Exam 2

Form A

Multiple Choice Questions

1. Suppose that $f(x) = \begin{cases} x+2 & \text{if } x \leq 2 \\ Ax+B & \text{if } 2 < x \leq 4 \\ 7x-12 & \text{if } x > 4 \end{cases}$

   Find the values of A and B which make $f(x)$ continuous everywhere.

   A. $A = 6,\ B = 8$

   B. $A = -6,\ B = 8$

   C. $A = 6,\ B = -8$

   D. $A = -6,\ B = -8$

   E. There is no solution.

Solution: We need $\lim_{x\to a} f(x) = f(a)$ at both $a = 2$ and $a = 4$. $\lim_{x\to 2^-} f(x) = 2+2 = 4$ and $\lim_{x\to 2^+} f(x) = 2A+B$ so we must have that $2A+B = 4$.

Likewise at $x = 4$, $\lim_{x\to 4^-} f(x) = 4A+B$ and $\lim_{x\to 4^+} f(x) = 7\cdot 4 - 12 = 16$. This gives us two equations in two unknowns in which we can solve for A and B.

$$\begin{aligned} 2A+B &= 4 \\ 4A+B &= 16 \\ 2A &= 12 \\ A &= 6 \\ 2\cdot 6 + B &= 4 \\ B &= -8 \end{aligned}$$

Thus, $A = 6$ and $B = -8$.

2. Find the horizontal asymptotes of $f(x) = \dfrac{e^x}{1+e^x}$.

   A. $y = 0$

   B. $y = 1$

   C. $y = 1/2$

   D. $y = 0$ and $y = 1$

   E. $y = -1$

Solution: We need to fine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

$$\lim_{x\to\infty} \frac{e^x}{1+e^x} = \lim x \to \infty \frac{e^x}{1+e^x}\frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1}{e^{-x}+1} = 1$$
$$\lim_{x\to-\infty} \frac{e^x}{1+e^x} = \frac{0}{1+0} = 0$$

Thus, there are two horizontal asymptotes: $y = 1$ and $y = 0$.

3. Which of the following statements is true if $f(x)$ is defined by

$$f(x) = \begin{cases} \sin x & \text{if } x \le -\pi/4 \\ \cos x & \text{if } x > -\pi/4 \end{cases}$$

   A. $f(x)$ is continuous at $x = -\pi/4$.

   B. $f(x)$ has a jump discontinuity at $x = -\pi/4$.

   C. $f(x)$ has an infinite discontinuity at $x = -\pi/4$.

   D. $f(x)$ is not defined at $x = -\pi/4$.

   E. There is not enough information to determine the continuity of $f(x)$ at $x = -\pi/4$.

Solution: Checking $\lim_{x\to-\pi/4} \sin x = -\frac{1}{\sqrt{2}}$ and $\lim_{x\to-\pi/4} \cos x = \frac{1}{\sqrt{2}}$. Thus, both limits exist but are not equal. This means that $f(x)$ has a jump discontinuity at $x = -\pi/4$.

4. Suppose that f is a continuous function on the interval $[0,5]$ and we know that

$$f(0) = 1,\ f(1) = -1,\ f(2) = 1,\ f(3) = -1,\ f(4) = 1, \text{ and } f(5) = -1.$$

Which of the following statements are true for any such f?

A. There are at least five solutions of the equation $f(x) = 0$ in the interval $[0,5]$.

B. There are at most five solutions of the equation $f(x) = 0$ in the interval $[0,5]$.

C. There are exactly five solutions of the equation $f(x) = 0$ in the interval $[0,5]$.

D. The equation $f(x) = -1$ has at most three solutions in the interval $[0,5]$.

E. The equation $f(x) = 1$ has exactly three solutions in the interval $[0,5]$.

Solution: The function changes sign 5 times on the interval $[0,5]$. Thus, by the Intermediate Value Theorem there must be at least five solutions to $f(x) = 0$ in that interval.

5. Find a formula for $\frac{dy}{dx}$ in terms of x and y, where $x^2 + xy + y^2 = 1$.

A. $\frac{dy}{dx} = -\frac{2x+y}{x+y}$

B. $\frac{dy}{dx} = -\frac{2x}{x+2y}$

C. $\frac{dy}{dx} = \frac{2x+y}{x+2y}$

D. $\frac{dy}{dx} = -\frac{2x+y}{2y}$

E. $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$

Solution:

$$\begin{aligned}
x^2 + xy + y^2 &= 1 \\
2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} &= 0 \\
(x+2y)\frac{dy}{dx} &= -2x - y \\
\frac{dy}{dx} &= -\frac{2x+y}{x+2y}
\end{aligned}$$

6. Find all values of x where $f'''(x) = 0$ when $f(x) = xe^{2x}$.

   A. $x = 3/2$

   B. $x = 2/3$

   C. $x = -3/2$

   D. $x = -2/3$

   E. $x = -1$

Solution:

$$\begin{aligned} f(x) &= xe^{2x} \\ f'(x) &= e^{2x} + 2xe^{2x} \\ f''(x) &= 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x} \\ f'''(x) &= 8e^{2x} + 4e^{2x} + 8xe^{2x} = (12 + 8x)e^{2x} \end{aligned}$$

Setting $f'''(x) = 0$ gives $(12 + 8x)e^{2x} = 0$ which means $12 + 8x = 0$ giving us that $x = -3/2$.

7. Find $f'(x)$ in terms of $g'(x)$ where $f(x) = [g(x)]^4$.

   A. $f'(x) = 4[g(x)]^3$

   B. $f'(x) = 4[g(x)]^3 g'(x)$

   C. $f'(x) = 4[g'(x)]^3$

   D. $f'(x) = 4[gx]^3[xg' + g]$

   E. $f'(x) = 4g'(x)$

Solution: From the Chain Rule,

$$f'(x) = 4[g(x)]^3 g'(x).$$

8. Find the derivative of $g(t) = \tan(\cos(2t))$.

A. $g'(t) = 2\sin(2t)\sec^2(\cos(2t))$

B. $g'(t) = \sin(2t)\sec^2(\cos(2t))$

C. $g'(t) = -\sin(2t)\sec^2(\cos(2t))$

D. $g'(t) = -2\sin(2t)\sec^2(\cos(2t))$

E. $g'(t) = -2\sec^2(\sin(2t))$

Solution: This is again using the Chain Rule.

$$g'(t) = \sec^2(\cos(2t))(-\sin(2t))(2) = -2\sin(2t)\sec^2(\cos(2t)).$$

9. Find the derivative of

$$g(x) = x^5 \ln(9x).$$

A. $g'(x) = x^4(1 + 5\ln(9x))$

B. $g'(x) = 1 + \dfrac{\ln(9x)}{9x}$

C. $g'(x) = x^4\left(\dfrac{1}{9} + 5\ln(9x)\right)$

D. $g'(x) = \dfrac{5}{9}x^3$

E. $g'(x) = x^4(5\ln(9x) - 1)$

Solution: This is the Product Rule.

$$g'(x) = 5x^4\ln(9x) + x^5\frac{1}{9x}(9) = 5x^4\ln(9x) + x^4 = x^4(1 + 5\ln(9x)).$$

10. Differentiate

$$f(x) = \frac{x^6}{1 - x^5}$$

A. $f'(x) = \dfrac{6x^5}{1 - 5x^4}$

B. $f'(x) = \dfrac{(x^5 - 1)^2}{x^5(6x - 5)}$

C. $f'(x) = \dfrac{x^5(6x - 5)}{(1 - x^5)^2}$

D. $f'(x) = \dfrac{x^5(1 - x^5)}{(6 - x^5)^2}$

E. $f'(x) = \dfrac{x^5(6 - x^5)}{(1 - x^5)^2}$

Solution: This is the Quotient Rule.

$$f'(x) = \frac{(1 - x^5)6x^5 - x^6(-5x^4)}{(1 - x^5)^2} = \frac{6x^5 - 6x^{10} + 5x^{10}}{(1 - x^5)^2} = \frac{x^5(6 - x^5)}{(1 - x^5)^2}.$$

11. Suppose that $F(x) = f(g(x))$ and $g(14) = 2$, $g'(14) = 5$, $f'(14) = 15$, and $f'(2) = 11$. Find $F'(14)$

A. $F'(14) = 140$

B. $F'(14) = 55$

C. $F'(14) = 24$

D. $F'(14) = 20$

E. $F'(14) = 17$

Solution: From the Chain Rule, $F'(x) = f'(g(x))g'(x)$, so

$$F'(14) = f'(g(14))g'(14) = f'(2)g'(14) = 11 \times 5 = 55.$$

12. If f and g are continuous functions with $f(9) = 6$ and $\lim_{x\to 9}[2f(x) - g(x)] = 9$, find $g(9)$.

A. $g(9) = 21$

B. $g(9) = 24$

C. $g(9) = 3$

D. $g(9) = 15$

E. $g(9) = 12$

Solution: We are given that $\lim_{x\to 9}[2f(x) - g(x)] = 9$. Since f and g are continuous and $f(9) = 6$, we have

$$\begin{aligned} \lim_{x\to 9}[2f(x) - g(x)] &= 9 \\ 2\lim_{x\to 9} f(x) - \lim_{x\to 9} g(x) &= 9 \\ 2f(9) - g(9) &= 9 \\ 12 - g(9) &= 9 \\ g(9) &= 3 \end{aligned}$$

Free Response Questions

13. Find the derivatives of the following functions.

(a) $f(x) = \ln(\cos(2x))$.

Solution: By the Chain Rule

$$f'(x) = \frac{\cos(2x)}{-\sin(2x)}(2) = -\frac{2\sin(2x)}{\cos(2x)} = -2\tan(2x).$$

(b) $g(x) = \frac{4}{x^5} - \frac{8}{x^4} - \frac{3}{x^3} + 700$.

Solution: First, rewrite $g(x)$.

$$\begin{aligned} g(x) &= 4x^{-5} - 8x^{-4} - 3x^{-3} + 700 \\ g'(x) &= -20x^{-6} + 32x^{-5} + 9x^{-4} \\ &= -\frac{20}{x^6} + \frac{32}{x^5} + \frac{9}{x^4} \end{aligned}$$

(c) $h(x) = 4\ln(x^2e^x)$.

Solution: First, rewrite $h(x)$.

$$\begin{aligned} h(x) &= 4\left(\ln(x^2) + \ln(e^x)\right) \\ &= 8\ln x + x \\ h'(x) &= \frac{8}{x} + 1 \end{aligned}$$

14. (a) Find the equation of the tangent line to $2xy^2 - 5x^2y + 192 = 0$ at the point $(4, 4)$.

Solution: We need to find $\left.\frac{dy}{dx}\right|_{(4,4)}$.

$$\begin{aligned} 2xy^2 - 5x^2y + 192 &= 0 \\ 2y^2 + 4xy\frac{dy}{dx} - 10xy - 5x^2\frac{dy}{dx} &= 0 \\ (4xy - 5x^2)\frac{dy}{dx} &= 10xy - 2y^2 \\ \frac{dy}{dx} &= \frac{10xy - 2y^2}{4xy - 5x^2} \end{aligned}$$

Now,

$$\left.\frac{dy}{dx}\right|_{(4,4)} = \frac{10(4)(4) - 2(4)^2}{4(4)(4) - 5(4)^2} = -8$$

So the equation of the tangent line is

$$y = 4 - 8(x - 4).$$

(b) Find $\lim_{x\to 0} \frac{\sin(3x)}{7x}$

Solution:

$$\lim_{x\to 0} \frac{\sin(3x)}{7x} = \frac{3}{7}\lim_{x\to 0} \frac{\sin(3x)}{3x} = \frac{3}{7} \times 1 = \frac{3}{7}.$$

15. Let $f(x) = \dfrac{x^2}{x+6}$.

(a) Find the derivative $f'(x)$.

Solution: This is the Quotient Rule.

$$f'(x) = \frac{(x+6)(2x) - x^2(1)}{(x+6)^2} = \frac{x^2+12x}{(x+6)^2}.$$

(b) Find the equation of the tangent line to $f(x)$ at the point where $x = 4$.

Solution: We need $f(4)$ and $f'(4)$. $f(4) = \dfrac{16}{10} = \dfrac{8}{5}$. Meanwhile,

$$f'(4) = \frac{16+48}{100} = \frac{16}{25}.$$

The equation of the tangent line is then

$$y - \frac{8}{5} = \frac{16}{25}(x-4).$$

16. This problem concerns the definition of the derivative using limits.

   (a) State the formal definition of the derivative of a function $f(x)$ at the point $x = a$. *Hint*: Your definition should involve a limit.

   Solution: The derivative of the function $f(x)$ at the point $x = a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{s\to a} \frac{f(x) - f(a)}{x - a}$$

   if this limit exists.

   (b) **Using the formal definition of derivative and the limit laws**, find the derivative of the function $f(x) = 2x^2 - 1$. An answer that is unsupported or uses differentiation rules will receive **no credit**.

   Solution:

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{2(a+h)^2 - 1 - (2a^2 - 1)}{h} \\ &= \lim_{h\to 0} \frac{2a^2 + 4ah + 2h^2 - 1 - 2a^2 + 1}{h} \\ &= \lim_{h\to 0} \frac{4ah + 2h^2}{h} \\ &= \lim_{h\to 0} 4a + 2h \\ f'(a) &= 4a \end{aligned}$$