MA 113— Calculus I
Exam 2

Fall 2011
October 18, 2011

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: ______________________

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		15
2		12
3		12
4		12
5		12
6		15
7		12
8		10
		100

(1) Find the derivatives of the following functions.

(a) $f(x) = \dfrac{x}{1+e^x} = x(1+e^x)^{-1}$

$$f'(x) = 1 \cdot (1+e^x)^{-1} - x(1+e^x)^{-2} \cdot e^x$$

or

$$f'(x) = \frac{1(1+e^x) - xe^x}{(1+e^x)^2}$$

(b) $g(x) = \arctan(x^2) = \arctan(u)$ with $u = x^2$

$$g'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx} = \frac{2x}{1+x^4}$$

(c) $h(x) = \ln(\sin(x^2)) = \ln(u)$ with $u = \sin v$ and $v = x^2$

$$h'(x) = \frac{1}{u} \frac{du}{dx} = \frac{1}{u} \cdot \frac{d(\sin v)}{dv} \frac{dv}{dx}$$

$$= \frac{1}{u} \cdot \cos v \cdot 2x = \frac{\cos v}{\sin v} \cdot 2x = \cot(x^2) \cdot 2x = 2x\cot(x^2).$$

(a) $f'(x) = \dfrac{1+e^x-xe^x}{(1+e^x)^2}$,

(b) $g'(x) = \dfrac{2x}{1+x^4}$,

(c) $h'(x) = 2x\cot(x^2)$

(2) Compute the following limits. You may use the known limit: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

(a) $L_1 = \lim_{x\to 0} \frac{\sin(4x)}{3x}$.

Let $t = 4x$, Then $t \to 0$ as $x \to 0$.

$$L_1 = \lim_{t\to 0} \frac{\sin(t)}{3t/4} = \frac{4}{3} \lim_{t\to 0} \frac{\sin t}{t} = \frac{4}{3}$$

(b) $L_2 = \lim_{x\to 0} x\left(\cot x - \frac{2}{x}\right)$.

$$x\left(\cot x - \frac{2}{x}\right) = x \frac{\cos x}{\sin x} - 2$$

$$L_2 = \lim_{x\to 0} \cos x \cdot \frac{1}{\frac{\sin x}{x}} - \lim_{x\to 0} 2 = \frac{1}{1} - 2 = -1.$$

(a) $L_1 =$ 4/3

(b) $L_2 =$ -1

(3) A population of bacteria grows in a petri dish according to $N(t) = N_0 \cdot e^{kt}$, where $N(t)$ is the population of bacteria after t minutes. The following observations are made.

- After 20 minutes: 12 bacteria are found in the petri dish.
- After 50 minutes: 96 bacteria are observed.

(a) Find the value of k.

$$\left.\begin{array}{l} N_0 e^{20k} = N(20) = 12 \\ N_0 e^{50k} = N(50) = 96 \end{array}\right\} \Rightarrow \frac{e^{50k}}{e^{20k}} = \frac{96}{12} = 8 \Rightarrow e^{30k} = 8$$

$$k = \frac{1}{30} \ln(2^3) = \frac{1}{10} \ln(2)$$

(b) Find the value of N_0.

$$12 = N(20) = N_0 e^{20 \cdot \left(\frac{1}{10} \ln 2\right)} = 4 N_0$$

$$N_0 = 3$$

(a) $k =$ $\frac{\ln 2}{10}$

(b) $N_0 =$ 3

(4) Find the equation for the tangent line to the graph of

$$y = e^{5\cos x}$$

at $x = \frac{\pi}{2}$. Give the equation in the form $y = mx + b$.

Let $f(x) = e^{5\cos x}$

$f\left(\frac{\pi}{2}\right) = e^{5\cos\frac{\pi}{2}} = e^{0} = 1.$

$f'(x) = 5\cdot(-\sin x)e^{5\cos x}.$

$f'\left(\frac{\pi}{2}\right) = -5\sin\frac{\pi}{2}\, e^{0} = -5.$

Tangent line:

$y - 1 = -5\left(x - \frac{\pi}{2}\right)$

$y = -5x + \frac{5\pi}{2} + 1$

Equation of the tangent line is y= $-5x + \frac{5\pi}{2} + 1$

(5) A particle moves along a line so that its position after t minutes is given by

$$s(t) = 2t^3 - 9t^2 + 12t \text{ meters.}$$

(a) Find the velocity, $v(t)$, and the acceleration, $a(t)$, at time t.

$$v(t) = s'(t) = 6t^2 - 18t + 12$$
$$a(t) = s''(t) = 12t - 18$$

(b) What is the total distance traveled by the particle in the time interval $0 \le t \le 2$?

$$v(t) = 6(t^2 - 3t + 2) = 6(t-1)(t-2)$$

		1		2	
$t-1$	$-$		$+$		$+$
$t-2$	$-$		$-$		$+$
v	$+$		$-$		$+$

total distance traveled: $|s(1) - s(0)| + |s(2) - s(1)|$

$$= |5 - 0| + |4 - 5| = 5 + 1 = 6$$

(a) $v(t) =$ $6t^2 - 18t + 12$ m/min.,
$a(t) =$ $12t - 18$ m/min.^{2}

(b) Distance traveled= 6 m

(6) A spherical snowball melts in such a way that the volume decreases at a rate of 5 cm^3/min. Find the rate at which the surface area is changing when the radius is 10 cm. [You may use that the volume of a sphere is given by $V = \frac{4\pi}{3}r^3$ and the surface area of a sphere is given by $S = 4\pi r^2$.]

$$\frac{dV}{dt} = -5$$

$$V = \frac{4\pi}{3} r^3 \rightsquigarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

and

$$S = 4\pi r^2 \rightsquigarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Hence, $-5 = 4\pi (10)^2 \frac{dr}{dt}$ at the moment when $r = 10$ cm.

$\therefore \frac{dr}{dt} = -\frac{1}{80\pi}$.

$\therefore \frac{dS}{dt} = 8\pi \cdot 10 \cdot \left(-\frac{1}{80\pi}\right)$ at the moment when $r = 10$ cm.

$= -1$.

Surface area decreases at rate of 1 cm^2/min.

(7) Consider the curve described by the equation

$$y^3 + xy = 2x^3.$$

Find the slope of the tangent line to this curve at the point $(1,1)$.

$$\frac{d}{dx}(y^3+xy) = \frac{d}{dx}(2x^3)$$

$$3y^2y' + y + xy' = 6x^2$$

$$y' = \frac{6x^2-y}{3y^2+x}$$

At $(1,1)$, $y' = \frac{6-1}{3+1} = \frac{5}{4}.$

The slope is $5/4$

(8) Select whether the following statements are true or false. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, it might be wise to skip a question rather than risking losing a point. However, your final score on this problem will not be negative! You need not justify your answer.

True	**False**	**Statement**
☐	☒	If $y = 3^x$, then $y' = x\,3^{x-1}$.
☐	☒	If f and g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.
☒	☐	$\frac{d}{dx}(\sin(\arcsin x)) = 1$.
☐	☒	An equation for the tangent line to $y = x^3$ at the point $(0,0)$ is $y = 3x^2$.
☒	☐	If $f'(x) = g(x)$ and $g'(x) = f(x)$, then $\frac{d}{dx}\left(f(x)^2 - g(x)^2\right) = 0$.