MA 113 — Calculus I Fall 2010
Exam 1 September 21, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Key

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		8
2		8
3		9
4		12
5		8
6		10
7		10
8		16
9		16
10		16
Free	3	3
		100

(1) Consider the functions $f(x) = \frac{1}{x-6}$ and $g(x) = x^2 + 2$.

(a) Compute $f(g(5))$ and $g(f(5))$. **Give exact answers.**

① [$f(g(5)) = f(5^2+2) = f(27) = \underline{\underline{\frac{1}{21}}}$.

① [$g(f(5)) = g\left(\frac{1}{5-6}\right) = g(-1) = \underline{\underline{3}}$.

(b) Let h be the composite function $h(x) = (f \circ g)(x)$. Find the domain of h. **As usual, justify your answer by showing your work.**

③ [
$$h(x) = f(g(x)) = f(x^2+2) = \frac{1}{x^2+2-6} = \frac{1}{x^2-4}$$

③ [Hence $h(x)$ is defined unless $x^2-4=0$, thus $x^2=4$, so $x=2$ or $x=-2$. Hence the domain of h is $\{x \mid x \neq \pm 2\} = (-\infty,-2)\cup(-2,2)\cup(2,\infty)$

(a) $f(g(5)) =$ $\frac{1}{21}$, $g(f(5)) =$ 3

(b) Domain of h is $\{x \mid x \neq \pm 2\}$

(2) (a) Solve the equation $5^{x+2} = 7$. **Show all steps of the computation and give the exact answer.**

(4)

Taking $\log_5$ we get $x + 2 = \log_5 7$,

thus $x = \log_5 7 - 2$.

(b) Express the quantity

$$\log_4(x^6 + 1) - \log_4(9x) + \frac{1}{5}\log_4(x)$$

as a single logarithm.

(4)

The laws of logarithm provide

$$\log_4(x^6+1) - \log_4(9x) + \frac{1}{5}\log_4(x)$$

$$= \log_4\left(\frac{x^6+1}{9x}\right) + \log_5\left(\sqrt[5]{x}\right)$$

$$= \log_4\left(\frac{(x^6+1)\sqrt[5]{x}}{9x}\right)$$

(a) Solution is $\log_5 7 - 2$

(b) $\log_4\left(\frac{(x^6+1)\sqrt[5]{x}}{9x}\right)$

(3) Consider the function

$$f(x) = \frac{2x-4}{5x+1}.$$

(a) Find the domain of f.

1 [$f(x)$ is defined unless $5x+1=0$, that is $x = -\frac{1}{5}$.

1 [Hence the domain of f is $\{x \mid x \neq -\frac{1}{5}\} = (-\infty, -\frac{1}{5}) \cup (-\frac{1}{5}, \infty)$

(b) Find the inverse function f^{-1} of f.

2 [Set $y = f(x) = \dfrac{2x-4}{5x+1}$ and solve for x:

3 [
$$y(5x+1) = 2x-4, \text{ thus}$$
$$5yx - 2x = -4 - y, \text{ so}$$
$$x(5y-2) = -y-4.$$

No deduction if not discussed. { It follows that $5y-2 \neq 0$ because otherwise $y = \frac{2}{5}$ and the left-hand side of the equation is zero whereas the right-hand side is $-\frac{2}{5}-4 \neq 0$, a contradiction. Hence, we can divide by $5y-2$ and obtain

1 [
$$x = -\frac{y+4}{5y-2}.$$

1 [Hence $f^{-1}(x) = -\dfrac{x+4}{5x-2}$.

(a) Domain of f is $\{x \mid x \neq -\frac{1}{5}\}$

(b) $f^{-1}(x) = -\dfrac{x+4}{5x-2}$

(4) Use the limit rules and continuity to determine each of the following limits if it exists. If a limit does not exist, but is ∞ or $-\infty$, then clearly indicate that.

(a) $\lim_{x\to 16} 2^{\sqrt{x}-5}$ $\overset{1}{=} 2^{\sqrt{16}-5} = 2^{-1} \overset{1}{=} \frac{1}{2}$ because the functions 2^x and $\sqrt{x}-5$ are continuous on their domains, thus so is their composition $2^{\sqrt{x}-5}$. 2

(b) $\lim_{h\to 0} \dfrac{(h+4)^2-16}{h}$ $\overset{1}{=} \lim_{h\to 0} \dfrac{h^2+8h+16-16}{h} \overset{1}{=} \lim_{h\to 0} \dfrac{h^2+8h}{h} = \lim_{h\to 0} \dfrac{h(h+8)}{h}$

$\overset{1}{=} \lim_{h\to 0} (h+8) = 0+8 \overset{1}{=} 8$ because the polynomial function $h+8$ is continuous.

(c) $\lim_{x\to 4} \dfrac{x^2+1}{x-4}$. Since x^2+1 and $x-4$ are continuous functions, we get

$\lim_{x\to 4} (x^2+1) = 17$ and $\lim_{x\to 4^+} (x-4) = 0^+$ as $x-4 \geq 0$

and $\lim_{x\to 4^-} (x-4) = 0^-$ as $x-4 \leq 0$.

It follows that $\lim_{x\to 4^+} \dfrac{x^2+1}{x-4} = \infty$ and $\lim_{x\to 4^-} \dfrac{x^2+1}{x-4} = -\infty$. 3

Since the one-sided limits are different, the limit does not exist. 1

(a) $\frac{1}{2}$

(b) 8

(c) DNE

(5) Let f and g be two functions such that the following limits exist

$$\lim_{x\to 2} g(x) = 7, \quad \lim_{x\to 2}\left[3^x f(x) - xg(x)\right] = 13.$$

Use the limit laws to compute the following limits.

(3)

(a) $\displaystyle\lim_{x\to 2}\frac{g(x)}{x^3-1} = \frac{\lim_{x\to 2} g(x)}{\lim_{x\to 2}(x^3-1)} = \frac{7}{2^3-1} = \underline{\underline{1}}$

by limit law for quotients

polynomial functions are continuous

(b) $\displaystyle\lim_{x\to 2} f(x)$. Set $L := \lim_{x\to 2} f(x)$. Then the limit laws provide

(3)

$$\underbrace{\left(\lim_{x\to 2} 3^x\right)}_{=3^2=9}\cdot\underbrace{\left(\lim_{x\to 2} f(x)\right)}_{=L} - \underbrace{\left(\lim_{x\to 2} x\right)}_{=2}\cdot\underbrace{\left(\lim_{x\to 2} g(x)\right)}_{7} = 13,$$

where the limits are obtained by direct substitution because 3^x and x are continuous.

(2)

Solving the equation $9L - 2\cdot 7 = 13$ for L we get

$$9L = 27$$

$$\underline{\underline{L = 3}}.$$

(a) 1

(b) 3

(6) A particle is moving on a straight line so that after t seconds it is $s(t) = 5t + 1$ meters to the right of a reference point.

(a) Find the average velocity of the particle over the time interval $1 \leq t \leq 2$.

3 [This is $\frac{s(2)-s(1)}{2-1} = \frac{11-6}{1} = 5\,\frac{m}{s}$

(b) Find the average velocity over the time interval $[1, t]$, where $t > 1$. **Simplify your answer.**

5 [$\frac{s(t)-s(1)}{t-1} \overset{1}{=} \frac{5t+1-6}{t-1} \overset{1}{=} \frac{5t-5}{t-1} = \frac{5(t-1)}{t-1} \overset{1}{=} 5\,\frac{m}{s}$

2

(c) Use your answer in (b) to find the instantaneous velocity of the particle after 1 second.

3 [This is $\underbrace{\lim_{t\to 1} \frac{s(t)-s(1)}{t-1}}_{1} \overset{(b)}{=} \lim_{t\to 1} 5 = 5\,\frac{m}{s}$

1 1

(a) Average velocity over [1, 2] is 5 $\frac{m}{s}$

(b) Average velocity over [1, t] is 5 $\frac{m}{s}$

(c) Instantaneous velocity at time $t = 1$ is 5 $\frac{m}{s}$

(7) Using the definition, find the equation of the tangent line to the graph of the function $f(x) = x^2 - 5x$ at $x = 2$. Write your answer in the form $y = mx + b$.

3 The slope of the tangent line at $x=2$ is

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 5(2+h) - (2^2 - 5\cdot 2)}{h}$$

1 $$= \lim_{h\to 0} \frac{4+4h+h^2-10-5h+6}{h}$$

1 $$= \lim_{h\to 0} \frac{h(h-1)}{h}$$

1 $$= \lim_{h\to 0} (h-1)$$

$$= -1 \quad \text{by continuity of polynomial functions.}$$

Hence the equation of the tangent line is

2 $$y - f(2) = f'(2)(x-2), \text{ so}$$

1 $$y - (-6) = -1\cdot(x-2), \text{ thus}$$

1 $$\underline{\underline{y = -x - 4}}$$

Equation of the tangent line: $y = -x - 4$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) Define what it means for a function f to be continuous at a. Use complete sentences.

(4) A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Let

$$f(x) = \begin{cases} c3^x - 16, & \text{if } x < 2, \\ 5, & \text{if } x = 2, \\ x^2 - c, & \text{if } x > 2. \end{cases}$$

As always, justify your answer to the following problems!

(b) Find all values for c such that $\lim_{x\to 2} f(x)$ exists.

Since $c3^x - 16$ and $x^2 - c$ are continuous functions in x, we get

(2) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (c\cdot 3^x - 16) = c\cdot 3^2 - 16 = 9c - 16$ and

(2) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 - c) = 2^2 - c = 4 - c$.

(3) The one-sided limits agree iff $9c - 16 \overset{(1)}{=} 4 - c$ iff $\underbrace{10c = +20 \text{ iff } c = 2}_{(2)}$

Hence $\lim_{x\to 2} f(x)$ exists iff $c = 2$.

(c) For which of the values for c found in (b) is the function f continuous at 2?

(3) If f is continuous at 2, then $\lim_{x\to 2} f(x)$ must exist. By part (b), this forces $c=2$.

But if $c=2$, then $\lim_{x\to 2} f(x) \overset{(b)}{=} 9\cdot 2 - 16 = 2 \neq 5 = f(2)$. Hence f is never continuous at 2.

(d) Find all values for c such that the function f is continuous at 1.

(2) Since 3^x is continuous, the function $c3^x - 16$ is continuous for every c.
Hence f is continuous at 1 for every number c

(b) 2 (c) none (d) $\mathbb{R}$

(9) (a) State the Intermediate Value Theorem. Use complete sentences.

7 If a function f is continuous on a closed interval $[a,b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c in the open interval (a,b) such that $f(c) = N$.

(b) **Explain in detail why and how** you can use this theorem to show that the equation

$$2^x - 3\sqrt{x} = 1$$

has a solution in the interval (1, 4).

2 Consider the function $f(x) = 2^x - 3\sqrt{x}$

2 Since exponential and root functions are continuous on their domains, f is continuous. In particular, f is continuous on $(1,4)$.

2 Now we compute $f(1) = 2^1 - 3\sqrt{1} = -1 < 1$

and $f(4) = 2^4 - 3\sqrt{4} = 10 > 1$.

2 Hence $N=1$ is strictly between $f(1)$ and $f(2)$, thus the IVT gives the existence of some c in $(1,4)$ such that $1 = f(c) = 2^c - 3\sqrt{c}$.

1 Therefore c is the desired solution.

(10) (a) State the definition of the derivative of a function at a point a. Use complete sentences.

3 The derivative of f at a is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ (or equivalently $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$),

1 provided the limit exists.

It is enough to mention one of the two limits.

(b) Using the definition, determine the derivative of the function

$$f(x) = \begin{cases} \dfrac{1}{3x+2} & \text{if } x \leq 0 \\ 7 & \text{if } x > 0. \end{cases}$$

at $x = -1$ and $x = 0$ if it exists.

3

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3(-1+h)+2} - \frac{1}{3 \cdot (-1)+2}}{h}$$ (h sufficiently close to zero)

3

$$= \lim_{h \to 0} \frac{-1 - (3h-1)}{h(3h-1)(-1)} = \lim_{h \to 0} \frac{-3h}{-h(3h-1)} = \lim_{h \to 0} \frac{3}{3h-1}$$

1 $= \underline{\underline{-3}}$ because a rational function is continuous on its domain.

Since rational and constant functions are continuous, we get

3

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{3x+2} = \frac{1}{2} \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 7 = 7$$

Since the one-sided limits are different, $\lim_{x \to 0} f(x)$ does not exist.

2 Hence f is not continuous at 0, thus f is not differentiable at 0.

$f'(-1) =$ -3 $\qquad f'(0) =$ DNE