(1) Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = 4 - x^2$.

(a) Determine $f(g(1))$ and $g(f(1))$.

(b) Find all numbers a such that $(f \circ g)(a)$ is defined.

(a) $f(g(x)) = f(4-x^2) = \frac{1}{4-x^2}$, thus $f(g(1)) = \underline{\underline{\frac{1}{3}}}$. 2

$g(f(x)) = g(\frac{1}{x}) = 4 - \frac{1}{x^2}$, thus $g(f(1)) = \underline{\underline{3}}$. 2

(b) $(f \circ g)(x) = \frac{1}{4-x^2}$ is not defined iff $4-x^2=0$ iff $x = \pm 2$.

Hence the domain of $f \circ g$ is $\{a \mid a \neq 2 \text{ and } a \neq -2\} =$
$(-\infty,-2) \cup (-2,2) \cup (2,\infty)$. 2

(a) $f(g(1)) =$ $\frac{1}{3}$, $g(f(1)) =$ 3

(b) $f \circ g$ is defined on: $\{a \mid a \neq \pm 2\}$

(2) Consider the rational function

$$f(x) = \frac{x+1}{x^2 - 2x - 3}.$$

(a) Use the limit rules to determine each of the following limits if it exists:

(i) $\lim_{x \to 2^+} f(x)$

(ii) $\lim_{x \to -1} f(x)$

(iii) $\lim_{x \to 3^-} f(x)$

(b) Which of the lines $x = 2$, $x = -1$, $x = 3$ are vertical asymptotes of the function f?

(a) $f(x) = \frac{x+1}{x^2-2x-3} = \frac{x+1}{(x+1)(x-3)}$

$= \frac{1}{x-3}$ if $x \neq -1$. Hence

(i) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{x-3} = \frac{1}{2-3} = -1.$ (3)

(ii) $\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{1}{x-3} = \frac{1}{-1-3} = -\frac{1}{4}$ (3)

(iii) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{1}{x-3} = \frac{1}{0^-} = -\infty$ (3)

(b) Since f is continuous at 2 and -1, only $x = 3$ is a vertical asymptote. (1)

(i) $\lim_{x \to 2^+} f(x) =$ -1 (ii) $\lim_{x \to -1} f(x) =$ $-\frac{1}{4}$ (iii) $\lim_{x \to 3^-} f(x) =$ ONE

(b) Vertical asymptote(s) are: $x = 3$

(3) Let f and g be two functions such that the following limits exist:

$$\lim_{x\to 10} g(x) = 5 \quad \text{and} \quad \lim_{x\to 10}[f(x) - 2g(x)] = -7.$$

Use the limit rules to show that the following limits exist and to calculate their value:

(a) $\lim_{x\to 10} \frac{x}{g(x)}$.

(b) $\lim_{x\to 10} f(x)$.

(a) The limit law for quotients gives

$$\lim_{x\to 10} \frac{x}{g(x)} \overset{2}{=} \frac{\lim_{x\to 10} x}{\lim_{x\to 10} g(x)} = \frac{10}{5} \overset{1}{=} 2.$$

(b) The limit laws for the product and the difference provide

$$-7 = \lim_{x\to 10}[f(x) - 2g(x)] \overset{2}{=} \lim_{x\to 10} f(x) - 2\cdot \lim_{x\to 10} g(x)$$

$$\overset{2}{=} \lim_{x\to 10} f(x) - 2\cdot 5.$$

Hence $\lim_{x\to 10} f(x) = -7 + 10 = 3.$ 1

(a) $\lim_{x\to 10} \frac{x}{g(x)} =$ 2

(b) $\lim_{x\to 10} f(x) =$ 3

(4) Consider the function $f(x) = (\frac{1}{\sqrt{x}} - 2)^{99}$.

(a) Determine the domain of f.

(b) Find all numbers a so that the function f is continuous at a.

(c) Determine $\lim_{x\to 1} f(x)$.

(a) $\sqrt{x}$ is defined if $x \geq 0$. (1)

Since $\frac{1}{0}$ is not defined, the domain of f is $\{x \mid x > 0\} = (0, \infty)$. (1)

(b) f is the composite of the following functions (1)

(1) $Y_1 = \sqrt{x}$ (root)

(1) $Y_2 = \frac{1}{Y_1} - 2$ (rational)

(1) $Y_3 = Y_2^{99}$ (power)

Since these functions are continuous on their domain and the composition of continuous functions is continuous, it follows that f is continuous on its domain. (1)

(c) Since f is continuous at 1, we get

$$\lim_{x\to 1} f(x) = f(1) = (-1)^{99} = \underline{\underline{-1}}.$$

(1) (1)

(a) The domain of f is $(0, \infty)$

(b) f is continuous on $(0, \infty)$

(c) $\lim_{x\to 1} f(x) =$ -1

(5) Let c be a number and consider the function

$$f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$$

(a) Find all numbers c such that the limit $\lim_{x \to 1} f(x)$ exists.

(b) Is there a number c such that f is continuous at 1? As always, justify your answer.

(a) $\lim_{x \to 1^-} f(x) \overset{(1)}{=} \lim_{x \to 1^-} (cx^2 - 5) \overset{(1)}{=} c - 5$

$\lim_{x \to 1^+} f(x) \overset{(1)}{=} \lim_{x \to 1^+} (\frac{1}{x} - 2c) \overset{(1)}{=} 1 - 2c$

The limit exists iff both one-sided limits agree, that is,

$c - 5 = 1 - 2c$ iff $3c = 6$

iff $\underline{\underline{c = 2}}$. (2)

(b) If f is continuous at 1, then $\lim_{x \to 1} f(x)$ must exist, which forces $c = 2$ by part (a). (1)

However, if $c = 2$, then

$\lim_{x \to 1} f(x) = 2 - 5 = -3 \neq 10 = f(1)$. (1)

Hence f is never continuous at 1. (1)

(a) $c =$ 2 (b) yes / no (*circle the correct answer*)

(6) A particle is moving in one direction along the x-axis so that its position in meters is given by $x(t) = t^2 + 3t$ after t seconds.

(a) Find the distance the particle traveled between 1 and 2 seconds.

(b) Determine the average velocity of the particle between 2 and 4 seconds.

(a) The distance is

$$\underbrace{x(2) - x(1)}_{2} \overset{1}{=} 2^2 + 3\cdot 2 - [1^2 + 3\cdot 1]$$
$$= 10 - 4$$
$$= \underline{\underline{6}}.\quad 1$$

(b) The average velocity is

$$\underbrace{\frac{x(4) - x(2)}{4-2}}_{2} \overset{1}{=} \frac{4^2 + 3\cdot 4 - 10}{2}$$
$$= \frac{28 - 10}{2}$$
$$= \underline{\underline{9}}.\quad 1$$

(a) The distance is 6 (b) The average velocity is 9

(7) Consider the function $f(x) = \dfrac{1}{x+2}$.

(a) Determine (as a function of a) the slope m_a of the secant line through the points $(2, f(2))$ and $(a, f(a))$ with $a \neq 2$ and $a \neq -2$. As always, simplify your answer.

(b) Find $\lim_{a \to 2} m_a$ if it exists.

(a) $$m_a \overset{2}{=} \frac{f(a) - f(2)}{a-2} \overset{1}{=} \frac{\frac{1}{a+2} - \frac{1}{4}}{a-2}$$

$$\overset{2}{=} \frac{\frac{4 - [a+2]}{4(a+2)}}{a-2}$$

$$\overset{1}{=} \frac{2-a}{4(a+2)(a-2)} = -\frac{a-2}{4(a+2)(a-2)}$$

$$\overset{1}{=} -\frac{1}{4(a+2)}.$$

(b) $$\lim_{a \to 2} m_a = \lim_{a \to 2} -\frac{1}{4(a+2)} = -\frac{1}{4(2+2)} = -\frac{1}{16}.$$ 2

(a) $m_a = -\dfrac{1}{4(a+2)}$ (b) $\lim_{a \to 2} m_a = -\dfrac{1}{16}$

(8) Using the definition, find the equation of the tangent line to the graph of the function $f(x) = \sqrt{x+3}$ at $x = 1$. Write your result in the form $y = mx + b$.

$$\text{②} \quad f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \overset{①}{=} \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

$$① = \lim_{h \to 0} \frac{4+h-2^2}{h\left[\sqrt{4+h}+2\right]}$$

$$① = \lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} \overset{①}{=} \frac{1}{4}.$$

The equation of the tangent line is

② $\quad y - f(1) = f'(1)(x-1)$, so

① $\quad y - 2 = \frac{1}{4}(x-1)$, thus

① $\quad y = \frac{1}{4}x + \frac{7}{4}.$

The equation of the tangent is $y = \frac{1}{4}x + \frac{7}{4}$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(9) (a) State the principle of mathematical induction. Use complete sentences.

(b) Prove by mathematical induction that, for all integers $n \geq 1$, the following equality is true:

$$\sum_{k=1}^{n} (6k+3) = 3n^2 + 6n.$$

(a) If a statement P_n depending on the integer $n \geq 1$ is true for $n = 1$ and,

5 for every integer $N \geq 1$, the assumption that P_N is true implies that P_{N+1} is

true, then P_n is true for every integer $n \geq 1$.

(b) Base step: $n = 1$

1 The left-hand side is $\sum_{k=1}^{1} (6k+3) = 6 \cdot 1 + 3 = 9$,

1 the right-hand side is $3 \cdot 1^2 + 6 \cdot 1 = 9$. Both sides agree, so the base step is established.

Induction step: Assume, for some $N \geq 1$, that $\sum_{k=1}^{N} (6k+3) = 3 \cdot N^2 + 6N$ is true.

1 To show: $\sum_{k=1}^{N+1} (6k+3) = 3(N+1)^2 + 6(N+1)$ $\quad (*)$

Separating the last summand on the left-hand side, it follows that

$$\sum_{k=1}^{N+1} (6k+3) \overset{1}{=} \underbrace{\sum_{k=1}^{N} (6k+3)}_{\text{by induction hypothesis}} + 6(N+1) + 3$$

$$2 \;= 3N^2 + 6N + 6N + 6 + 3 \overset{1}{=} 3N^2 + 12N + 9$$

Simplifying the right-hand side of $(*)$ we get

2 $3(N+1)^2 + 6(N+1) = 3(N^2 + 2N + 1) + 6N + 6 = 3N^2 + 12N + 9$.

Now we see that indeed both sides of $(*)$ agree, as desired.

(10) (a) State the Intermediate Value Theorem. Use complete sentences.

(b) Explain why and how you can use this theorem to show that the equation

$$x^5 - 6x - 2 = 0$$

has a root between -1 and 0.

(a) If a function f is continuous on the closed intervall $[a,b]$ and N is a number strictly between $f(a)$ and $f(b)$, then there is some number c in the open intervall (a,b) such that $f(c) = N$.

5

2 1

(b) The function $f(x) = x^5 - 6x - 2$ is polynomial, thus continuous on $\mathbb{R}$, so in particular on $[-1, 0]$.

It is $f(-1) = -1 + 6 - 2 = 3 > 0$, 1

$f(0) = -2 < 0$. 1

1 1

Hence, applying the IVT with $N = 0$, it follows that there is some c in $(0,1)$ with $f(c) = 0$, that is, c is the desired root.

1

(11) (a) State the definition of the derivative of a function f at a point a. Use complete sentences.

(b) Using the definition, determine the derivative of the function

$$f(x) = \begin{cases} 0 & \text{if } x \le 3 \\ \sqrt{x^2 - 9} & \text{if } x > 3 \end{cases}$$

(It is enough to mention one of the two limits.)

at 5 and 3 if it exists.

(a) The derivative of f at a is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ (or $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$), provided the limit exists.

3

(b) Consider the function $g(x) = \sqrt{x^2 - 9}$. If $h \neq 0$, then

$$\frac{g(a+h) - g(a)}{h} = \frac{\sqrt{(a+h)^2 - 9} - \sqrt{a^2 - 9}}{h}$$

$$2 \quad = \frac{a^2 + 2ah + h^2 - 9 - (a^2 - 9)}{h\left[\sqrt{(a+h)^2 - 9} + \sqrt{a^2 - 9}\right]}$$

$$2 \quad = \frac{2a + h}{\sqrt{(a+h)^2 - 9} + \sqrt{a^2 - 9}}.$$

Hence $f'(5) = g'(5) = \lim_{h \to 0} \frac{2 \cdot 5 + h}{\sqrt{(5+h)^2 - 9} + \sqrt{5^2 - 9}} = \frac{10}{2 \cdot \sqrt{16}} = \frac{5}{4}$. 2

Moreover, since $f(3) = g(3) = 0$, it follows that

1 $\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} \overset{1}{=} \lim_{h \to 0^+} \frac{g(3+h) - g(3)}{h} = \frac{2 \cdot 3 + h}{\sqrt{(3+h)^2 - 9} + \sqrt{3^2 - 9}} = \frac{6}{0^+} \overset{1}{=} \infty.$

Hence $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ does not exist, thus f is not differentiable at 3. 2

(b) (i) $f'(5) =$ $\frac{5}{4}$ (ii) $f'(3) =$ DNE