MA 113 — Calculus I Spring 2008
Exam 3 April 8, 2008

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please:

1. check answers when possible,
2. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
3. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question. Also when appropriately record your answers at the bottom of the page.
You are to answer *two of the last three questions*. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name: ______________________________

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		9
2		8
3		9
4		8
5		9
6		9
7		8
8		9
9		14
10		14
11		14
Free	3	3
		100

(1) Calculate the following limits:

(a)

$$\lim_{x\to\infty} \frac{\sqrt{7+5x+3x^5}}{5+7x+2x^3}.$$

(b)

$$\lim_{x\to\infty} \frac{4+8\sqrt{x}+3x}{4x+8\sqrt{x}+3}.$$

(c)

$$\lim_{x\to\infty} (\sqrt{x^2+x}-x).$$

In each case show the algebraic operations needed to get the correct answer.

Solution

(a)

$$\lim_{x\to\infty} \frac{\sqrt{7+5x+3x^5}}{5+7x+2x^3} = \lim_{x\to\infty} \frac{x^{\frac{5}{2}}\sqrt{\frac{7}{x^5}+\frac{5}{x^4}+3}}{x^3(\frac{5}{x^3}+\frac{7}{x^2}+2)} = (\lim_{x\to\infty} \frac{1}{x^{\frac{1}{2}}}) \cdot \frac{\sqrt{3}}{2} = 0.$$

Alternatively,

$$\lim_{x\to\infty} \frac{\sqrt{7+5x+3x^5}}{5+7x+2x^3} = \lim_{x\to\infty} \sqrt{\frac{7+5x+3x^5}{(5+7x+2x^3)^2}} = \sqrt{\lim_{x\to\infty} \frac{...+3x^5}{...+4x^6}} = 0.$$

(b)

$$\lim_{x\to\infty} \frac{4+8\sqrt{x}+3x}{4x+8\sqrt{x}+3} = \lim_{x\to\infty} \frac{\frac{4}{x}+\frac{8}{\sqrt{x}}+3}{4+\frac{8}{\sqrt{x}}+\frac{3}{x}} = \frac{3}{4}.$$

(c)

$$\lim_{x\to\infty} (\sqrt{x^2+x}-x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}.$$

(a) 0

(b) $\frac{3}{4}$

(c) $\frac{1}{2}$

(2) Consider the polynomial function

$$f(x) = 2x^3 - 9x^2 + 12x + 5.$$

(a) Find the critical numbers of $f(x)$ and show your work.
(b) Use (a) and either the first or second derivative test to find the local extrema of f.

Solution

(a)

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2) = 0$$

The critical numbers are $x = 1$ and $x = 2$.
(b)

$$f''(x) = 12x - 18$$

By the second derivative test, $f''(1) = 12 - 18 = -6 < 0$ implies that

$$f(1) = 2 - 9 + 12 + 5 = 10$$

is a local maximum and $f''(2) = 24 - 18 = 6 > 0$ implies that

$$f(2) = 16 - 36 + 24 + 5 = 9$$

is a local minimum.
Alternatively, by the first derivative test, $f(1)$ is a local maximum since f' changes at 1 from negative to positive and $f(2)$ is a local minimum since f' changes at 2 from negative to positive.

(a) The critical number(s): 0, 1

(b) The local maximum: 10 The local minimum: 9

(3) Find the absolute maximum value of the function $f(x) = \sin^2 x + \sqrt{3}\cos x$ on the closed interval $[0, \pi]$. Justify your answer.

Solution

$$f'(x) = 2\sin x\cos x - \sqrt{3}\sin x = \sin x(2\cos x - \sqrt{3}) = 0$$

$$\sin x = 0 \text{ or } \cos x = \frac{\sqrt{3}}{2}$$

The critical number in the open interval $(0, \pi)$ is $x = \frac{\pi}{6}$.
Evaluate f at the critical number:

$$f(\frac{\pi}{6}) = \sin^2(\frac{\pi}{6}) + \sqrt{3}\cos\frac{\pi}{6} = \frac{1}{4} + \frac{3}{2} = \frac{7}{4} = 1.75$$

Evaluate f at the endpoints of the interval $[0, \pi]$:

$$f(0) = \sin^2(0) + \sqrt{3}\cos 0 = \sqrt{3} = 1.73...$$

$$f(\pi) = \sin^2(\pi) + \sqrt{3}\cos\pi = -\sqrt{3}$$

By the closed interval method, the desired absolute maximum value is 1.75
Alternatively, $\sin x > 0$ on $(0, \pi)$ and $2\cos x - \sqrt{3} > 0$ on $(0, \pi/6)$ and < 0 on $(\pi/6, \pi)$ imply f' is positive on $(0, \pi/6)$ and negative on $(\pi/6, \pi)$. Hence, $f(\pi/6)$ is an absolute maximum on $[0, \pi]$.

The absolute maximum value is: 1.75

(4) Consider the polynomial function

$$f(x) = x^4 - 6x^3 + 12x^2 + 10x + 14.$$

(a) Find the interval(s) of concavity of $f(x)$ and show your work.

(b) Find the point(s) of inflection on the graph of f and justify your answer.

Solution

(a)

$$f'(x) = 4x^3 - 18x^2 + 24x + 10$$

$$f''(x) = 12x^2 - 36x + 24 = 12(x^2 - 3x + 2) = 12(x-1)(x-2) = 0$$

if and only if

$$x = 1 \text{ or } x = 2$$

Also, $f''(1.5) = -3$, $f''(0) = f''(3) = 24$.
Therefore, f'' is positive on the intervals $(-\infty, 1)$ and $(2, \infty)$ and negative on the interval $(1, 2)$. By the concavity test, f is concave up on $(-\infty, 1)$ and $(2, \infty)$ and concave down on $(1, 2)$.
(b)

$$f(1) = 1 - 6 + 12 + 10 + 14 = 31$$

$$f(2) = 16 - 48 + 48 + 20 + 14 = 50$$

By (a), $(1, 31)$ is an inflection point since f changes at 1 from concave up to concave down. Also $(2, 50)$ is an inflection point since f changes at 2 from concave down to concave up.

(a) $f(x)$ is concave up on: $(-\infty, 1) \cup (2, \infty)$

$f(x)$ is concave down on: $(1, 2)$

(b) The point(s) of inflection $(x, y) = (1, 31)$, $(2, 50)$

(5) Find the most general antiderivative for each of the following functions:

(a) $f(x) = x^7 - 15x^4 + 3x - 8$;

(b) $g(x) = 5\sin(x) - 2\cos(x)$;

(c) $h(x) = 4\sqrt[3]{x} + \dfrac{2}{x^2\sqrt{x}}$.

Solution

(a) General anti-derivative of $f(x)$ is

$$\frac{x^8}{8} - 3x^5 + \frac{3}{2}x^2 - 8x + C$$

(b) General anti-derivative of $g(x)$ is

$$-5\cos x - 2\sin x + C$$

(c) General anti-derivative of $h(x)$ is

$$4\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + 2\frac{x^{-2-\frac{1}{2}+1}}{-2-\frac{1}{2}+1} + C = 3x^{4/3} - \frac{4}{3}x^{-3/2} + C$$

(a) General antiderivative of f: $\frac{x^8}{8} - 3x^5 + \frac{3}{2}x^2 - 8x + C$

(b) General antiderivative of g: $-5\cos x - 2\sin x + C$

(c) General antiderivative of h: $3x^{4/3} - \frac{4}{3}x^{-3/2} + C$

(6) A particle is moving along the x-axis so that its acceleration in $\frac{ft}{s^2}$ is given by $a(t) = 6t+3$ after t seconds. After 1 second the velocity of the particle is zero, and the particle is 3 feet from the origin in the positive direction after 2 seconds.

(a) Find the velocity $v(t)$ after t seconds.

(b) Find the position $x(t)$ after t seconds.

Solution

(a) Since $v'(t) = a(t) = 6t + 3$ we get using anti-derivatives

$$v(t) = 3t^2 + 3t + C$$

Using $v(1) = 0$ we get

$$3 + 3 + C = 0, \quad C = -6.$$

Hence,

$$v(t) = 3t^2 + 3t - 6$$

(b) Since $x'(t) = v(t) = 3t^2 + 3t - 6$ we get using anti-derivatives

$$x(t) = t^3 + \frac{3}{2}t^2 - 6t + D$$

Using $x(2) = 3$ we get

$$8 + 6 - 12 + D = 3, \quad D = 1$$

Hence,

$$x(t) = t^3 + \frac{3}{2}t^2 - 6t + 1$$

(a) The velocity in feet per second is $v(t) = 3t^2 + 3t - 6$

(b) The position in feet is $x(t) = t^3 + \frac{3}{2}t^2 - 6t + 1$

(7) (a) Sketch the region (in particular, specify the function and the interval you are using), whose area is given by $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{(1+\frac{i}{n})^2}$.

(b) Calculate the following definite integral by first interpreting it using areas and then applying area formulas from geometry: $\int_0^1 (3x-2)dx$.

Solution

(a) Using right endpoints, the Riemann sum R_n becomes

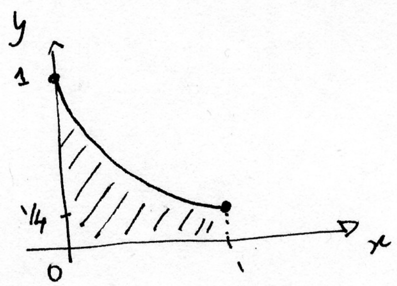
$$R_n = \Delta x \sum_{i=1}^{n} f(a+i\Delta x) = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i).$$

Hence, if we take

$$\Delta x = \frac{b-a}{n} = \frac{1}{n}, \quad x_i = a + i\Delta x = \frac{i}{n}, \quad f(x_i) = \frac{1}{(1+\frac{i}{n})^2} = \frac{1}{(1+x_i)^2}$$

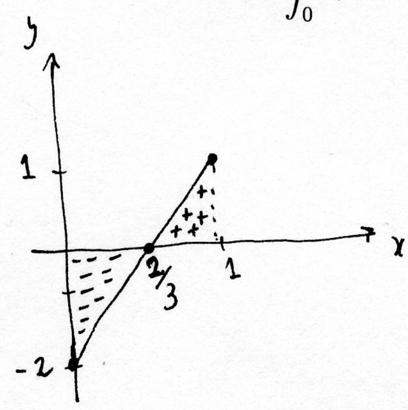
then we get the interval $[a,b]$ and the function $f(x)$:

$$a=0, \quad b=1, \quad f(x) = \frac{1}{(1+x)^2}.$$



(b) The graph $y = 3x-2$ is a straight line from $(0,-2)$ to $(1,1)$ intersecting the x-axis at $x = 2/3$. Thus, the definite integral is the area of a triangle with base $1/3$ and height 1 from which we subtract the area of a triangle with base $2/3$ and height 2:

$$\int_0^1 (3x-2)dx = \frac{1}{2}(1/3)(1) - \frac{1}{2}(2/3)(2) = -\frac{1}{2}.$$



(a) Function used: $f(x) = \frac{1}{(1+x)^2}$ Interval: $[0,1]$

(b) $-\frac{1}{2}$

(8) Find the two nonnegative numbers x and y whose sum is 9 such that x^2y+24x is maximal. Carefully justify your answer.

Solution

We are given $x+y=9$, thus $y=9-x$ where $0 \le x \le 9$ and hence,

$$x^2y+24x = x^2(9-x)+24x = 9x^2-x^3+24x = f(x)$$
$$f'(x) = 18x-3x^2+24 = 3(6x-x^2+8) = 0$$

The critical number in the interval $(0,9)$ is given by the quadratic formula

$$x = 3+\sqrt{17}$$

because $3-\sqrt{17} < 0$. Evaluate f at the critical number

$$f(3+\sqrt{17}) = 266.18...$$

Evaluate f at the endpoints of the interval $[0,9]$:

$$f(0) = 0$$

$$f(9) = 216$$

By the closed interval method we conclude that the maximum value of x^2y+24x is obtained for $x=3+\sqrt{17}$ and $y=6-\sqrt{17}$.
Alternatively, $f'(0)=24>0$ and $f'(9)=-57<0$ imply that f' is positive on $(3-\sqrt{17}, 3+\sqrt{17})$ and negative on $(3+\sqrt{17}, \infty)$. Hence, $f(3+\sqrt{17})$ is an absolute maximum on $[0,9]$.

$x=3+\sqrt{17}$ $\quad y=6-\sqrt{17}$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(9) (a) State the Mean Value Theorem. Use complete sentences.

(b) Is the Mean Value Theorem applicable to the function $f(x) = |x^2 - 9|$ on the interval $[0, 4]$? Justify your answer.

(c) Use the Mean Value Theorem to explain why one should not drive 9 miles in 15 minutes on a street where the speed limit is 35 miles per hour.

Solution

(a) If f is continuous on $[a, b]$ and differentiable on (a, b) then there is c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(b) The function $f(x) = |x^2 - 9|$ is not differentiable at $x = 3$ in $(0, 4)$. So f is not differentiable on $(0, 4)$. Hence, the theorem does not apply.

(c) The theorem says that the average speed $9/(1/4) = 36$ mph (15 minutes equal $1/4$ of an hour) equals the instantaneous speed at some instant during the trip. So the driver is violating the speed limit of 35 mph.

(10) Let $ABCD$ be a square of sidelength 1 mile. Find a point P on the side AB such that the time traveling from A to C via the line segments AP and PC is minimized, assuming the speed limit on AP is 1/2 mile per minute and on PC is 1/3 mile per minute (see the sketch). Specify the point P by giving its distance to B. Carefully justify why your answer is correct by using calculus.

Solution

Let $|PB| = x$ where $0 \le x \le 1$. By the Pythagorean theorem, we get

$$|PC| = \sqrt{1 + x^2}$$

and since P is between A and B we get

$$|AP| = 1 - x.$$

Hence, the time needed to travel along APC is

$$t = \frac{|AP|}{1/2} + \frac{|PC|}{1/3} = 2(1 - x) + 3\sqrt{x^2 + 1} = f(x).$$

$$f'(x) = -2 + \frac{3x}{\sqrt{x^2+1}} = 0, \quad \frac{3x}{\sqrt{x^2+1}} = 2,$$

$$9x^2 = 4x^2 + 4, \quad 5x^2 = 4, \quad x = \pm 2/\sqrt{5}.$$

The critical number in $(0, 1)$ is $x = 2/\sqrt{5}$. Evaluate f at the critical number:

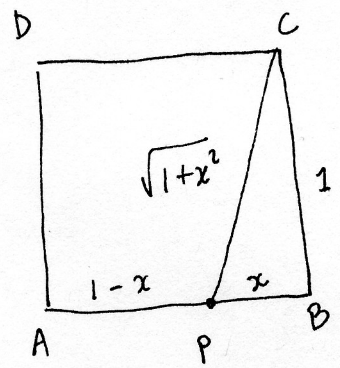
$$f(2/\sqrt{5}) = 4.23...$$

Evaluate f at the endpoints of the interval $[0, 1]$:

$$f(0) = 5$$

$$f(1) = 4.24...$$

By the closed interval method, the point P is $2/\sqrt{5}$ miles apart from B.
Alternatively, $f'(0) = -2 < 0$ and $f'(1) = 0.12... > 0$ imply that f' is negative on $(-2/\sqrt{5}, 2/\sqrt{5})$ and positive on $(2/\sqrt{5}, \infty)$. Hence, $f(2/\sqrt{5})$ is an absolute minimum on $[0, 1]$.



The distance $|PB| = 2/\sqrt{5} = 0.89...$ miles.

(11) Sketch the graph of a function $f(x)$ defined for $x > 0$ such that

(a) $\lim_{x\to 0+} f(x) = 3$,

(b) $f(2) = f(4) = 2$, $f(3) = 4$,

(c) $\lim_{x\to\infty} f(x) = f(1) = 1$,

(d) $f''(x)$ exists and is continuous for all $x > 0$,

(e) $f'(1) = f'(3) = f''(2) = f''(4) = 0$, and $f'(x)$ and $f''(x)$ are not zero for all other values of x.

Label all the important points on the graph by their x- and y-values and sketch the graph such that the intervals of increase/decrease and of the same concavity can be read off.

Solution

On $(0, 1)$ the derivative f' is either positive or negative. If f' is negative then f must decrease from $\lim_{x\to 0} f(x) = 3$ to $f(1) = 1$ but this is impossible. So the derivative must be negative. On $(1, 2)$ the derivative f' must be positive by a similar argument. In particular f has a local minimum at 1.
On $(0, 2)$ the second derivative f'' is either positive or negative. If f'' is negative then $f''(1) < 0$ would imply that f has a local maximum at 1 by the 2nd derivative test. This is impossible since f has a local minimum at 1 by the previous discussion. So the second derivative must be positive. Similar arguments work on the other intervals.
In conclusion, f is not defined at 0, decreasing on $(0, 1)$ and $(3, \infty)$, increasing on $(1, 3)$, concave up on $(0, 2)$ and $(4, \infty)$ and concave down on $(2, 4)$. Also, the graph of f is approaching the horizontal asymptote $y = 1$ at infinity.