Exam 1

Form A Solutions

Multiple Choice Questions

1. Find $\lim_{x \to -\infty} \dfrac{12x^5 + 13x^4 - 14x}{12x^4 - 31x^2 + 12}$.
   A. 1
   B. 12
   C. $-11/7$
   D. ∞
   E. $-\infty$

Solution:

$$\lim_{x\to-\infty} \frac{12x^5 + 13x^4 - 14x}{12x^4 - 31x^2 + 12} = \lim_{x\to-\infty} \left(\frac{12x^5 + 13x^4 - 14x}{12x^4 - 31x^2 + 12} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \right)$$

$$= \lim_{x\to-\infty} \frac{12x + 13 - \frac{14}{x^3}}{12 - \frac{31}{x^2} + \frac{12}{x^4}}$$

$$= -\infty$$

2. Find $\lim_{x \to -\infty} \dfrac{x + |x|}{x + 1}$.
   A. 0
   B. 1
   C. 2
   D. -2
   E. $-\infty$

Solution: First note that for $x < 0$, $x + |x| = 0$. Then,

$$\lim_{x\to-\infty} \frac{x + |x|}{x + 1} = \lim_{x\to-\infty} \frac{0}{x+1} = 0$$

3. Find the inverse function of $f(x) = \dfrac{2x+1}{3x+2}$.

   A. $\dfrac{3x+2}{2x+1}$

   B. $\dfrac{\frac{1}{2}x+1}{\frac{1}{3}x+\frac{1}{2}}$

   C. $\dfrac{2x}{2-3x}$

   D. $\dfrac{2x-1}{2-3x}$

   E. $\dfrac{2x-1}{3x-2}$

Solution: Start with $y = \dfrac{2x+1}{3x+2}$ and solve for x, then swap the x and y variables.

$$\begin{aligned} y &= \frac{2x+1}{3x+2} \\ y(3x+2) &= 2x+1 \\ 3xy - 2x &= 1 - 2y \\ x(3y-2) &= 1-2y \\ x &= \frac{1-2y}{3y-2} = \frac{2y-1}{2-3y} \end{aligned}$$

Now, swap x and y

$$y = \frac{2x-1}{2-3x}$$

4. Evaluate the limit

$$\lim_{x\to 1} \left(x^3+4\right)^2 \left(x^2+9\right)$$

   A. 25

   B. 50

   C. 250

   D. 500

   E. 2500

Solution:

$$\lim_{x\to 1}\left(x^3+4\right)^2\left(x^2+9\right) = (1+4)^2(1+9) = 250$$

5. Given that $\lim_{x\to a} f(x) = -3$, $\lim_{x\to a} g(x) = -4$, and $\lim_{x\to a} h(x) = 2$, find

$$\lim_{x\to a}\left((3h(x))^2 - 2f(x)g(x)\right).$$

A. -12

B. 12

C. 36

D. 43

E. 60

Solution:

$$\lim_{x\to a}\left((3h(x))^2 - 2f(x)g(x)\right) = \left(\left(3\lim_{x\to a} h(x)\right)^2 - 2\lim_{x\to a} f(x)\lim_{x\to a} g(x)\right)$$
$$= 6^2 - 2(-3)(-4) = 12$$

6. If $2x+2 \le f(x) \le x^2+2x+2$, for all x, find $\lim_{x\to 0} f(x)$.

A. 2

B. 4

C. 5

D. 10

E. Does not exist

Solution: Since $2x+2 \le f(x) \le x^2+2x+2$ for all x, we then have that

$$\lim_{x\to 0} 2x+2 \le \lim_{x\to 0} f(x) \le \lim_{x\to 0} x^2+2x+2$$

The two outer limits go to 2 thus by the Squeeze Theorem $\lim_{x\to 0} f(x) = 2$.

7. By the Intermediate Value Theorem the equation $x^3 + 5x - 10 = 0$ has a root in which interval?

   A. $[-1, 0]$

   B. $[0, 1]$

   C. $[1, 2]$

   D. $[2, 3]$

   E. $[3, 4]$

Solution: Let $f(x) = x^3 + 5x - 10$. We just need to check the endpoints of each interval to see where the function changes sign. Briefly, $f(1) = -4$ while $f(2) = 8$ so by the Intermediate Value Theorem, $f(x)$ must be zero somewhere in the interval $[1, 2]$.

8. Find the equation of the line passing through the points $(-1, 2)$ and $(3, 10)$.

   A. $y = 2x - 2$

   B. $y = 2x + 4$

   C. $y = 2x - 10$

   D. $y = \frac{1}{2}x - \frac{3}{2}$

   E. $y = \frac{1}{2}x + \frac{5}{2}$

Solution: The slope is $m = \dfrac{10 - 2}{3 - (-1)} = 2$. Using the point-slope form of the equation of a line we find that $y - 2 = 2(x + 1)$ or $y = 2x + 4$.

9. Find $\arcsin\left(\sin\left(\dfrac{7\pi}{6}\right)\right)$.

   A. $\dfrac{7\pi}{6}$

   B. $-\dfrac{\pi}{6}$

   C. $\dfrac{\pi}{6}$

   D. $\dfrac{5\pi}{6}$

   E. $-\dfrac{5\pi}{6}$

Solution: $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$ and $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

10. Find the horizontal asymptote(s) for $f(x) = \dfrac{\sqrt{4x^2+7}}{8x+6}$.

A. $y = \frac{1}{4}$
B. $y = \frac{1}{2}$
C. $y = -\frac{1}{2}$ and $y = \frac{1}{2}$
D. $y = -\frac{1}{4}$ and $y = \frac{1}{4}$
E. The function has no horizontal asymptotes.

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+7}}{8x+6} = \lim_{x\to\infty} \frac{\sqrt{4x^2}}{8x} = \lim_{x\to\infty} \frac{2|x|}{8x} = \frac{1}{4}$$

Meanwhile,

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+7}}{8x+6} = \lim_{x\to-\infty} \frac{\sqrt{4x^2}}{8x} = \lim_{x\to-\infty} \frac{2|x|}{8x} = \lim_{x\to-\infty} \frac{-2x}{8x} = -\frac{1}{4}$$

Therefore, the two horizontal asymptotes for $f(x)$ are $y = \frac{1}{4}$ and $y = -\frac{1}{4}$.

11. Find $\displaystyle\lim_{x\to-1} \frac{2x^2+3x+1}{x^2-2x-3}$

A. $-\frac{3}{4}$
B. $-\frac{1}{4}$
C. $\frac{1}{4}$
D. $\frac{1}{2}$
E. $\frac{3}{4}$

Solution:

$$\lim_{x\to-1} \frac{2x^2+3x+1}{x^2-2x-3} = \lim_{x\to-1} \frac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim_{x\to-1} \frac{(2x+1)}{(x-3)} = \frac{1}{4}$$

12. A stone is tossed in the air from ground level. Its height at time t is $h(t) = 45t - 4.9t^2$ meters. Compute the average velocity of the stone over the time interval [1.5, 3.5].

   A. 41 m/s

   B. 30.3 m/s

   C. 20.5 m/s

   D. 10.7 m/s

   E. None of the above

Solution:

$$\text{Avg velocity} = \frac{h(3.5) - h(1.5)}{3.5 - 1.5} = \frac{97.475 - 56.475}{2} = 20.5$$

Free Response Questions

13. (a) What does it mean for a function $f(x)$ to be continuous at a point $x = a$? Use complete sentences.

Solution: The function $f(x)$ is continuous at $x = a$ if the following three conditions are met:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

(b) Consider the piecewise defined function

$$f(x) = \begin{cases} 5 & \text{if } 0 \le x < 1 \\ ax + 3 & \text{if } 1 \le x < 2 \\ x^2 - 2x + b & \text{if } 2 \le x \le 3 \end{cases}$$

where a and b are constants. Find the values of a and b for which $f(x)$ is continuous on $[0, 3]$.

Solution: By its definition, it is continuous everywhere except possibly at $x = 1$ and at $x = 2$ where the branches of the function change. We need to assure that it is continuous at these two points. We need the limit at $x = 1$ to exist so we need

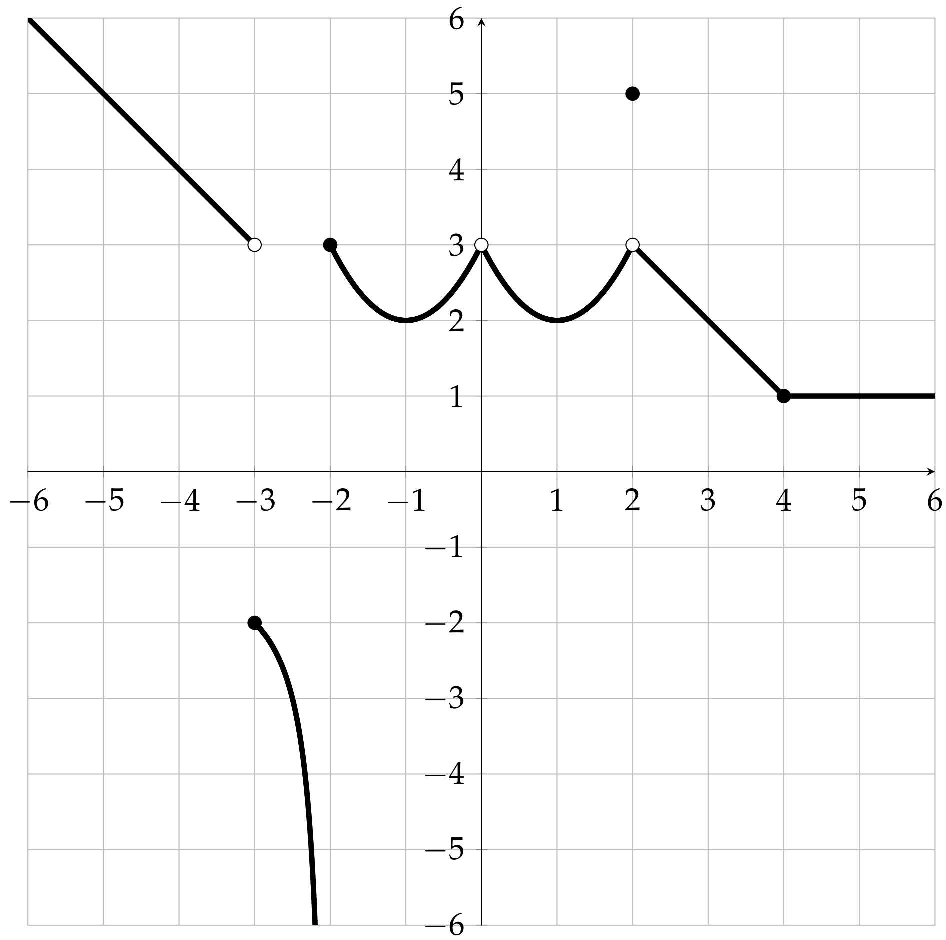
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 5 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax + 3$$

This gives us at $a + 3 = 5$ or $a = 2$.

We need the limit at $x = 2$ to exist also so we need

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax + 3 = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - 2x + b.$$

This gives us that $2a + 3 = 2 \cdot 2 + 3 = 7 = 4 - 4 + b$ or $b = 7$.



14. The graph of $f(x)$ is shown above. Find the following limits if they exist.

(a) $\lim_{x\to -3^+} f(x)$

Solution:

$$\lim_{x\to -3^+} f(x) = -2$$

(b) $\lim_{x\to -2^+} f(x)$

Solution:

$$\lim_{x\to -2^+} f(x) = 3$$

(c) $\lim_{x\to 2} f(x)$

Solution:

$$\lim_{x\to 2} f(x) = 3$$

(d)

Solution:

$$\lim_{x\to 0} f(x) = 3$$

(e) What are the x-values at which $f(x)$ is not continuous on $[-6, 6]$.

Solution: $f(x)$ is not continuous at $x = -3$ (jump discontinuity), at $x = -2$ (infinite discontinuity), at $x = 0$ because $f(0)$ is not defined, and at $x = 2$ which is a removable discontinuity.

15. Find the limits or state that the limit does not exist. In each case, show your work.

(a) $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$

Solution:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

(b) $\lim_{x\to 1} \dfrac{x^3 - x}{x - 1}$

Solution:

$$\lim_{x\to 1} \frac{x^3 - x}{x - 1} = \lim_{x\to 1} \frac{x(x+1)(x-1)}{x-1} = \lim_{x\to 2} x(x+1) = 2$$

(c) $\lim_{x\to 5} 3x - 4 + \dfrac{2x+4}{x-3}$

Solution:

$$\lim_{x\to 5} 3x - 4 + \frac{2x+4}{x-3} = 3(5) - 4 + \frac{2(5)+4}{5-3} = 18$$

(d) $\lim_{x\to \frac{\pi}{2}} 3\sin(2x) - 4e^{2\cos x}$

Solution:

$$\lim_{x\to \frac{\pi}{2}} 3\sin(2x) - 4e^{2\cos x} = 3\sin(\pi) - 4e^{2\cos\frac{\pi}{2}} = 3\times 0 - 4\times e^0 = -4$$

16. Find the limits or state that the limit does not exist. In each case, justify your answer.

(a) $\lim_{x\to +\infty} \dfrac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$

Solution:

$$\begin{aligned}
\lim_{x\to +\infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)} &= \lim_{x\to +\infty} \frac{4x^4+4x^2+1}{x^4-x^3-x^2+x} \\
&= \lim_{x\to +\infty} \left(\frac{4x^4+4x^2+1}{x^4-x^3-x^2+x} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \right) \\
&= \lim_{x\to +\infty} \frac{4+\frac{4}{x^2}+\frac{1}{x^4}}{1-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}} \\
&= 4
\end{aligned}$$

(b) $\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$

Solution:

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} &= \lim_{x\to-\infty} \frac{\sqrt{4x^6}}{-x^3} \\ &= \lim_{x\to-\infty} \frac{2|x|^3}{-x^3} = \lim_{x\to-\infty} \frac{2(-x^3)}{-x^3} \\ &= 2\end{aligned}$$

(c) $\lim_{x\to+\infty} \frac{x-3x^2+x^4}{x^3-x+2}$

Solution: $\lim_{x\to+\infty} \frac{x-3x^2+x^4}{x^3-x+2} = +\infty$

(d) $\lim_{x\to+\infty} e^{-2x}\cos x$

Solution: $\lim_{x\to+\infty} e^{-2x}\cos x = 0$