Exam 4

Solutions

Multiple Choice Questions

1. Assuming that $\int_0^5 f(x)\,dx = 5$ and $\int_0^5 g(x) = 12$, find

$$\int_0^5 \left(3f(x) - \frac{1}{3}g(x)\right)\,dx.$$

   A. 19
   B. 11
   C. 15
   D. 17
   E. 4

Solution:

$$\int_0^5 \left(3f(x) - \frac{1}{3}g(x)\right)\,dx = 3\int_0^5 f(x)\,dx - \frac{1}{3}\int_0^5 g(x)\,dx = 3\cdot 5 - \frac{1}{3}\cdot 12 = 11$$

2. Find

$$\int_3^0 (4t^3 - 2t^2)\,dt.$$

   A. 63
   B. 90
   C. -54
   D. -63
   E. -90

Solution:

$$\int_3^0 (4t^3 - 2t^2)\,dt = -\int_0^3 4t^3 - 2t^2\,dt = -\left(t^4 - \frac{2}{3}t^3\right]_0^3 = -63$$

3. The traffic flow rate past a certain point on a highway is $f(t) = 1500 + 2500t - 270t^2$ where t is in hours and $t = 0$ is 8 AM. How many cars pass by during the time interval from 8 to 11 AM?

   A. 13320

   B. 2040

   C. 47960

   D. 34640

   E. 45920

Solution:

$$\int_0^3 (1500 + 2500t - 270t^2)\,dt = \left(1500t + 1250t^2 - 90t^3\right]_0^3 = 13,320$$

4. Find the equation of the tangent line to $f(x) = xe^x + \cos x$ at $x = 0$.

   A. $y = x - 1$

   B. $y = 1 - x$

   C. $y = -1 - x$

   D. $y = 1 + x$

   E. None of these.

Solution: The slope of the tangent line is the derivative of the function at $x = 0$, and the equation of the tangent line is $y = f(0) + f'(0)(x - 0)$. So, $f(0) = 0 + 1 = 1$. $f'(x) = e^x + xe^x - \sin x$ and $f'(0) = 1$. Thus, the equation of the tangent line to $f(x)$ at $x = 0$ is $y = 1 + x$.

5. Let $g(x)$ be a differentiable function such that $\lim_{x\to 0} g(x) = 1$ and $\lim_{x\to 0} g'(x) = 4$. Compute the limit

$$\lim_{x\to 0} \frac{x\cos x}{g(x)-1}.$$

A. 0

B. $\frac{1}{4}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

E. ∞

Solution: Since $\lim_{x\to 0} \frac{x\cos x}{g(x)-1} = \frac{0}{0}$, we have an indeterminant form of the form $\frac{0}{0}$, so we can use l'Hospital's Rule and

$$\lim_{x\to 0} \frac{x\cos x}{g(x)-1} = \lim_{x\to 0} \frac{\cos x - x\sin x}{g'(x)} = \frac{1}{4}$$

6. Find the sixth-degree Taylor polynomial approximation of $\cos x$ centered at $a = 0$. Recall that the N-th degree Taylor polynomial for $f(x)$ at $x = a$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A. $x - \frac{x^3}{6} + \frac{x^5}{120}$

B. $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$

C. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$

D. $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$

E. None of the above

Solution: We need to find the first six derivatives of $f(x) = \cos x$ and then evaluate these at $a = 0$.

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\ f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -1 \end{aligned}$$

So, by what we are given and what we just computed,

$$\begin{aligned} T_6(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \end{aligned}$$

7. Suppose $G(x) = \int_1^{x^2} t \ln(t)\, dt$. Find $G'(2)$.

   A. $2\ln(2)$
   B. $4\ln(4)$
   C. $1 + \ln(2)$
   D. $8\ln(4) - \frac{15}{4}$
   E. $16\ln(4)$

Solution: By the Fundamental Theorem of Calculus and the Chain Rule, $G'(x) = \left(x^2 \ln(x^2)\right)(2x)$, so $G'(2) = 16\ln(4)$.

8. Find

$$\int_0^1 \frac{e^x + 1}{e^x + x}\,dx.$$

A. 1

B. -1

C. $\dfrac{1}{e+1} - 1$

D. $\dfrac{1}{(e+1)^2}$

E. $\ln(e+1)$

Solution: This is done by substitution. Let $u = e^x + x$, then $du = (e^x + 1)dx$, $u = 1$ when $x = 0$, and $u = e + 1$ when $x = 1$.

$$\int_0^1 \frac{e^x + 1}{e^x + x}\,dx = \int_1^{e+1} \frac{1}{u}\,du = \ln(u)|_1^{e+1} = \ln(e+1)$$

9. The linearization for $f(x) = \sqrt{x+3}$ at $x = 1$ is

A. $L(x) = 2 + \frac{1}{2}(x-1)$

B. $L(x) = 4 + \frac{1}{4}(x-1)$

C. $L(x) = 2 + \frac{1}{4}(x-1)$

D. $L(x) = 4 + \frac{1}{8}(x-1)$

E. $L(x) = 2 + \frac{1}{4}x - 1$

Solution: Recall that the linearization of a function at a point is just the equation of the tangent line to the curve at that point. Thus, we need $f(1)$ and $f'(1)$. $f(1) = 2$ and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$ so $f'(1) = \frac{1}{4}$. Then the linearization at $x = 1$ is $L(x) = 2 + \frac{1}{4}(x-1)$.

10. Find the slope of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2, 1)$.

A. $\frac{3}{4}$

B. $\frac{3}{2}$

C. $\frac{4}{3}$

D. 0

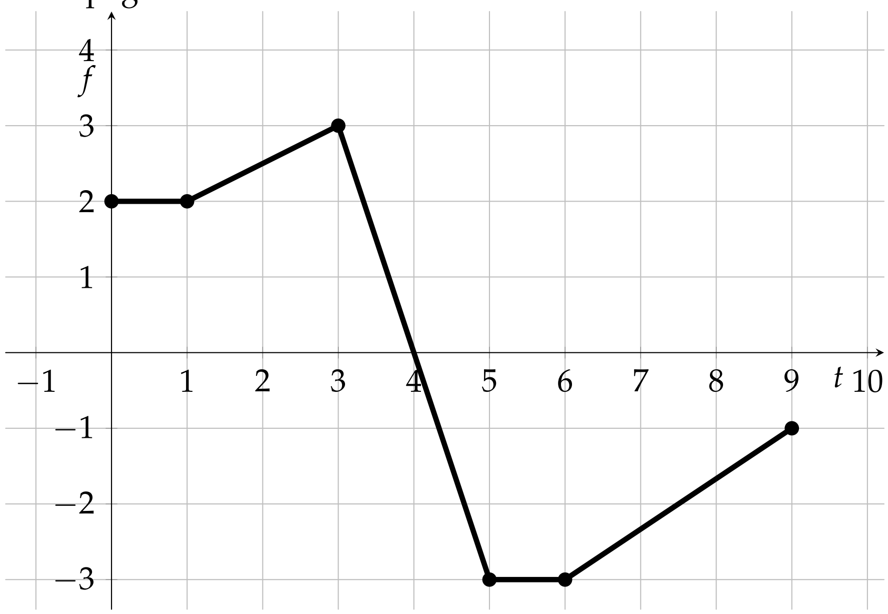
E. $-\frac{3}{4}$

Solution: Use implicit differentiation to find the derivative, $\frac{dy}{dx}$.

$$\begin{aligned} x^2 - xy - y^2 &= 1 \\ 2x - \left(y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 2x - y \\ \frac{dy}{dx} &= \frac{2x - y}{x + 2y} \end{aligned}$$

So, at $x = 2$ and $y = 1$, this gives us that $\frac{dy}{dx} = \frac{3}{4}$.

Let $F(x) = \int_0^x f(t)\,dt$ for x in the interval $[0,9]$, where $f(t)$ is the function with the graph shown in the following picture. This function F will be used for Problems 11 and 12 on this page.



11. Find $F(6)$.

   A. -3

   B. 3

   C. 4

   D. 12

   E. 13

12. Which of the following statements is true?

   A. F is decreasing on the interval $[3,5]$.

   B. F is increasing on the interval $[6,9]$.

   C. F has a local minimum at 5.

   D. F has a local maximum at 4.

   E. $F(5) = F(6)$.

Free Response Questions
Show all of your work

13. Compute the following antiderivatives. These are also called indefinite integrals.

(a) $\displaystyle\int \frac{2e^x}{(5e^x+3)^4}\,dx$

Solution: Let $u = 5e^x + 3$, then $du = 5e^x\,dx$. Then,

$$\int \frac{2e^x}{(5e^x+3)^4}\,dx = \frac{2}{5}\int u^{-4}\,du = -\frac{2}{15}u^{-3} + C = -\frac{2}{15}(5e^x+3)^{-3} + C$$

(b) $\displaystyle\int x\sqrt{x+1}\,dx$

Solution: Let $u = x+1$, then $du = dx$ and $x = u-1$. Then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du = \int (u^{3/2} - u^{1/2})\,du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C\end{aligned}$$

(c) $\displaystyle\int \frac{\cos(x)}{(1+\sin(x))^3}\,dx$

Solution: Let $u = 1 + \sin x$, then $du = \cos x\,dx$.

$$\int \frac{\cos x}{(1+\sin x)^3}\,dx = \int \frac{1}{u^3}\,du = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C = -\frac{1}{2}(1+\sin x)^{-2} + C$$

14. (a) What does it mean for a function $f(x)$ to be continuous at a point $x = a$? Use complete sentences.

Solution: A function $f(x)$ is continuous at a point $x = a$ if the following three conditions hold:
(1) $f(a)$ exists; i.e., the function has a value at $x = a$;
(2) $\lim_{x\to a} f(x)$ exists; i.e., the function has a limit at $x = a$;
(3) $\lim_{x\to a} f(x) = f(a)$; i.e., the limit at $x = a$ and the functional value $f(a)$ are the same.

(b) Consider the piecewise defined function

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } 0 \le x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } 3 \le x \le 5 \end{cases}$$

where a and b are constants. Find the values of a and b for which $f(x)$ is continuous on $[0,5]$.

Solution: We need to check for continuity at $x = 2$ and at $x = 3$. Elsewhere, the function is continuous.

At $x = 2$ we need the left-hand limit to equal the right-hand limit.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 - bx + 3 = 4a - 2b + 3$$

Thus, $4a - 2b + 3 = 4$ or $4a - 2b = 1$.

At $x = 3$ we need the left-hand limit to equal the right-hand limit.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} ax^2 - bx + 3 = 9a - 3b + 3$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2x - a + b = 6 - a + b$$

Thus, $9a - 3b + 3 = 6 - a + b$ or $10a - 4b = 3$.

Solving the system of equations:

$$4a - 2b = 1$$
$$10a - 4b = 3$$

gives $a = \dfrac{1}{2}$ and $b = \dfrac{1}{2}$.

15. Let a and b be positive numbers whose product is 8. Find the minimum value of $a^2 + 2b$. Determine the values of a and b for which the minimum is attained.

Solution: We are given that $ab = 8$ and we want to minimize $a^2 + 2b$. Solving the first equation for b gives $b = \dfrac{8}{a}$. Plugging this into the second equation we have a function of one variable, $f(a) = a^2 + \frac{16}{a}$. We need to find the critical points for f.

$$f(a) = a^2 + \frac{16}{a}$$
$$f'(a) = 2a - \frac{16}{a^2}$$

Setting $f'(a) = 0$ gives us that $a = 2$ as the single critical point for f. The second derivative for f is

$$f''(a) = 2 + \frac{32}{a^3}.$$

For a positive, the second derivative is always positive and hence concave up. This gives us an absolute minimum at $a = 2$. Thus, the values for which the minimum is attained are $a = 2$ and $b = 4$.

16. A particle moves in a straight line so that its velocity at time t seconds is $v(t) = 12 - 2t$ feet per second.

(a) Find the displacement of the particle over the time interval $[0, 8]$.

Solution:

$$\text{displacement} = \int_0^8 12 - 2t\,dt = 12t - t^2\Big|_0^8 = 96 - 64 = 32 \text{ feet}$$

(b) Find the total distance traveled by the particle over the time interval $[0, 8]$.

Solution: $12 - 2t = 0$ at $t = 6$ so the function changes sign in the interval. Thus, the distance traveled is given by

$$\begin{aligned}\text{distance} &= \int_0^6 12 - 2t\,dt + \left|\int_6^8 12 - 2t\,dt\right| \\ &= 12t - t^2\Big|_0^6 + \left|12t - t^2\right|_6^8 \\ &= 36 + |96 - 64 - 72 + 36| \\ &= 36 + 4 \\ &= 40 \text{ feet}\end{aligned}$$