MA 113 — Calculus I Fall 2007
Exam 2 October 16, 2007

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please:

1. check answers when possible,
2. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
3. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name: ______________________________

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		9
2		6
3		8
4		8
5		9
6		9
7		10
8		10
9		14
10		14
11		14
Free	3	3
		100

(1) For each of the following functions, compute the derivative and simplify whenever possible.

(a) $f(x) = \dfrac{3x}{x^2 - 4x + 2}$

(b) $g(x) = \dfrac{1}{\sqrt[3]{x^2 - 3x + 1}}$

(c) $h(x) = \cos(4\pi x^3) - 5\sin(3x + 1)$

2 (a) $f'(x) = \dfrac{3(x^2-4x+2) - 3x(2x-4)}{(x^2-4x+2)^2}$

1 $= \dfrac{-3x^2+6}{(x^2-4x+2)^2}$

(b) $g'(x) = -\frac{1}{3}(x^2-3x+1)^{-4/3} \cdot (2x-3)$

3 $= \dfrac{3-2x}{3\sqrt[3]{(x^2-3x+1)^4}}$

(c) $h'(x) = -\sin(4\pi x^3) \cdot 12\pi x^2$

2 $-5\cos(3x+1) \cdot 3$

$= -12\pi x^2 \sin(4\pi x^3)$

1 $-15\cos(3x+1)$

(a) $f'(x) =$ $\dfrac{-3x^2+6}{(x^2-4x+2)^2}$

(b) $g'(x) =$ $-\frac{1}{3}(2x-3)(x^2-3x+1)^{-4/3}$

(c) $h'(x) =$ $-12\pi x^2 \sin(4\pi x^3) - 15\cos(3x+1)$

(2) Use the limit laws to find the following limits.

(a) $\lim_{x\to 0} \frac{2\sin(4x)}{3x}$

(b) $\lim_{x\to 0} \frac{\tan x}{x}$.

3

(a) $\lim_{x\to 0} \frac{2\sin(4x)}{3x} = \lim_{x\to 0} \frac{8}{3} \frac{\sin(4x)}{4x}$

$= \frac{8}{3} \lim_{t\to 0} \frac{\sin t}{t} = \frac{8}{3}\cdot 1 = \underline{\underline{\frac{8}{3}}}$

$t = 4x$, if $x \to 0$ then $t \to 0$.

3

(b) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos(0)}$

since $\cos(x)$ is continuous.

$= \underline{\underline{1}}$

(a) $\frac{8}{3}$ (b) 1

(3) A particle is traveling on a straight line such that its position after t seconds is given by $s(t) = t^3 - 12t^2 + 50t + 4$ centimeters.

(a) Find the velocity and the acceleration of the particle after 5 seconds.

(b) Is the particle slowing down or speeding up after 5 seconds? Justify your answer.

(2) (a) $v(t) = s'(t) = 3t^2 - 24t + 50$ is the velocity at time t.
Hence $v(5) = 75 - 120 + 50 = \underline{\underline{5}}$

(3) $a(t) = v'(t) = s''(t) = 6t - 24$ is the acceleration at time t.
Hence $a(5) = 30 - 24 = \underline{\underline{6}}$

(2) (b) The particle is speeding up since $v(5)$ and $a(5)$ have the same sign.

(1) point for putting down the units.

Note: In (b) 1 point for the correct answer and 1 point for the justification.

(a) velocity is 5 cm/sec, acceleration is 6 cm/sec^2

(b) slowing down/ speeding up (circle your answer).

(4) Let x be the number in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\sin x = -\frac{3}{5}$. Compute the following values. As usual show your work and give exact values for your answers.

(a) $\cos x$.

(b) $\cot x$.

3

(a) $\cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-\frac{9}{25}}$

$= \pm \frac{4}{5}$

2

Since $\cos x$ is nonnegative on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, we get

$\cos x = \frac{4}{5}$

3

(b) $\cot x = \frac{\cos x}{\sin x} = \frac{4/5}{-3/5} = -\frac{4}{3}$

(a) $\cos x =$ $4/5$ (b) $\cot x =$ $-4/3$

(5) Consider the function $f(x) = x - \cos x$ for $0 \le x \le 2\pi$. Find the point (a, b) on the graph of $f(x)$ such that a is between 0 and 2π and such that the tangent line to the graph of $f(x)$ at $x = a$ is horizontal. Give exact values for a and b.

2 [$f'(x) = 1 + \sin x$

1 [Wanted: $f'(a) = 0$

$1 + \sin a = 0$

2 [$\sin a = -1$ and a in $[0, 2\pi]$

2 [Then $a = \frac{3\pi}{2}$.

2 [Then $b = f(a) = \frac{3\pi}{2} - \cos\frac{3\pi}{2} = \frac{3\pi}{2}$

$(a, b) = \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

(6) Consider the function $f(x) = \sqrt{x+2}$.

(a) Find the linear approximation $L(x)$ of $f(x)$ at 34. Put your answer in the form $L(x) = mx + b$. Give precise values for m and b.

(b) Use the linear approximation of part (a) to estimate $\sqrt{35}$. Give your answer as a fraction.

1 (a) $f(34) = 6$

2 $f'(x) = \dfrac{1}{2\sqrt{x+2}}$, $f'(34) = \dfrac{1}{2\sqrt{36}} = \dfrac{1}{12}$

2 $L(x) = 6 + \dfrac{1}{12}(x - 34)$

1 $= \dfrac{1}{12}x + \dfrac{38}{12} = \dfrac{1}{12}x + \dfrac{19}{6}$ (*)

3 (b) $\sqrt{35} = \sqrt{33+2} = f(33) \approx L(33)$

$= \dfrac{33+38}{12} = \dfrac{71}{12}$

(*) Note: Either of these two solutions deserve full credit.

(a) $L(x) = \frac{1}{12}x + \frac{19}{6}$ (b) $\sqrt{35} \approx \frac{71}{12}$

(7) (a) Compute the second derivative $f''(x)$ of the function $f(x) = (\cos x)^2$.

(b) Compute the second derivative $g''(x)$ of the function $g(x) = \sqrt{2x+3}$.

2 (a) $f'(x) = 2\cos x\,(-\sin x)$

$= -2\cos x \cdot \sin x$

3 $f''(x) = -2\big((-\sin x)\sin x + \cos x \cos x\big)$

$= -2(\cos^2 x - \sin^2 x)$

(b) $g(x) = (2x+3)^{\frac{1}{2}}$

2 $g'(x) = \frac{1}{2}(2x+3)^{-1/2} \cdot 2 = (2x+3)^{-\frac{1}{2}}$

3 $g''(x) = -\frac{1}{2}(2x+3)^{-3/2} \cdot 2$

$= -(2x+3)^{-3/2}$

(*) Notice that we did not ask to simplify the answer.

(a) $f''(x) =$ $-2(\cos^2 x - \sin^2 x)$

(b) $g''(x) =$ $-(2x+3)^{-3/2}$

(8) Consider the curve given by the equation $x^2 + 2xy - y^2 + x = 2$. Find the equation of the tangent line to the curve at the point $(1, 2)$. Put your answer in the form $y = mx + b$.

implicit differentiation:

3 $$2x + 2y + 2xy' - 2yy' + 1 = 0$$

2 $$y' = \frac{2x + 2y + 1}{2y - 2x}$$

At the point $(1, 2)$ we get

2 $$y'(1) = \frac{2 + 4 + 1}{4 - 2} = \frac{7}{2} = \text{slope}$$

Thus, the tangent line has the equation

2 $$y - 2 = \frac{7}{2}(x - 1)$$

1 $$y = \frac{7}{2}x - \frac{3}{2}$$

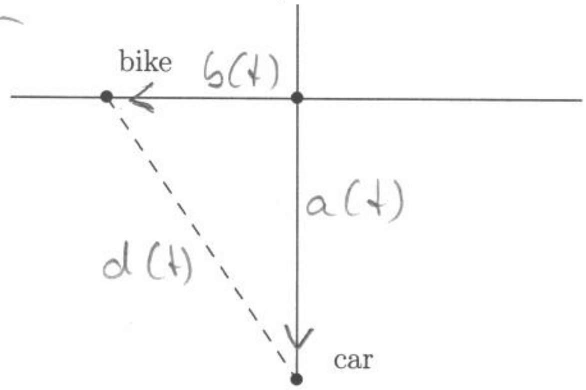
Note: Substituting $f(x)$ for y and then running the same type of computations is correct and deserves full credit. The tangent line, of course, is with y and not $f(x)$.

Equation of the tangent line is: $y = \frac{7}{2}x - \frac{3}{2}$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(9) A car and a bicycle start moving from the same point at the same time. The car travels south at 60 mi/h and the bike goes west at 25 mi/h. At what rate does the distance between the car and the bicycle increase two hours later?
(Please, introduce and explain your notation.)

2 for notation



$b(t)$ = distance of the bike to the intersection at time t.

$a(t)$ = distance of the car to the intersection at time t.

$d(t)$ = distance between car and bike at time t.

At time $t=0$ both are at the intersection.

2 Wanted: $d'(2)$. Known: $a'(t)=60$, $b'(t)=25$.

2 We have $d^2(t) = b^2(t) + a^2(t)$, thus

$$d(t)\,d'(t) = b(t)\,b'(t) + a(t)\,a'(t)$$

3 hence
$$d'(t) = \frac{b(t)b'(t)+a(t)a'(t)}{d(t)}$$

1
$$= \frac{25\,b(t) + 60\,a(t)}{d(t)}$$

2 At time $t=2$ we have $\boxed{b(2)=50}$, $\boxed{a(2)=120}$, thus $\boxed{d(2)} = \sqrt{(5^2+12^2)\cdot 100} = \boxed{130}$

1 Thus $d'(2) = \dfrac{25\cdot 50 + 60\cdot 120}{130} = \dfrac{845}{13} = 65$

1 Answer: 65 mi/h

(10) (a) Explain Newton's method to locate a root of the equation $2x^3 - 5x = -2$. In particular, give the formula relating x_n and x_{n+1}, and specify the function to be considered.

(2) $f(x) = 2x^3 - 5x + 2$, $f'(x) = 6x^2 - 5$

Wanted: solution of $f(x) = 0$.

(4) If x_1 is the given starting value, then one computes successively

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ where $f(x)$ is as above.

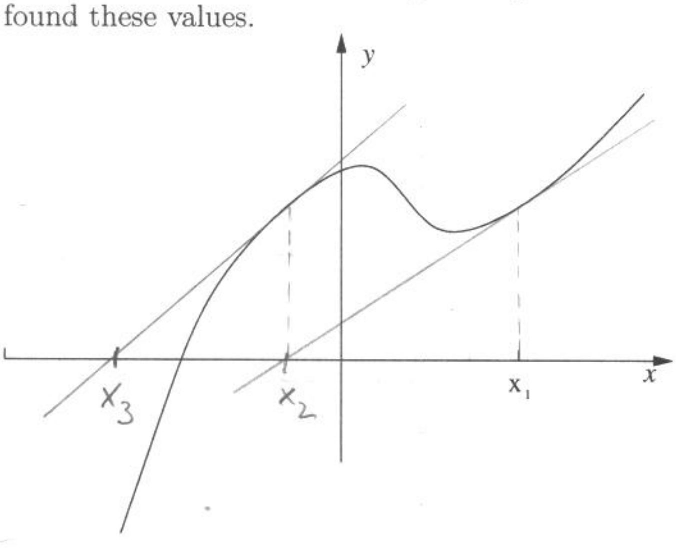
(b) Use $x_1 = 1$ and carry out 2 steps of Newton's method for the equation in part (a), that is, compute x_2 and x_3.

(2) $$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{1} = 2$$

(2) $$x_3 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{8}{19} = \frac{30}{19}$$

$x_2 =$ 2 $\qquad x_3 =$ $\frac{30}{19}$

(4) (c) The picture shows the graph of some function $f(x)$ and a value x_1. In the graph, sketch two steps of Newton's method starting with the given value x_1. In your solution, you should mark the locations of x_2 and x_3 on the x-axis, and you should show how you found these values.



(2) points for x_2

(2) points for x_3

(11) Consider the function $f(x) = 2x^2 + 4$.

(a) Find the equation of the tangent line to the graph of $f(x)$ at $x = a$.

(b) Find all points $(a, f(a))$ on the graph of $f(x)$ such that the tangent line to the graph at $x = a$ passes through the point $(3, 20)$. As usual, show your work to support your answer.

(c) Find the point $(c, f(c))$ on the graph of $f(x)$ such that the tangent line to the graph at $x = c$ is perpendicular to the line with equation $y = \frac{1}{8}x - 5$.

4 (a) $f(a) = 2a^2+4$, $f'(x) = 4x$, $f'(a) = 4a$

$y - 2a^2 - 4 = 4a(x-a)$

$\underline{y = 4ax - 2a^2 + 4}$

2 (b) $(3,20)$ on the line yields

$20 = 4 \cdot a \cdot 3 - 2a^2 + 4$

2 hence $2a^2 - 12a + 16 = 0$

$a^2 - 6a + 8 = 0$

$(a-4)(a-2) = 0$

$a = 4, 2$

2 $(2, f(2)) = \underline{(2, 12)}$, $(4, f(4)) = \underline{(4, 36)}$

3 (c) Wanted: slope $= -8$. Hence $4c = -8$ and thus $c = -2$.

1 $(c, f(c)) = (-2, 12)$

(a) Equation of the tangent line: $y = 4ax - 2a^2 + 4$

(b) The points are: $(2,12)$, $(4,36)$

(c) The point is: $(-2, 12)$