MA 113 Calculus I — Fall 2013

Exam 2 — Tuesday, 22 October 2013

Name: ____________________

Section: ____________________

Last 4 digits of student ID #: ________

This exam has ten multiple choice questions (five points each) and five free response questions (ten points each). Additional blank sheets are available if necessary for scratch work. No books or notes may be used. Turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities.

On the multiple choice problems:

- Select your answer by placing an X in the appropriate square of the multiple choice answer box on the front page of the exam.
- Carefully check your answers. No credit will be given for answers other than those indicated on the *multiple choice answer box*.

On the free response problems:

- Clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
- Give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each free response question is followed by space to write your answer. Please write your solutions neatly in the space below the question.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

[B,C,A,B,B, E,C,B,D,B]

Exam Scores

Question	Score	Total
MC		50
11		10
12		10
13		10
14		10
15		10
Total		100

Record the correct answer to the following problems on the front page of this exam.

1. The tangent line to the graph of the function f at $(2, f(2))$ is $y = 2x - 1$. Give the values of $f(2)$ and $f'(2)$.

   (A) $f(2) = 2$ and $f'(2) = 2$.

   (B) $f(2) = 3$ and $f'(2) = 2$

   (C) $f(2) = 3$ and $f'(2) = -1$

   (D) $f(2) = -1$ and $f'(2) = 2$

   (E) $f(2) = 2$ and $f'(2) = 3$

2. Let $f(x) = 3\sin(\pi x)$. Then $f^{(5)}(1)$, the fifth derivative of f evaluated at $x = 1$, is:

   (A) 243

   (B) $3\pi^5$

   (C) $-3\pi^5$

   (D) -243

   (E) $3\pi^4$

Record the correct answer to the following problems on the front page of this exam.

3. Let y be implicitly defined by $\sin y + \cos y = x^3$. Then $\dfrac{dy}{dx}$ evaluated at the point $(x, y) = (-1, \pi)$ is:

(A) -3

(B) 0

(C) 1

(D) -1

(E) $-\frac{3}{2}$

4. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Suppose that when the radius is 3 cm, the radius is decreasing at a rate of 2 cm/s. Here cm/s is the abbreviation for centimeters per second. Find the rate of change of the volume with respect to time when the radius is 3 cm.

(A) 72π cm^3/s

(B) -72π cm^3/s

(C) 36π cm^3/s

(D) -36π cm^3/s

(E) None of the above.

Record the correct answer to the following problems on the front page of this exam.

5. Find the first derivative of $\arctan(e^{3x})$. The function arctan is also denoted by $\tan^{-1}$.

   (A) $3e^{3x}\sec^2(e^{3x})$

   (B) $\dfrac{3e^{3x}}{1+e^{6x}}$

   (C) $-\dfrac{e^{3x}}{1+e^{6x}}$

   (D) $\dfrac{1}{1+e^{6x}}$

   (E) $\sec(e^{3x})\cdot\tan(e^{3x})$

6. Below are graphs of a function f, its first derivative f', and its second derivative f''. Matching f and its derivatives with the correct graphs gives:

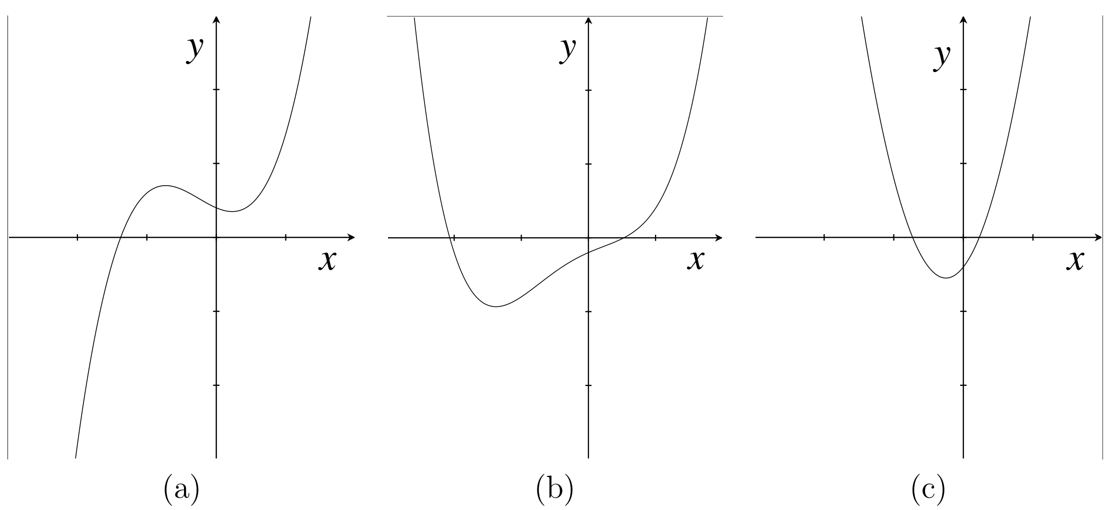
   (A) (a) is f, (b) is f', and (c) is f''.

   (B) (c) is f, (a) is f', and (b) is f''.

   (C) (a) is f, (c) is f', and (b) is f''.

   (D) (b) is f, (c) is f', and (a) is f''.

   (E) (b) is f, (a) is f', and (c) is f''.



Record the correct answer to the following problems on the front page of this exam.

7. Find the equation of the tangent line to the graph of the function $f(x) = \dfrac{x^3}{\ln(x)}$ at the point $(e, f(e))$.

   (A) $y = 2e^2x - 2e$
   (B) $y = x - 3$
   (C) $y = 2e^2x - e^3$
   (D) $y = x - e^2 + 2e$
   (E) $y = e^3x + 4$

8. Suppose that $f(-1) = 0$, $f(0) = 1$, $f(1) = 2$, $f'(-1) = -2$, $f'(0) = -1$, and $f'(1) = -3$, and

$$g(x) = f\left(\arcsin\left(\frac{x}{2}\right) - 1\right).$$

   The function arcsin is also denoted by $\sin^{-1}$. The value of $g'(0)$ is

   (A) -2
   (B) -1
   (C) 0
   (D) 1
   (E) 2

Record the correct answer to the following problems on the front page of this exam.

9. Let g and h be functions that are differentiable at 1 and let $f(x) = g(x) \cdot h(x)^2$. We are given that $g(1) = -1$, $h(1) = 2$, $g'(1) = 3$ and $h'(1) = 4$. Find $f'(1)$.

   (A) -1

   (B) -2

   (C) -3

   (D) -4

   (E) -5

10. Find the second derivative of $f(x) = e^{-2x^2}$.

   (A) $f''(x) = 16x^2\, e^{-2x^2}$

   (B) $f''(x) = (16x^2 - 4)\, e^{-2x^2}$

   (C) $f''(x) = (4x^4 + 2x^2)\, e^{-2x^2-2}$

   (D) $f''(x) = (-4x + 16)\, e^{-2x^2}$

   (E) None of the above answers are correct.

Free Response Questions: Show your work!

11. Find the equation of the tangent line to the curve given by

$$x^2y^3 + 4y^2 + x = 25$$

at the point $(x, y) = (1, 2)$. Give the equation in the form $y = mx + b$.

$$\begin{aligned} \frac{d}{dx}(x^2y^3 + 4y^2 + x) &= \frac{d}{dx}(25) \\ 2xy^3 + x^2(3y^2\frac{dy}{dx}) + 4(2y\frac{dy}{dx}) + 1 &= 0 \\ 2xy^3 + 3x^2y^2\frac{dy}{dx} + 8y\frac{dy}{dx} + 1 &= 0 \\ (3x^2y^2 + 8y)\frac{dy}{dx} &= -2xy^3 - 1 \end{aligned}$$	4 points: Correct differentiation.
$$\frac{dy}{dx} = \frac{-2xy^3 - 1}{3x^2y^2 + 8y}$$	1 point: Solve for $\frac{dy}{dx}$.
$$\frac{dy}{dx}\Big\vert_{(1,2)} = \frac{-2(1)(8) - 1}{3(1)(4) + 8(2)} = -\frac{17}{28}$$	3 points: Correct slope of the tangent line.
$$\begin{aligned} y - 2 &= -\frac{17}{28}(x - 1) \\ y &= -\frac{17}{28}x + \frac{17}{28} + 2 \\ y &= -\frac{17}{28}x + \frac{73}{28} \end{aligned}$$	2 points: Correct equation of the tangent line. Deduct 1 point if the equation is not in slope-intercept form.

If the student does differentiation incorrectly, but solves correctly for $\frac{dy}{dx}$ and continues to use this form for the remainder of the problem, the student should lose the 4 differentiation points, but be eligible to receive all other points.

As long as the student computes the slope from their solution of the derivative, they should be eligible to receive points for the equation of the tangent line.

12. (a) Give the definition of $f'(a)$, the derivative of a function f at the point a.

The derivative of f at a is $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, if the limit exists OR The derivative of f at a is $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, if the limit exists	3 points total: 2 points: Correct difference quotient. 1 point: Answer includes the statement "if the limit exists."

(b) Let f be the function defined by

$$f(x) = \begin{cases} ax + b, & x < 2 \\ x^2 + 1, & x \geq 2 \end{cases}$$

Use the definition of the derivative and properties of differentiable functions to find a and b so that f is differentiable at 2.

If f is differentiable at $x = 2$ then f is continuous at $x = 2$, so $2a + b = 5$ or $b = 5 - 2a$. $\begin{aligned} \lim\limits_{x \to 2^-} \tfrac{ax+b-5}{x-2} &= \lim\limits_{x \to 2^-} \tfrac{ax+(5-2a)-5}{x-2} \\ &= \lim\limits_{x \to 2^-} \dfrac{a(x-2)}{x-2} = a. \end{aligned}$	2 points: Correct evaluation of the limit from the left
$\begin{aligned} \lim\limits_{x \to 2^+} \tfrac{x^2+1-5}{x-2} &= \lim\limits_{x \to 2^+} \tfrac{x^2-4}{x-2} \\ &= \lim\limits_{x \to 2^+} x + 2 = 4 \end{aligned}$	2 points: Correct evaluation of the limit from the right
$a = 4$	2 points: Equate the left and right limits, correct value for a
$b = 5 - 2(4) = -3$	1 point: Correct value for b

Free Response Questions: Show your work!

13. The height of an object falling with constant acceleration is given by a quadratic polynomial $h(t) = at^2 + bt + c$. An object is dropped from a building that is 64 meters tall with 0 initial velocity. The building is not on the Earth, but is on a planet where the acceleration of gravity is -8m/s^2.

(a) Find a function $h(t)$ that gives the height of the object t seconds after it is dropped until it hits the ground.

$$\begin{aligned} h(t) &= \frac{-8}{2}t^2 + 0t + 64 \\ &= -4t^2 + 64 \end{aligned}$$

4 points:
Correct formula for $h(t)$. Students who do not successfully answer part (a) can still earn points for parts (b) and (c).

(b) Find the time when the object hits the ground.

$$\begin{aligned} 0 &= -4t^2 + 64 \\ 4t^2 &= 64 \\ t^2 &= 16 \\ t &= 4 \end{aligned}$$

Only the positive root of $t^2 = 16$ makes sense in the context of the problem, so the time when the object hits the ground is $t = 4\,s$.

3 points total:
Set $h(t) = 0$ and correctly solve for t.

(c) Find the velocity of the object at the moment when it hits the ground.

$v(t) = h'(t) = -8t$
Then $v(4) = -32$ m/s.

3 points:
Correctly solve for $v(4)$. Include units.

14. At exactly 6:42 pm during the combine demolition derby on the fourth day of the County Fair, Combine A finds itself 30 meters south of the center of the fairgrounds while traveling north at a speed of 2 m/s, while at the same time Combine B is 40 meters west of the center of the fairgrounds and traveling east at a speed of 3 m/s.

(a) Make a sketch to represent the given information.

x = distance between B and the origin in meters y =distance between A and the origin in meters At 6:42 p.m., given that $x = 40\,\mathrm{m}$, $y = 30\,\mathrm{m}$, and $\frac{dx}{dt} = -3\,\mathrm{m/s}$ and $\frac{dy}{dt} = -2\,\mathrm{m/s}$ z = distance between A and B in meters	3 points: Correct picture, includes all of the given information.

(b) Find the rate at which the distance between the combines is changing at exactly 6:42 pm. Be sure to explain your work. Use the sketch in part (a) to indicate the meaning of any variables you introduce in your solution.

Constraint equation: $x^2 + y^2 = z^2$.	2 points: Correct constraint equation. Information about variables may be found in part (a) or (b).
$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt}$ $\dfrac{dz}{dt} = \dfrac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$	2 points: Correct differentiation.
At 6:42 p.m., $x = 40$ m and $y = 30$ m, $z = \sqrt{(40^2) + (30^2)} = 50\,\mathrm{m}$	1 point: Correct computation for z at 6:42 p.m.
$\dfrac{dz}{dt} == \dfrac{40(-3) + 30(-2)}{50} = -\dfrac{18}{5}$ m/s	2 points: Correct computation for $\frac{dz}{dt}$.

Free Response Questions: Show your work!

15. (a) Write down a statement (theorem) in complete sentence form which gives a formula for computing the derivative of an inverse function $f^{-1}(x)$. Your statement should include conditions which guarantee that f has a differentiable inverse.

Assume that $f(x)$ is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If b belongs to the domain of g and $f'(g(b)) \neq 0$, then $$g'(b) = \frac{1}{f'(g(b))}.$$	6 points total, 1 point each for Stating that f must be differentiable stating that f must be one-to-one stating the $f'(g(b)) \neq 0$ stating that b must be in the domain of g Stating the correct formula grammar and using logical, complete sentences.

(b) Let f be a function which is one to one and let f^{-1} be the inverse function. We are given that $f(0) = 1$, $f(1) = 0$, $f(2) = -1$, $f'(0) = 3$, $f'(1) = -2$, and $f'(2) = -1$. Find ${f^{-1}}'(0)$.

Let $g(x) = f^{-1}(x)$. Then $$g'(0) = \frac{1}{f'(g(0))}$$ $g(0) = 1$ since $f(1) = 0$ $$g'(0) = \frac{1}{f'(1)} = \frac{1}{-2} = -\frac{1}{2}$$	4 points total, 1 point each for: correct evaluation of $g(0) = f^{-1}(0)$ evaluating $f'(g(0))$ evaluating $1/f'(g(0))$ getting the correct answer.