Exam 4

Form A

Multiple Choice Questions

1. If $\lim_{x \to 2} f(x) = 3$ and $\lim_{x \to 2} g(x) = 1$, then find the value of $\lim_{x \to 2} \frac{f(x) + g(x)}{\sqrt{g^2(x) + 3}}$.

   A. 1

   B. 2

   C. 3

   D. 4

   E. 5

2. Find the value of $\lim_{x \to \infty} \arcsin\left(\frac{x^2 - 3x + 5}{2x^2 + x + 9}\right)$.

   A. $-\pi/2$

   B. 0

   C. $\pi/6$

   D. $\pi/3$

   E. $\pi/2$

3. Find the values of A and B such that the function $f(x) = \begin{cases} -2x^2+5, & x \leq -1 \\ Ax+B, & -1 < x < 2 \\ 3x^2-3, & 2 \leq x \end{cases}$

   is continuous.

   A. $A = 0,\ B = 9$

   B. $A = 1,\ B = 4$

   C. $A = 1,\ B = 8$

   D. $A = 2,\ B = 5$

   E. $A = 2,\ B = 7$

4. By Intermediate Value Theorem, which interval contains a solution of the equation $2^x + 3 = 3^x$?

   A. $[-1, 0]$

   B. $[0, 1]$

   C. $[1, 2]$

   D. $[2, 3]$

   E. $[3, 4]$

5. Find the derivative of $f(x) = \dfrac{x^2+x}{e^x}$.

   A. $f'(x) = e^{-x}(-x^2+x+1)$.

   B. $f'(x) = \dfrac{2x+1}{e^x}$

   C. $f'(x) = \dfrac{x^2-x-1}{e^x}$

   D. $f'(x) = \dfrac{2x+1}{xe^{x-1}}$

   E. $f'(x) = 2xe^{-x} + x^2e^{-x} + 1$

6. Find the equation of the tangent line to the curve $x^2 + xy - y^2 + 1 = 0$ at the point $(1, 2)$.

   A. $y = -10x + 12$

   B. $y = -\dfrac{4}{5}x + \dfrac{14}{5}$

   C. $y = \dfrac{2}{3}x + \dfrac{4}{3}$

   D. $y = \dfrac{4}{3}x + \dfrac{2}{3}$

   E. $y = \dfrac{5}{3}x + \dfrac{1}{3}$

7. Find the derivative of $f(x) = \ln(x^4 + 2x)$, where $x > 0$.

   A. $f'(x) = \dfrac{1}{x^4 + 2x}$

   B. $f'(x) = \dfrac{4x^3 + 2}{x^4 + 2x}$.

   C. $f'(x) = \dfrac{1}{4x^3 + 2}$

   D. $f'(x) = \ln(4x^3 + 2)$

   E. $f'(x) = \dfrac{x^4 + 2x}{4x^3 + 2}$

8. Let $f(x) = |x^2 - 4x + 3|$. Find the point(s) c so that $f'(c)$ does not exist.

   A. $c = 2$

   B. $c = 1$ and $c = 2$

   C. $c = 1$ and $c = 3$

   D. $c = 2$ and $c = 3$

   E. $c = 1$, $c = 2$, and $c = 3$

9. Find a function F which is an anti-derivative of x^3 and satisfies $F(1) = 1$.

   A. $F(x) = 3x^2 - 2$

   B. $F(x) = x^4$

   C. $F(x) = \frac{x^3}{3} + \frac{2}{3}$

   D. $F(x) = x^3$

   E. $F(x) = \frac{x^4}{4} + \frac{3}{4}$

10. Suppose that f is a differentiable function on $(0, 4)$, that $f'(x) > 0$ for x in each of the intervals $(0, 1)$, $(1, 2)$ and $(3, 4)$ and that $f'(x) < 0$ on the interval $(2, 3)$.

    Select the correct statement.

    A. f has a local minimum at 1 and no local maximum.

    B. f has a local minimum at 2 and a local maximum at 3.

    C. f has a local maximum at 2 and a local minimum at 3.

    D. f has local minima at 1 and 3 and a local maximum at 2.

    E. f has local maxima at 1 and 3 and a local minimum at 2.

11. The side length ℓ of a square is increasing. When $\ell = 5$, the derivative $\frac{d\ell}{dt} = 6$. Let A be the area of the square and find $\frac{dA}{dt}$ when $\ell = 5$.

    A. 25

    B. 30

    C. 50

    D. 60

    E. 72

12. Use L'Hôpital's rule to find the limit $\lim_{x \to 0} \frac{\cos(3x) - 1}{x^2}$.

    A. $-9/2$

    B. $-3/2$

    C. $+3/2$

    D. $+3$

    E. $+9/2$

13. $\int_0^{\pi/4} \sin^5 x \cos x \, dx = \frac{1}{a}$ where $a =$ ________

A. 16

B. 24

C. 48

D. 49

E. 50

14. $\int_1^5 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx =$

A. $12 + \ln(5)$

B. $16 + \ln(5)$

C. $18 + \ln(5)$

D. $20 + \ln(5)$

E. $22 - \ln(5)$

15. If $f(x) = \int_2^{x^2} \cos\sqrt{t}\,dt$ and $x \geq 0$, find $f'(x)$.

A. $f'(x) = \cos(x)$

B. $f'(x) = 2x\sin(x)$

C. $f'(x) = 2x\cos(x)$.

D. $f'(x) = \sin(x)$

E. $f'(x) = \cos(x) - 2$

16. Find the linearization, $L(x)$, of the function $f(x) = \dfrac{2}{1+e^x}$ at $a = 0$.

A. $L(x) = 1 - \frac{1}{4}x$

B. $L(x) = 1 + \frac{1}{2}x$

C. $L(x) = 1 - \frac{1}{2}x$

D. $L(x) = 1 + x$

E. $L(x) = 1 - x$

Free Response Questions
Show all of your work

17. The tangent line to the graph of a function f at the point $(1, 2)$ is $y = 3x - 1$.

(a) What is $f(1)$?

Solution: $f(1) = 2$

(b) What is $f'(1)$?

Solution: $f'(1) = 3$

(c) If $g(x) = f(x^3)$, then find $g'(1)$. Show your work.

Solution:

$$g(x) = f(x^3)$$
$$g'(x) = f'(x^3) \times 3x^2$$
$$g'(1) = f'(1) \times 3(1^2)$$
$$g'(1) = 3 \times 3 = 9$$

18. Find the ***third*** derivatives of the following two functions. Show your work.

(a) $f(x) = \sin(x) + \cos(x)$

Solution:

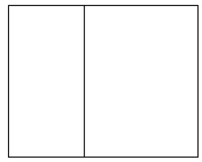
$$\begin{aligned} f(x) &= \sin(x) + \cos(x) \\ f'(x) &= \cos(x) - \sin(x) \\ f''(x) &= -\sin(x) - \cos(x) \\ f'''(x) &= -\cos(x) + \sin(x) \end{aligned}$$

(b) $g(x) = x^2 \ln(x)$

Solution:

$$\begin{aligned} g(x) &= x^2 \ln(x) \\ g'(x) &= 2x \ln(x) + x^2 \times \frac{1}{x} = 2x \ln(x) + x \\ g''(x) &= 2 \ln(x) + 2x \times \frac{1}{x} + 1 = 2 \ln(x) + 3 \\ g'''(x) &= \frac{2}{x} \end{aligned}$$

19. We have 240 meters of fencing and form a rectangular pen that is divided in two by a fence parallel to two of the sides. Find the area and dimensions of the pen which encloses the largest area.



Solution: If the horizontal dimension of the large rectangle is x and the vertical dimension is y, then the total amount of fencing using is $2x + 3y$.

Since the total amount of fencing is 240 meters, we have $2x + 3y = 240$. Solving for x gives $x = 120 - 3y/2$.

The area is $A = xy$ and we can eliminate y to write this in terms of x as $A(y) = y(120 - 3y/2) = 120y - 3y^2/2$ and we have $0 \leq y \leq 80$ since $y \geq 0$ and $x \geq 0$.

The derivative is $A'(y) = 120 - 3y$ and find a critical number at $y = 40$. The maximum value of A is 2400 and occurs when $y = 40$ and $x = 60$.

The maximum area that can be enclosed is 2400 meters^2 and the dimensions of the largest pen are 60 meters and 40 meters.

20. A particle is moving along a line with acceleration function $a(t) = 2t + 3$ meters/second^2 and initial velocity $v(0) = -4$ meters/second.

(a) Find the velocity at time t. Show your work.

Solution: Since acceleration is the derivative of velocity, we have that $v(t) = \int a(t)\,dt = \int 2t + 3\,dt$ with $v(0) = -4$.

$$\begin{aligned} v(t) &= \int (2t+3)\,dt \qquad v(0) = -4 \\ &= t^2 + 3t + C \\ -4 = v(0) &= 0^2 + 3 \times 0 + C \Rightarrow C = -4 \text{ so} \\ v(t) &= t^2 + 3t - 4 \end{aligned}$$

(b) Find the displacement of the particle over the time interval $0 \le t \le 3$. Show your work.

Solution: The displacement is given by $\int_0^3 v(t)\,dt$, so

$$\begin{aligned} \text{Displacement} &= \int_0^3 t^2 + 3t - 4\,dt \\ &= \tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t\Big|_0^3 \\ &= \frac{21}{2} \text{ meters} \end{aligned}$$

(c) Find the total distance traveled during the time interval $0 \le t \le 3$. Show your work.

Solution: The velocity is 0 when $t = 1$ and when $t = -4$. The time $t = -4$ is not in the interval in question, but $t = 1$ is and the particle changes direction at $t = 1$. Thus, the distance traveled by the particle is given by:

$$\begin{aligned} \text{Distance} &= \left|\int_0^1 v(t)\,dt\right| + \left|\int_1^3 v(t)\,dt\right| \\ &= \left|\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t\Big|_0^1\right| + \left|\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t\Big|_1^3\right| \\ &= \left|-\frac{13}{6}\right| + \left|\frac{38}{3}\right| \\ &= \frac{89}{6} \text{ meters} \end{aligned}$$