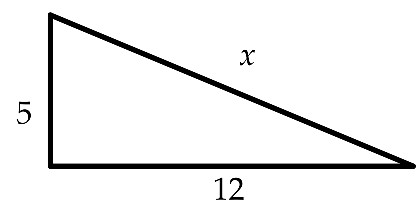
Solution:

$$\begin{aligned} v_{average} &= \frac{h(3.5) - h(1.5)}{3.5 - 1.5}\ m/s \\ &= \frac{97.475 - 56.475}{2}\ m/s \\ &= 20.5\ m/s \end{aligned}$$

Free Response Questions

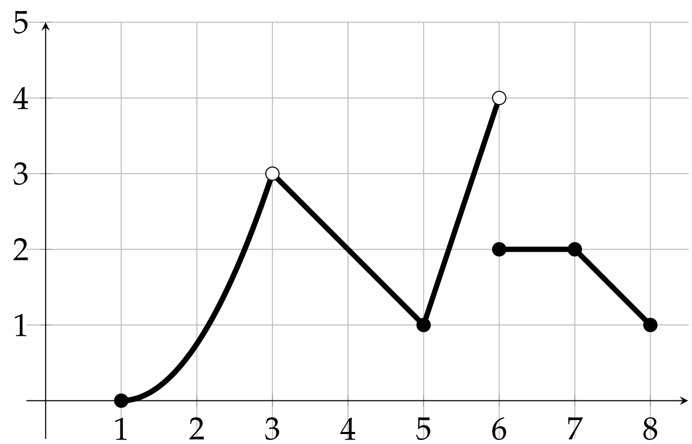
13. Given that $\tan(\theta) = \frac{5}{12}$ and $0 \leq \theta \leq \frac{\pi}{2}$, find $\sin(\theta)$, $\cos(\theta)$, $\sec(\theta)$, $\sin(2\theta)$ and $\cos(2\theta)$.

Solution:



Since the tangent is $\frac{5}{12}$, we have that the two legs are 5 units and 12 units. The hypotenuse is then $\sqrt{5^2+12^2} = \sqrt{169} = 13$. Then we can compute the other ratios.

$$\begin{aligned}
\sin(\theta) &= \frac{5}{13} \\
\cos(\theta) &= \frac{12}{13} \\
\sec(\theta) &= \frac{1}{\cos(\theta)} = \frac{13}{12} \\
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) = \frac{120}{169} \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}
\end{aligned}$$



14. The graph of $f(x)$ is shown above. Find the following limits if they exist.

(a) $\lim_{x\to 6^-} f(x)$

Solution:

$$\lim_{x\to 6^-} f(x) = 4.$$

(b) $\lim_{x\to 6^+} f(x)$

Solution:

$$\lim_{x\to 6^+} f(x) = 2$$

(c) $\lim_{x\to 6} f(x)$

Solution:

$$\lim_{x\to 6} f(x) \text{ does not exist.}$$

(d) $\lim_{x\to 3} f(x)$

Solution:

$$\lim_{x\to 3} f(x) = 3$$

(e) $\lim_{x\to 5} f(x)$

Solution:

$$\lim_{x\to 5} f(x) = 1$$

15. Find the limits or state that the limit does not exist. In each case, justify your answer. (Students who guess the answer based on a few values of the function will not receive full credit.)

(a) $\lim_{x \to 2} \frac{x^3 - 4x}{x - 2}$

Solution:

$$\begin{aligned} \lim_{x \to 2} \frac{x^3 - 4x}{x - 2} &= \lim_{x \to 2} \frac{x(x+2)(x-2)}{x-2} \\ &= \lim_{x \to 2} \frac{x(x+2)}{1} \\ &= 8 \end{aligned}$$

(b) $\lim_{x \to 3} 3x - 4 + \frac{x^2}{x-2}$

Solution:

$$\lim_{x \to 3} 3x - 4 + \frac{x^2}{x-2} = 3(3) - 4 + \frac{3^2}{3-1} = 14$$

(c) $\lim_{x \to 2^+} f(x)$ if $f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x \le 2 \\ -4x + 12 & \text{if } x > 2 \end{cases}$

Solution: Since we are taking the limit from the right, we need to look at the branch of the function for $x > 2$. So we have

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-4x + 12) = -4(2) + 12 = 4$$

16. Assume that the position of an object after t seconds is given by $f(t) = 10t^2 + 3t$ meters.

(a) Write an expression for the average velocity on the interval $[2, 2+h]$. Include units!

Solution:

$$\begin{aligned} v_{avg} &= \frac{f(2+h) - f(2)}{2+h-2}\ m/s = \frac{10(2+h)^2 + 3(2+h) - 46}{h}\ m/s \\ &= \frac{40h + 10h^2 + 3h}{h}\ m/s = 43 + 10h\ m/s \end{aligned}$$

(b) Compute the average velocity over the time intervals $[1.999, 2]$ and $[2, 2.001]$ to estimate the instantaneous velocity. Include units!

Solution: We plug in $h = -0.001$ and $h = +0.001$ into the formula we found in (a).

$$v_{avg}[1.999, 2] = 43 + 10(-0.001)\ m/s = 42.99\ m/s$$
$$v_{avg}[2, 2.001] = 43 + 10(0.001)\ m/s = 43.01\ m/s$$

(c) Take the limit as h approaches 0 of the expression you found in part (a) to find the instantaneous velocity of the object at time $t = 2$ seconds. Include units!

Solution: $v_{instantaneous} = \lim_{h \to 0}(43 + 10h)\ m/s = 43\ m/s$